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PROPERTIES AND APPLICATIONS OF A TWO-PARAMETER INVERSE EXPONENTIAL DISTRIBUTION WITH A DECREASING FAILURE RATE

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ABSTRACT

This research introduces a two-parameter generalization of the inverse exponential distribution known as "odd Lindley-Inverse Exponential distribution" with a decreasing failure rate and greater flexibility for survival modeling of random events better than both exponential and inverse exponential distributions. The article derived and discussed some mathematical and statistical properties of the new distribution which include ordinary moments, quantile and generating functions, survival and hazard functions, order statistics and other useful measures. We discuss the maximum likelihood estimation of the model parameters and provide a simulation study to examine the behavior of different estimates under varying sample sizes and initial parameter values. We also illustrate the importance of the proposed distribution over the Lindley, Exponential and inverse exponential distributions by means of two real life datasets of different nature.

KEYWORDS

Inverse exponential; Lindley; Properties; Estimation; Simulation; Applications.

1. INTRODUCTION

The Exponential distribution is a continuous probability distribution considered in Poisson processes and modeling of the time between events. It is applied mostly in life testing experiments. It has memoryless property with a constant failure rate making it unfit for analyzing real life problems.

Besides the applications of Exponential distribution and its attractive properties, its usage has been limited in modeling real life situations due to the fact that it has a constant failure rate (Lemonte, 2013). One other problem with exponential distribution is in its memoryless property, this is because the memoryless assumption is hardly obtained in real life situations. To deal with these associated limitations, Keller and Kamath (1982) proposed another form of the Exponential distribution, called the Inverse Exponential distribution which has been discussed by Lin *et al.*, (1989).

The Inverse Exponential distribution was found adequate for modeling datasets with inverted bathtub failure rates (Keller and Kamath, 1982) but it also has a limitation which is its inability to efficiently analyze datasets that are highly skewed (Abouammoh and Alshingiti (2009)). This therefore gives room for introducing skewness and flexibility into the Inverse Exponential distribution to enable it adequately model heavily skewed datasets.

Nowadays, many families of probability distributions have been proposed in the literature and it has been shown that they are useful for adding skewness and flexibility to other models. Some of these families include the Weibull-X by Alzaatreh *et al*. (2013), the Exponentiated-G (EG) family by Cordeiro *et al*. (2013), the Exponentiated T-X family by Alzaghal *et al*. (2013), the Weibull-G family by Bourguignon *et al*. (2014), the Logistic-G family by Torabi and Montazari (2014), the Gamma-X family by Alzaatreh *et al*., (2014), the Lomax-G family by Cordeiro *et al.* (2014), a new generalized Weibull-G family by Cordeiro *et al.* (2015), Beta Marshall-Olkin family of distributions by Alizadeh *et al*. (2015), the Lindley-G family by Cakmakyapan and Ozel (2016), Logistic-X family by Tahir *et al*. (2016), a new Weibull-G family by Tahir *et al.* (2016), the odd Lindley-G family by Gomes-Silva et al., (2017), the Gompertz-G family by Alizadeh *et al*., (2017) and the odd Lomax generator of distributions (Odd Lomax-G family) by Cordeiro *et al.*, (2019).

In an attempt to make a much better model from the Inverse Exponential distribution, many authors have utilized the aforementioned families to proposed different extensions of the inverse exponential distribution and these among others include the transmuted inverse exponential distribution by Oguntunde and Adejumo (2015), the Weibull-Exponential distribution by Oguntunde *et al.*, (2015), the transmuted exponential distribution by Owoloko *et al.*, (2015), the odd generalized exponential-exponential distribution by Maiti and Pramanik (2015), a new Lindley-Exponential distribution by Oguntunde *et al.*, (2016), the transmuted Weibull-exponential distribution by Yahaya and Ieren (2017), the Exponential Inverse Exponential distribution by Oguntunde *et al*., (2017a), the Kumaraswamy Inverse Exponential distribution by Oguntunde *et al.*, (2017b), the exponentiated generalized Inverse Exponential distribution by Oguntunde *et al*., (2017c), the Lomax-exponential distribution by Ieren and Kuhe (2018), the transmuted odd generalized exponential-exponential distribution by Abdullahi *et al.*, (2018) and a transmuted Lindley-Exponential distribution by Umar *et al*., (2019).

In the work of Gomes-Silva et al., (2017), the odd Lindley-Weilbull distribution which is based on the odd Lindley-G family has been found to fit real dataset much better than other extensions of the Weibull distribution such as exponentiated Weibull distribution, beta Weibull distribution, Kumaraswamy Weibull distribution and the conventional Weibull distribution. Hence, our interest in this article is to develop a new extension of the Inverse exponential distribution using the odd Lindley-G family of probability distributions proposed by Gomes-Silva et al., (2017).

The cumulative distribution function (cdf) and probability density function (pdf) of the Inverse Exponential distribution (INExD) with scale parameter θ are respectively given as:

$$G(x) = e^{-\frac{\theta}{x}} \tag{1}$$

and

$$g(x)=\frac{\theta}{x^2}e^{-\frac{\theta}{x}} \tag{2}$$

where $x>0,\theta>0$ in which θ represents a scale parameter of the model.

The following arrangement is considered for remaining sections of this article, they are: proposition, validity check and plots of new distribution is done in section 2. Section 3 presents the proposed properties of the new distribution. Estimation and simulation for the unknown parameters of the new distribution under maximum likelihood estimation method is carried out in section 4 and section 5 respectively. The proposed distribution is being applied to two real life datasets together with other related distributions in section 6. Finally, a concise summary and conclusion is presented in section 7.

2. ODD LINDLEY INVERSE EXPONENTIAL DISTRIBUTION (OLINEXD)

2.1 Definition of the New Model

According to Gomes-Silva *et al*. (2017), the cdf and pdf of the Odd Lindley-G family of distributions are defined as:

$$F(x)=\int_{-\infty}^{\frac{G(x)}{1-G(x)}}\frac{\alpha^2}{\alpha+1}(1+t)e^{-\alpha t}dt=1-\frac{\alpha+(1-G(x))}{(1+\alpha)(1-G(x))}\exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]\right\} \tag{3}$$

and

$$f(x)=\frac{\alpha^2 g(x)}{(1+\alpha)(1-G(x))^3}\exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]\right\} \tag{4}$$

respectively, where $g(x)$ and $G(x)$ are the *pdf* and the *cdf* of any continuous distribution to be modified respectively and $\alpha>0$ is the shape parameter of the family responsible for additional skewness and flexibility in the modified model.

Making appropriate substitutions of (1) and (2) in (3) and (4) and using simple algebra in the result, the cdf and pdf of the OLINExD are obtained respectively as:

$$F(x)=1-\frac{\alpha+\left(1-e^{-\frac{\theta}{x}}\right)}{(1+\alpha)\left(1-e^{-\frac{\theta}{x}}\right)}\exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\} \tag{5}$$

and

$$f(x)=\frac{\alpha^2\theta x^{-2}e^{-\frac{\theta}{x}}}{(1+\alpha)\left[1-e^{-\frac{\theta}{x}}\right]^3}\exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\} \tag{6}$$

where, $x>0,\alpha>0,\theta>0$ α in which α represents the shape parameter of OLINExD and θ stands for the scale parameter of OLINExD.

2.3 Graphical representation of Pdf and Cdf of OLINExD

The pdf and cdf of the OLINExD are displayed in Figures 1 and 2 for some selected parameter values as follows.

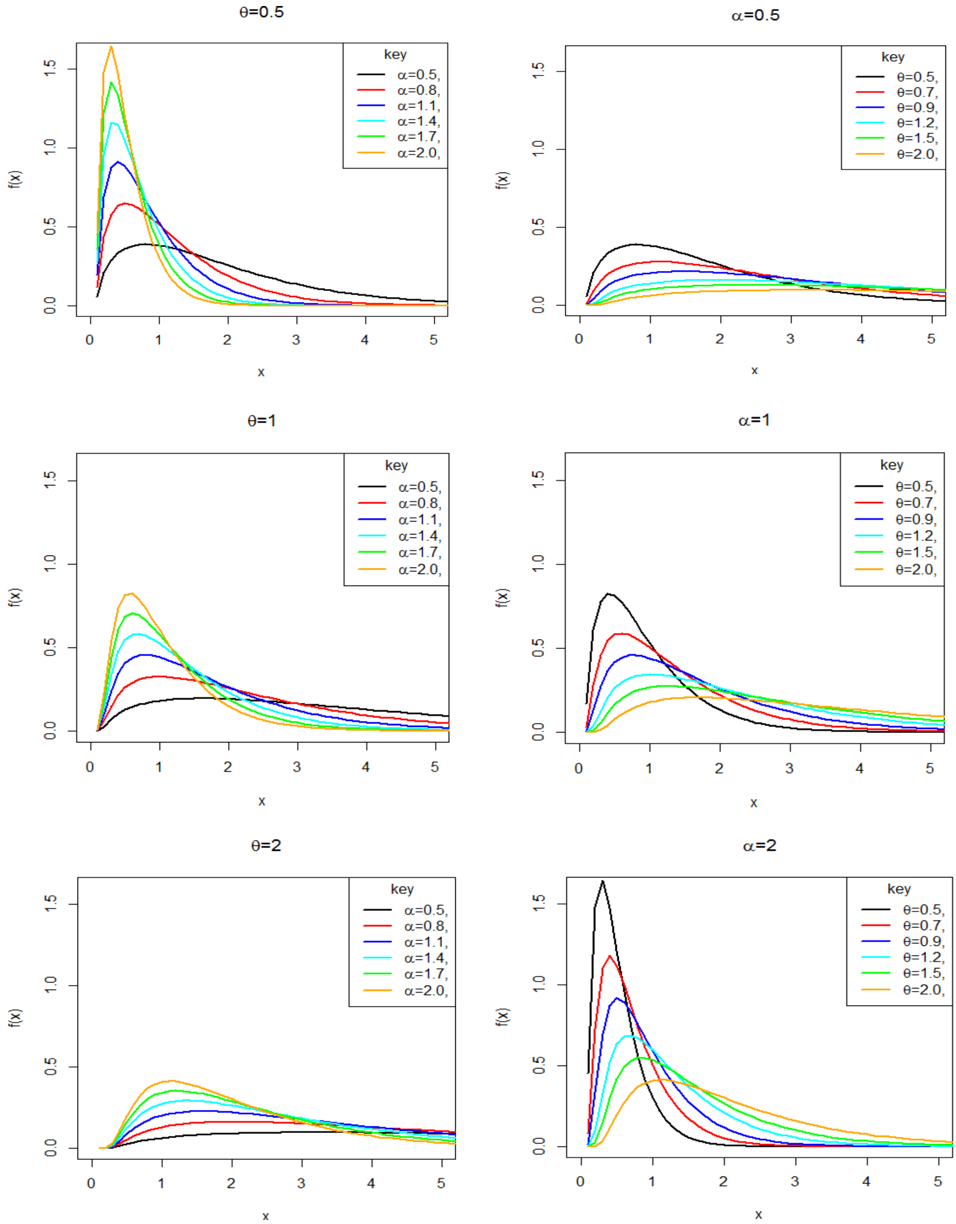


Figure 1: Plots of the PDF of the OLINExD for Selected Parameter Values

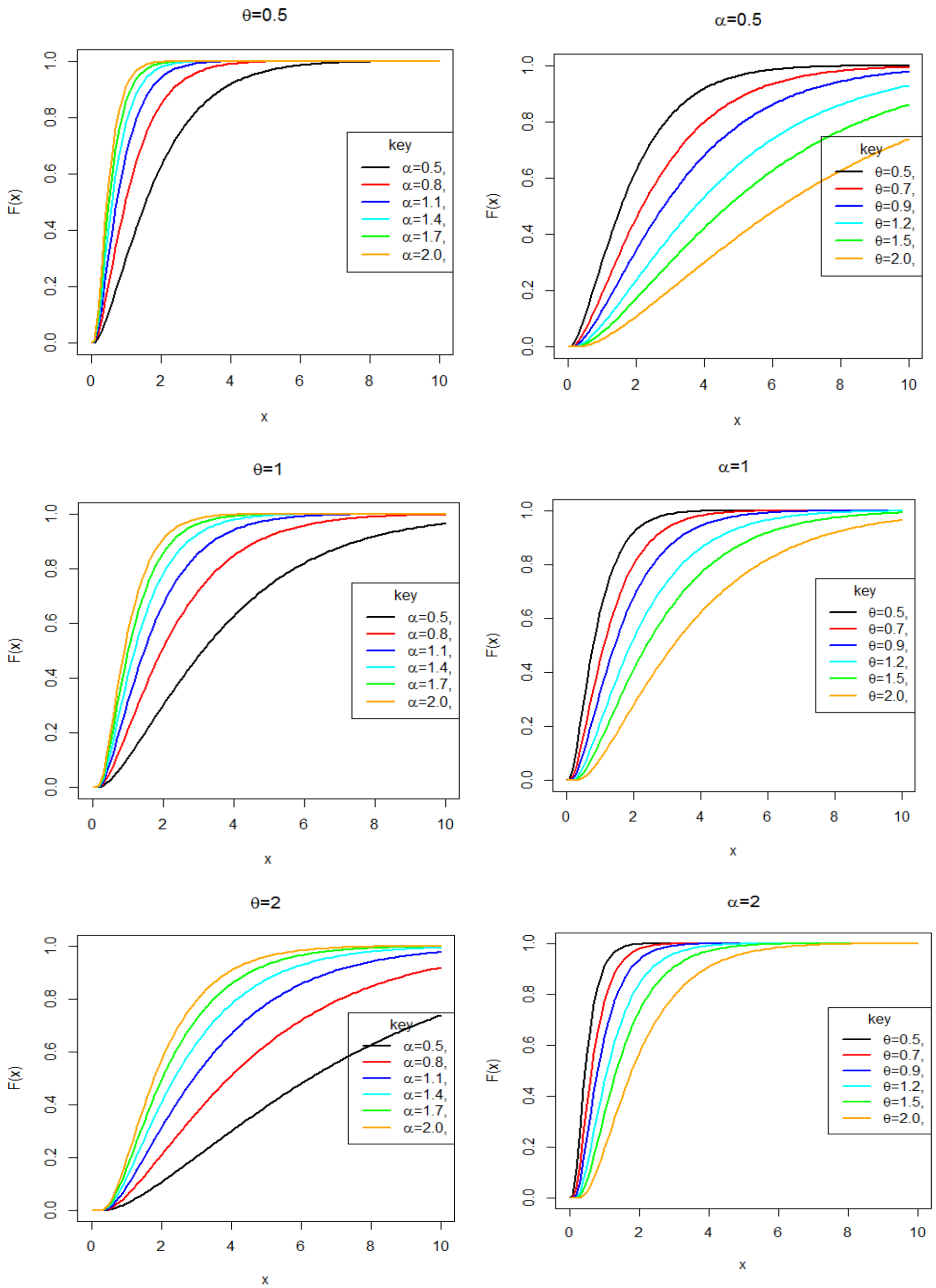


Figure 2: Plots of the CDF of the OLINExD for Selected Parameter Values

3. MATHEMATICAL AND STATISTICAL PROPERTIES OF OLINEXD

This section presents some properties of the OLINExD distribution. They include the following:

3.1 Moments

Let X denote a continuous random variable, the n^{th} moment of X is given by;

$$\mu_n^{'} = E\left(X^n\right) = \int_0^\infty x^n f(x)dx \tag{7}$$

where f(x) the pdf of the OLINExD and is stated from (6) as:

$$f(x) = \frac{\alpha^2 \theta x^{-2} e^{-\frac{\theta}{x}}}{(1+\alpha)\left[1-e^{-\frac{\theta}{x}}\right]^3} \exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\} \tag{8}$$

Prior to substitution in (8), the expansion and simplification of the pdf is done as follows:

First, by expanding the exponential term in (8) using power series, it gives:

$$\exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\} = \sum_{i=0}^{\infty} \frac{(-1)^i \alpha^i}{i!}\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)^i \tag{9}$$

Using (9) above and simplifying, (8) becomes

$$f(x) = \sum_{i=0}^{\infty} \frac{\alpha^{2+i}\theta}{i!(1+\alpha)} x^{-2} e^{-\frac{\theta}{x}(i+1)} \left[1-e^{-\frac{\theta}{x}}\right]^{-(i+3)} \tag{10}$$

Also, using the generalized binomial theorem, the last term from the above result can be written as:

$$\left[1-e^{-\frac{\theta}{x}}\right]^{-(i+3)} = \sum_{j=0}^{\infty} \frac{\Gamma(j+i+3)}{j!\Gamma(i+3)} e^{-\frac{\theta}{x}j} \tag{11}$$

Using (11) above in equation (10) and simplifying gives:

$$f(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\Gamma(j+i+3)(-1)^i \alpha^{i+2}\theta}{i!\,j!\,\Gamma(i+3)(1+\alpha)} x^{-2} \mathrm{e}^{-\frac{\theta}{x}(i+j+1)} \tag{12}$$

Consequently, the pdf in (12) can also be written in its simplest form as:

$$f(x) = W_{i,j} x^{-2} \mathrm{e}^{-\frac{\theta}{x}(i+j+1)} \tag{13}$$

where $W_{i,j} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\Gamma(j+i+3)(-1)^i \alpha^{i+2}\theta}{i!\,j!\,\Gamma(i+3)(1+\alpha)}$.

Now, using the linear form of the pdf of the OLINExD in equation (13), the n^{th}ordinary moment of the OLINExD is derived as follows:

$$\mu_n^{'} = E\left(X^n\right) = \int_0^\infty x^n f(x)dx = \int_0^\infty W_{i,j} x^{n-2} e^{-\frac{\theta}{x}(i+j+1)} dx \tag{14}$$

Making use of integration by substitution method in equation (14) leads to the following operations:

Let $u = \frac{\theta}{x}(i+j+1) \Rightarrow x = u^{-1}\theta(i+j+1)$ which implies that $\frac{du}{dx} = -\frac{\theta(i+j+1)}{x^2} \Rightarrow dx = -\frac{x^2 du}{\theta(i+j+1)}$.

Substituting for x, u and dx in equation (14) and simplifying; we have:

$$\mu_n^{'} = E\left(X^n\right) = W_{i,j}\left(\frac{(\theta(i+j+1))^n}{\theta(i+j+1)}\right)\int_0^\infty u^{-n} e^{-u} du = W_{i,j}\left(\frac{(\theta(i+j+1))^n}{\theta(i+j+1)}\right)\int_0^\infty u^{1-n-1} e^{-u} du \tag{15}$$

Hence, recall that $\int_0^\infty t^{k-1} e^{-t} dt = \Gamma(k)$ and that $\int_0^\infty t^k e^{-t} dt = \int_0^\infty t^{k+1-1} e^{-t} dt = \Gamma(k+1)$

Thus we obtain the n^{th} ordinary moment of X for the OLINExD as follows:

$$\mu_n^{'} = E\left(X^n\right) = \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{\Gamma(j+i+3)(-1)^{i+1}\alpha^{i+2}\theta\Gamma(1-n)}{i!\,j!\,\Gamma(i+3)(1+\alpha)\left[\theta(i+j+1)\right]^{1-n}} \tag{16}$$

Meanwhile, some measures such as variation, skewness and kurtosis could be derived from the non-central moments using familiar formulas.

3.2 Moment Generating Function

The moment generating function of a random variable X can be obtained as

$$M_x(t) = E\left[e^{tx}\right] = \int_{-\infty}^\infty e^{tx} f(x)dx \tag{17}$$

Recall that by power series expansion,

$$e^{tx} = \sum_{r=0}^\infty \frac{(tx)^r}{r!} = \sum_{r=0}^\infty \frac{t^r}{r!} x^r \tag{18}$$

Therefore, the moment generating function can also be expressed as:

$$M_x(t) = \sum_{r=0}^\infty \frac{t^r}{r!}\int_{-\infty}^\infty x^r f(x)dx = \sum_{r=0}^\infty \frac{t^r}{r!} E\left(X^r\right) = \sum_{r=0}^\infty \frac{t^r}{r!}\int_{-\infty}^\infty x^r f(x)dx$$

$$= \sum_{r=0}^\infty \frac{t^r}{r!} E\left(X^r\right) = \sum_{r=0}^\infty \frac{t^r}{r!}\left[\mu_r^{'}\right]$$

Using the result in (18) and simplifying the integral in (17) gives:

$$M_x(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma\left(j+i+3\right)\left(-1\right)^{i+1}\alpha^{i+2}\theta\Gamma\left(1-n\right)}{i!\,j!\,\Gamma\left(i+3\right)\left(1+\alpha\right)\left[\theta\left(i+j+1\right)\right]^{1-n}}\right] \tag{19}$$

3.3 Characteristics Function

A representation for the characteristics function is given by

$$\varphi_x(t)=E\left[e^{itx}\right]=E\left[\cos(tx)+i\sin(tx)\right]=E\left[\cos(tx)\right]+E\left[i\sin(tx)\right] \tag{20}$$

Recall from power series expansion that

$$\cos(tx)=\sum_{r=0}^{\infty}\frac{\left(-1\right)^r t^{2r}}{\left(2r\right)!}x^{2r}\ \text{ and } E\left[\cos(tx)\right]=\sum_{r=0}^{\infty}\frac{\left(-1\right)^r t^{2r}}{\left(2r\right)!}\mu'_{2r}$$

and also that

$$\sin(tx)=\sum_{r=0}^{\infty}\frac{\left(-1\right)^r t^{2r+1}}{\left(2r+1\right)!}x^{2r+1}\ \text{ and } E\left[\sin(tx)\right]=\sum_{r=0}^{\infty}\frac{\left(-1\right)^r t^{2r+1}}{\left(2r+1\right)!}\mu'_{2r+1}$$

Hence, simple algebra and use of power series expansion above produces the following results:

$$\phi_x(t)=\sum_{r=0}^{\infty}\frac{\left(-1\right)^r t^{2r}}{\left(2r\right)!}\mu'_{2r}+i\sum_{n=0}^{\infty}\frac{\left(-1\right)^r t^{2r+1}}{\left(2r+1\right)!}\mu'_{2r+1} \tag{21}$$

where μ'_{2r} and μ'_{2r+1} are obtained as ordinary moments of X for $n=2r$ and $n=2r+1$ respectively and can be computed from μ'_n in equation (16).

3.4 Quantile Function

According to Hyndman and Fan (1996), the quantile function for any distribution with cdf, $F(x)$ is defined in the form, $Q\left(u\right)=X_q=F^{-1}\left(u\right)$, where $Q\left(u\right)$ is the quantile function of $F(x)$ for $0<u<1$

Using the cdf of the OLINExD and inverting it as above gives the quantile function as follows:

$$F(x)=1-\frac{\alpha+\left(1-e^{-\frac{\theta}{x}}\right)}{\left(1+\alpha\right)\left(1-e^{-\frac{\theta}{x}}\right)}\exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\}=u \tag{22}$$

Collecting like terms and simplifying equation (22) above gives:

$$-\left(\alpha+1\right)\left(1-u\right)e^{-\left(\alpha+1\right)}=-\frac{\alpha+1-e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}e^{-\frac{\alpha+1-e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}} \tag{23}$$

From (23), it is observed that $-\frac{\alpha+1-e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}$ is the Lambert function of the real argument $-\left(\alpha+1\right)\left(1-u\right)e^{-\left(\alpha+1\right)}$ since the Lambert function is defined as: $w\left(x\right)e^{w\left(x\right)}=x$

Also note that the Lambert function has two branches with a branching point located at $\left(-e^{-1},1\right)$. The lower branch, $W_{-1}\left(x\right)$ is defined in the interval $\left[-e^{-1},1\right]$ and has a negative singularity for $x\to0^{-1}$. The upper branch, $W_{0}\left(x\right)$, is defined for $x\in\left[-e^{-1},\infty\right]$. Hence, (23) can be expressed as:

$$W\left(-\left(\alpha+1\right)\left(1-u\right)e^{-\left(\alpha+1\right)}\right)=-\frac{\alpha+1-e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}} \tag{24}$$

Note that for any $\alpha>0$ and $u\in\left(0,1\right)$, it follows that $\frac{\alpha+1-e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}>1$ and $\left(\left(\alpha+1\right)\left(1-u\right)e^{-\left(\alpha+1\right)}\right)<0$. Such that looking at the lower branch of the Lambert function, (24) is expressed as:

$$W_{-1}\left(-\left(\alpha+1\right)\left(1-u\right)e^{-\left(\alpha+1\right)}\right)=-\frac{\alpha+1-e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}} \tag{25}$$

Simplifying (25), the quantile function of the OLINExD is expressed as:

$$Q\left(u\right)=\left\{-\frac{1}{\theta}\log\left[\alpha\left(1+W_{-1}\left(-\left(\alpha+1\right)\left(1-u\right)e^{-\left(\alpha+1\right)}\right)\right)^{-1}+1\right]\right\}^{-1} \tag{26}$$

where $W_{-1}\left(.\right)$ stands for the negative branch of the Lambert function and u is uniform interval (0,1).

From (26), the median of X based on the OLINExD is calculated by letting u=0.5 and this gives:

$$MD=\left\{-\frac{1}{\theta}\log\left[\alpha\left(1+W_{-1}\left(-\left(\alpha+1\right)\left(0.5\right)e^{-\left(\alpha+1\right)}\right)\right)^{-1}+1\right]\right\}^{-1} \tag{27}$$

Consequently, random samples could be obtained from OLINExD from (26) by letting $Q(u) = X$ which is known as inverse transformation method of simulation. Hence it gives the representation:

$$X = \left\{ -\frac{1}{\theta} \log\left[\alpha\left(1 + W_{-1}\left(-(\alpha+1)(1-u)e^{-(\alpha+1)}\right)\right)^{-1} + 1 \right] \right\}^{-1} \tag{28}$$

From (26) and according to Kennedy and Keeping (1962), the Bowley's measure of skewness is defined as:

$$SK = \frac{Q\left(3/4\right) + 2Q\left(1/2\right) + Q\left(1/4\right)}{Q\left(3/4\right) - Q\left(1/4\right)} \tag{29}$$

Similarly, Moors (1988) defined Moors' kurtosis based on octiles from (26) as:

$$KT = \frac{Q\left(7/8\right) - Q\left(5/8\right) - Q\left(3/8\right) + \left(1/8\right)}{Q\left(6/8\right) - Q\left(1/8\right)} \tag{30}$$

where $Q(.)$ is calculated from equation (26).

3.5 Reliability analysis of the OLINExD

The Survival function describes the likelihood that a system or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S(x) = 1 - F(x) \tag{31}$$

Applying the *cdf* of the OLINExD in (31), the survival function for the OLINExDis obtained as:

$$S(x) = \frac{\alpha + \left(1 - e^{-\frac{\theta}{x}}\right)}{(1+\alpha)\left(1 - e^{-\frac{\theta}{x}}\right)} \exp\left\{ -\alpha \left[\frac{e^{-\frac{\theta}{x}}}{1 - e^{-\frac{\theta}{x}}} \right] \right\} \tag{32}$$

The figure below presents the behavior of the survival function of the OLINExD for some selected parameter values;

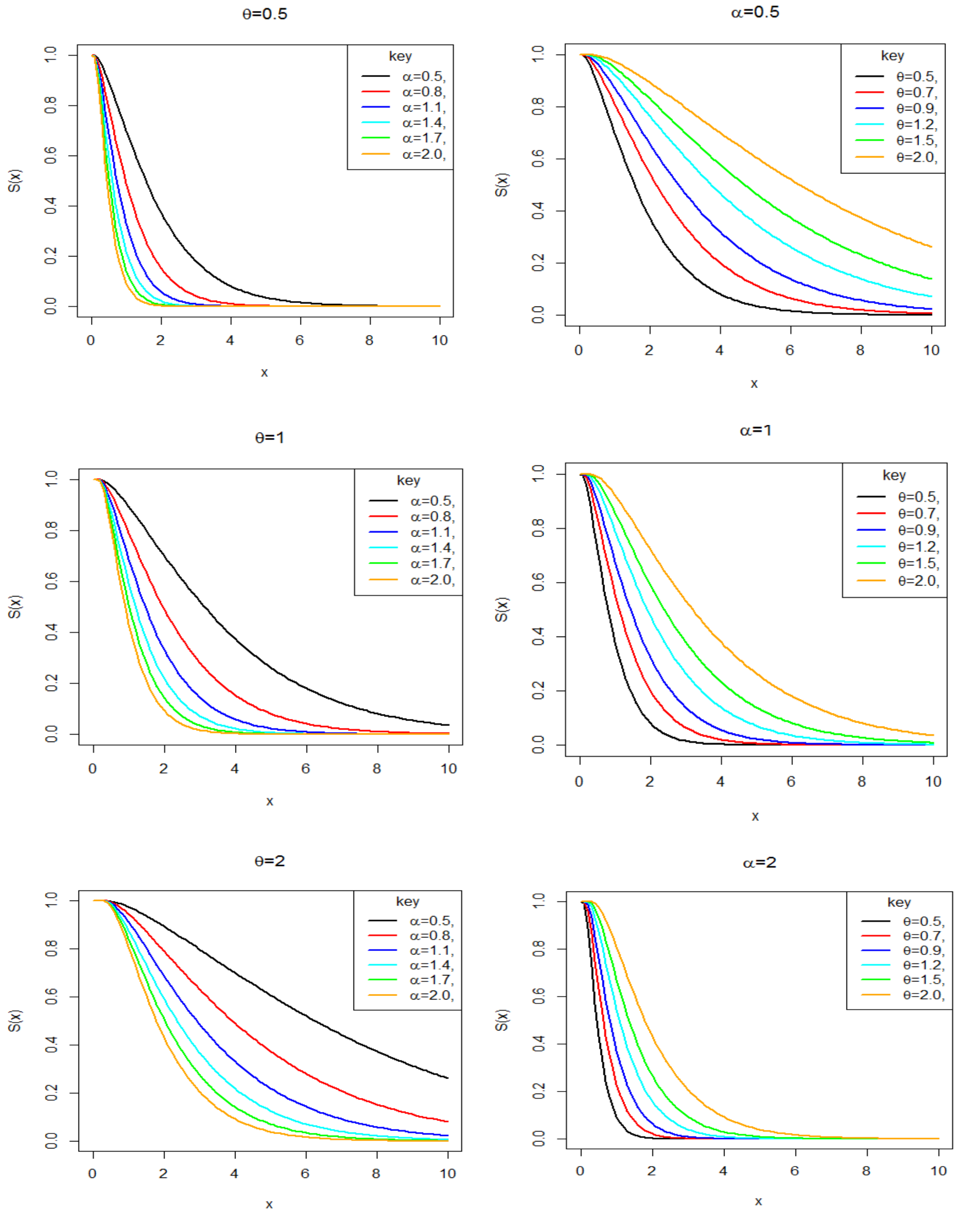


Figure3: Plots of the Survival function of OLINExD for Selected Values of the Parameters

Hazard function is a function that describes the chances that a product or component will breakdown over an interval of time. It is mathematically defined as:

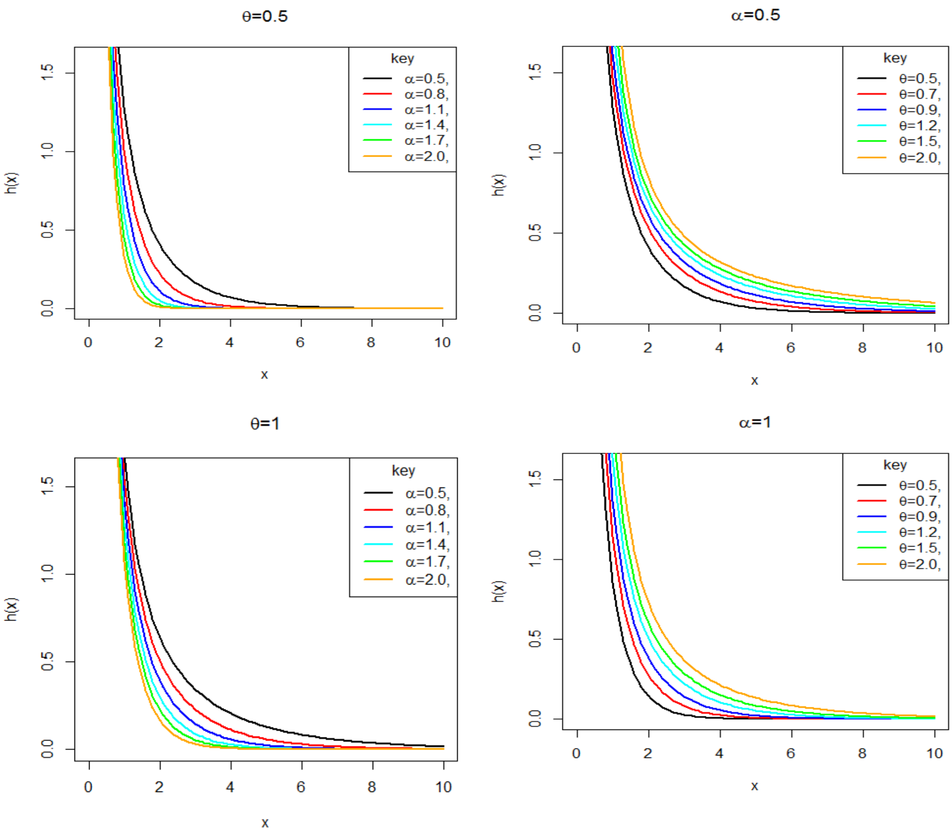
$$h(x)=\frac{f(x)}{S(x)}=\frac{f(x)}{1-F(x)} \tag{33}$$

Therefore, our definition of the hazard rate of the OLINExD is given by

$$h(x)=\frac{\alpha^{2}\theta x^{-2}e^{-\frac{\theta}{x}}}{\left[\alpha+\left(1-e^{-\frac{\theta}{x}}\right)\right]\left[1-e^{-\frac{\theta}{x}}\right]^{2}} \tag{34}$$

where $\alpha,\theta>0$.

The plot showing the behavior of the hazard function of OLINExD based on arbitrary parameter values is as follows:



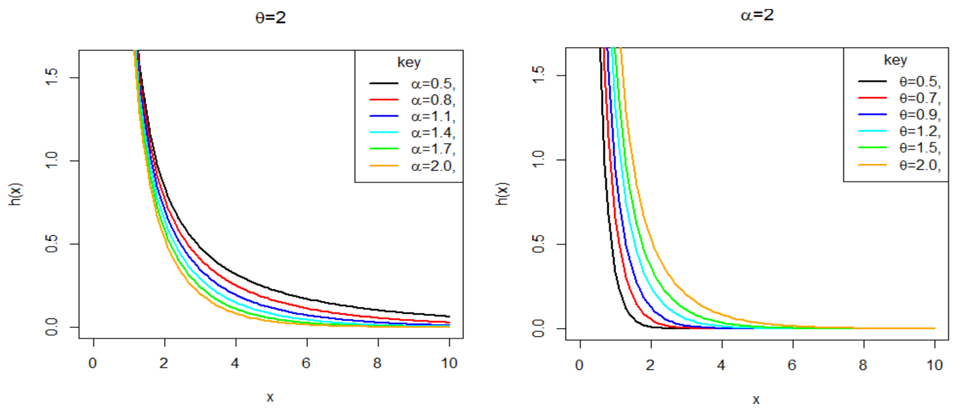


Figure4: Plots of the Hazard function of OLINExD for Selected Values of the Parameters.

3.6 Distribution of order Statistics

Given that $X_1, X_2, ..., X_n$ is a random sample from OLINExD and $X_{1:n}, X_{2:n}, ..., X_{i:n}$ denote the associated order statistic from the same sample. The pdf, $f_{i:n}(x)$ of the i^{th} order statistic can be obtained by:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} f(x) F(x)^{k+i-1} \tag{35}$$

Using (5) and (6), the *pdf* of the i^{th} order statistics$X_{i:n}$, can be expressed from (35) as:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} \left[\frac{\alpha^2 \theta x^{-2} e^{-\frac{\theta}{x}}}{(1+\alpha)\left[1-e^{-\frac{\theta}{x}}\right]^3} \exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\} \right] \left[1 - \frac{\alpha + \left(1-e^{-\frac{\theta}{x}}\right)}{(1+\alpha)\left(1-e^{-\frac{\theta}{x}}\right)} \exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\} \right]^{i+k-1} \tag{36}$$

The pdf of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the OLINExD are respectively given as:

$$f_{1:n}(x) = n\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\left[\frac{\alpha^2\theta x^{-2}\mathrm{e}^{-\frac{\theta}{x}}}{(1+\alpha)\left[1-\mathrm{e}^{-\frac{\theta}{x}}\right]^3}\exp\left\{-\alpha\left[\frac{\mathrm{e}^{-\frac{\theta}{x}}}{1-\mathrm{e}^{-\frac{\theta}{x}}}\right]\right\}\right]$$
$$\left[1-\frac{\alpha+\left(1-\mathrm{e}^{-\frac{\theta}{x}}\right)}{(1+\alpha)\left(1-\mathrm{e}^{-\frac{\theta}{x}}\right)}\exp\left\{-\alpha\left[\frac{\mathrm{e}^{-\frac{\theta}{x}}}{1-\mathrm{e}^{-\frac{\theta}{x}}}\right]\right\}\right]^k \quad (37)$$

and

$$f_{n:n}(x) = n\left[\frac{\alpha^2\theta x^{-2}\mathrm{e}^{-\frac{\theta}{x}}}{(1+\alpha)\left[1-\mathrm{e}^{-\frac{\theta}{x}}\right]^3}\exp\left\{-\alpha\left[\frac{\mathrm{e}^{-\frac{\theta}{x}}}{1-\mathrm{e}^{-\frac{\theta}{x}}}\right]\right\}\right]$$
$$\left[1-\frac{\alpha+\left(1-\mathrm{e}^{-\frac{\theta}{x}}\right)}{(1+\alpha)\left(1-\mathrm{e}^{-\frac{\theta}{x}}\right)}\exp\left\{-\alpha\left[\frac{\mathrm{e}^{-\frac{\theta}{x}}}{1-\mathrm{e}^{-\frac{\theta}{x}}}\right]\right\}\right]^{n-1}. \quad (38)$$

4. ESTIMATION OF UNKNOWN PARAMETERS OF OLINExD

Let $X_1, X_2, ..., X_n$ be a sample of size "n" independently and identically distributed random variables from the OLINExD with unknown parameters α and θ defined previously.

The likelihood function is given by:

$$L\left(\underline{X} \mid \alpha,\theta\right) = \frac{\left(\alpha^2\theta\right)^n \prod_{i=1}^{n}\left(x_i^{-2}e^{-\frac{\theta}{x_i}}\right)}{(1+\alpha)^n \prod_{i=1}^{n}\left[1-e^{-\frac{\theta}{x}}\right]^3} e^{-\alpha\sum_{i=1}^{n}\left[\frac{e^{-\frac{\theta}{x_i}}}{1-e^{-\frac{\theta}{x_i}}}\right]}$$

Let the log-likelihood function be $l = \log L\left(\underline{X} \mid \alpha,\theta\right)$ therefore

$$l = 2n\log\alpha + n\log\theta - n\log(1+\alpha) - 2\sum_{i=1}^{n}\log x_i - \theta\sum_{i=1}^{n} x_i^{-1}$$
$$-3\sum_{i=1}^{n}\log\left(1-e^{-\frac{\theta}{x}}\right) - \alpha\sum_{i=1}^{n}\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right) \quad (39)$$

Differentiating l partially with respect to α and θ respectively gives;

$$\frac{\partial l}{\partial \alpha}=\frac{2n}{\alpha}-\frac{n}{(\alpha+1)}-\sum_{i=1}^{n}\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right) \tag{40}$$

$$\frac{\partial l}{\partial \theta}=\frac{n}{\theta}-\sum_{i=1}^{n}x_i^{-1}-3\sum_{i=1}^{n}\left\{\frac{x_i^{-1}e^{-\frac{\theta}{x_i}}}{\left[1-e^{-\frac{\theta}{x^i}}\right]}\right\}+\alpha\sum_{i=1}^{n}\left\{\frac{x_i^{-1}e^{-\frac{\theta}{x_i}}}{\left[1-e^{-\frac{\theta}{x^i}}\right]^2}\right\} \tag{41}$$

Equating (40) and (41) to zero (0) and solving for the solution of the non-linear system of equations will give the maximum likelihood estimates (MLEs) of parameters α and θ.

5. SIMULATION STUDY

In this section, we perform a simulation study in order to evaluate some frequentist properties of the MLEs of α and θ. The observations are simulated from the OLINExD by inverse transformation method with the help of the quantile function of the distribution. We consider the following values for the parameters: $\alpha = 1.5$, $\theta = 4.5$ and $\alpha = 3.5$, $\theta = 0.5$ for the OLINExD. All simulations are performed using the R software. The results are obtained from 10000 Monte Carlo simulations. In each replication, a random sample of size n is drawn from the distribution and the parameters are estimated by maximum likelihood. The sample sizes are taken as n = 10, 15, 25, 35, 55, 75, 125, 150 and 200.

Table 1
The MLEs, Biases and MSEs of the Parameters of the OLINExD

Sample Size	Parameter (True value)	Estimates (Mean)	Absolute Bias	MSE	Parameter (True value)	Estimates (Mean)	Absolute Bias	MSE
n=10	α (1.5) θ (4.5)	1.5575 4.5484	0.0575 0.0484	0.1059 0.4236	α (3.5) θ (0.5)	3.6107 0.5281	0.1107 0.0281	0.4643 0.0201
n=15	α (1.5) θ (4.5)	1.5446 4.5293	0.0446 0.0293	0.0696 0.3091	α (3.5) θ (0.5)	3.5876 0.5181	0.0876 0.0181	0.3094 0.0133
n=25	α (1.5) θ (4.5)	1.5216 4.5284	0.0216 0.0284	0.0389 0.1963	α (3.5) θ (0.5)	3.5418 0.5126	0.0418 0.0126	0.1767 0.0080
n=35	α (1.5) θ (4.5)	1.5164 4.5164	0.0164 0.0164	0.0275 0.1415	α (3.5) θ (0.5)	3.5321 0.5083	0.0321 0.0083	0.1257 0.0055
n=55	α (1.5) θ (4.5)	1.5137 4.5096	0.0137 0.0096	0.0173 0.0920	α (3.5) θ (0.5)	3.5275 0.5051	0.0275 0.0051	0.0794 0.0035
n=75	α (1.5) θ (4.5)	1.5088 4.5083	0.0088 0.0083	0.0126 0.0697	α (3.5) θ (0.5)	3.5176 0.5038	0.0176 0.0038	0.0578 0.0026
n=125	α (1.5) θ (4.5)	1.5047 4.5043	0.0047 0.0043	0.0073 0.0426	α (3.5) θ (0.5)	3.5094 0.5027	0.0094 0.0027	0.0338 0.0016
n=150	α (1.5) θ (4.5)	1.5037 4.5051	0.0037 0.0051	0.0063 0.0350	α (3.5) θ (0.5)	3.5073 0.5021	0.0073 0.0021	0.0290 0.0013
n=200	α (1.5) θ (4.5)	1.5034 4.5027	0.0034 0.0027	0.0047 0.0267	α (3.5) θ (0.5)	3.5055 0.5015	0.0055 0.0015	0.0211 0.0010

The result in Table 1 provides the averages of the MLEs (Mean), their biases (Absolute Bias) and mean square errors (MSEs) for the parameters of OLINExD distribution. Based on the figures from Table 1, it is clear that the average estimates tend to be closer to the true parameters when sample size increases and the biases and mean square errors all decrease as sample size increases which is in agreement with first-order asymptotic theory.

6. APPLICATIONS

This particular section presents two applications to real life data to illustrate the flexibility of the OLINExD distribution defined in Section 2. The MLEs of the model parameters are determined and some goodness-of-fit statistics for this distribution are compared with other competitive models.

The model selection is carried out based upon the value of the log-likelihood function evaluated at the MLEs, ℓ, Akaike Information Criterion, *AIC,* Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC) and Hannan Quin Information Criterion (HQIC).

It is also done using other goodness-of-fit measures such as Anderson-Darling (A*), Cramѐr-Von Mises (W*) and Kolmogorov-smirnov (K-S) statistics. The details about the statistics A*, W* and K-S are discussed in Chen and Balakrishnan (1995). Meanwhile, the smaller these statistics are, the better the fit of the distribution is. The required computations are carried out using the R package “AdequacyModel” which is freely available from http://cran.r-project.org/web/packages/AdequacyModel/AdequacyModel.pdf.

Dataset I: This dataset is on 59 samples of the monthly actual taxes revenue in Egypt (in 1,000 million Egyptian pounds) between January 2006 and November 2010. It was used by Owoloko *et al*., (2015), Oguntunde *et al*. (2015) and Ieren *et al*., (2018).

Dataset II: This dataset is on the strength of 1.5cm glass fibers initially collected by members of staff at the UK national laboratory. The data was considered in several studies including Smith and Naylor (1987), Barreto-Souza *et al*. (2011), Bourguignon *et al*. (2014), Oguntunde e*t al*. (2015), Afify and Aryal (2016), Ieren and Yahaya (2017) and Ieren *et al*. (2018).

Table 2
Descriptive Summary of Dataset I and II

Dataset	*n*	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
I	59	4.10	8.45	10.60	16.85	13.49	39.20	64.8266	1.6083	2.256
II	63	0.550	1.375	1.590	1.685	1.507	2.240	0.105	-0.8786	3.9238

Based on the descriptive statistics in Table 2 above, it is clear that the first dataset (dataset I) is skewed to the right or positively skewed and the second dataset (dataset II) is negatively skewed, that is, skewed to the left and would be good for flexible models like OLINExD.

For these datasets, we compare the fits of the OLINExD distribution with those of the Inverse exponential distribution (INExD), Lindley distribution (LIND) and the conventional exponential distribution (ExD).

Table 3
Maximum Likelihood Parameter Estimates for Dataset I and II

Distribution	**Dataset I**		**Dataset II**	
	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$
OLINExD	9.588458	1.572982	3.838106	9.809366
LIND	-	0.1395136	-	0.9966814
ExD	0.07611176	-	0.663383	-
INExD	3.876527		1.410422	

Table 4
The Statistics ℓ, AIC, CAIC, BIC and HQIC for Dataset I and II

Distribution	$\hat{\ell}$	*AIC*	*CAIC*	*BIC*	*HQIC*	Ranks
Dataset I						
OLINExD	195.3761	394.7521	394.9664	398.9072	396.3741	1st
LIND	200.6295	403.2591	403.3293	405.3366	404.0701	2nd
ExD	212.5273	427.0547	427.1248	429.1322	427.8657	3rd
INExD	232.3908	466.7816	466.8518	468.8592	467.5926	4th
Dataset II						
OLINExD	37.59052	79.18103	79.37776	83.4988	80.88202	1st
LIND	82.58536	167.1707	167.2352	169.3296	168.0212	2nd
ExD	90.26897	182.5379	182.6025	184.6968	183.3884	3rd
INExD	90.88183	183.7637	183.8282	185.9225	184.6141	4th

Table 5
The A^*, W^*, K-S Statistic and P-values for Dataset I and II

Distribution	A^*	W^*	*K-S*	*P-Value (K-S)*	Ranks
Dataset I					
OLINExD	0.9577852	0.1549982	0.17104	0.06337	1st
LIND	1.30299	0.2091359	0.19294	0.02473	2nd
ExD	1.20913	0.1945511	0.31092	2.222e-05	3rd
INExD	0.3392768	0.05879799	0.46754	1.256e-11	4th
Dataset II					
OLINExD	3.825918	0.7027762	0.23295	0.001925	1st
LIND	3.078748	0.5629456	0.38894	7.794e-09	2nd
ExD	3.232996	0.5914413	0.42008	3.098e-10	3rd
INExD	4.807349	0.8887843	0.48863	1.067e-13	4th

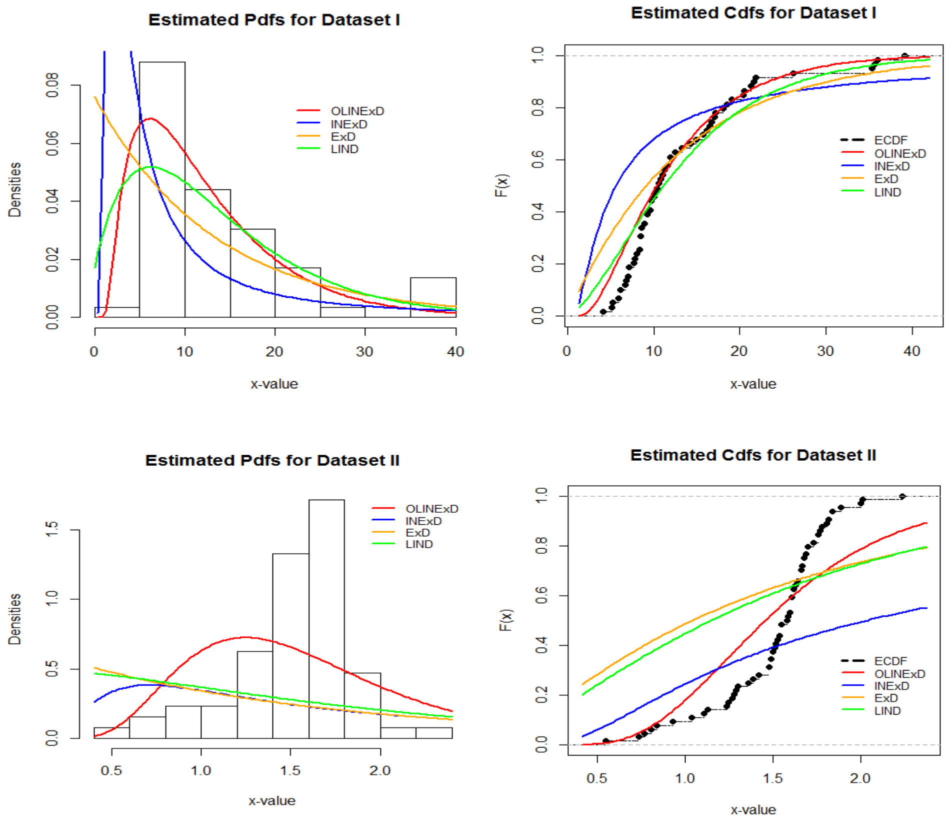


Figure 6: Histogram and Estimated Densities and Cdfs of the Fitted Distributions Based on Datasets I and II

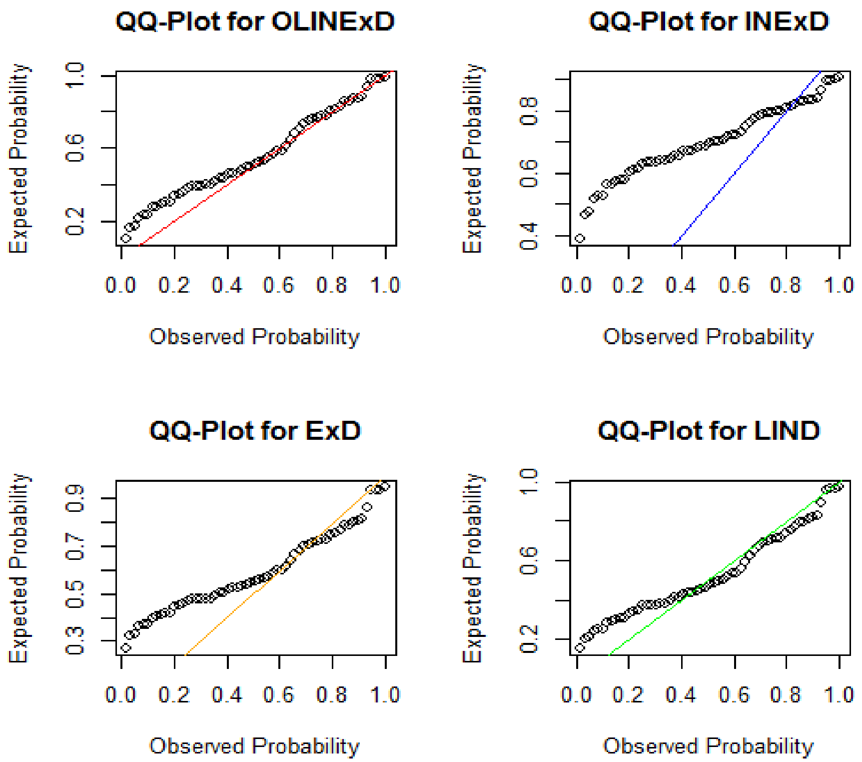


Figure 7: Probability Plots for the Fit of OLINExD, INExD, ExD & LIND Based on Dataset I

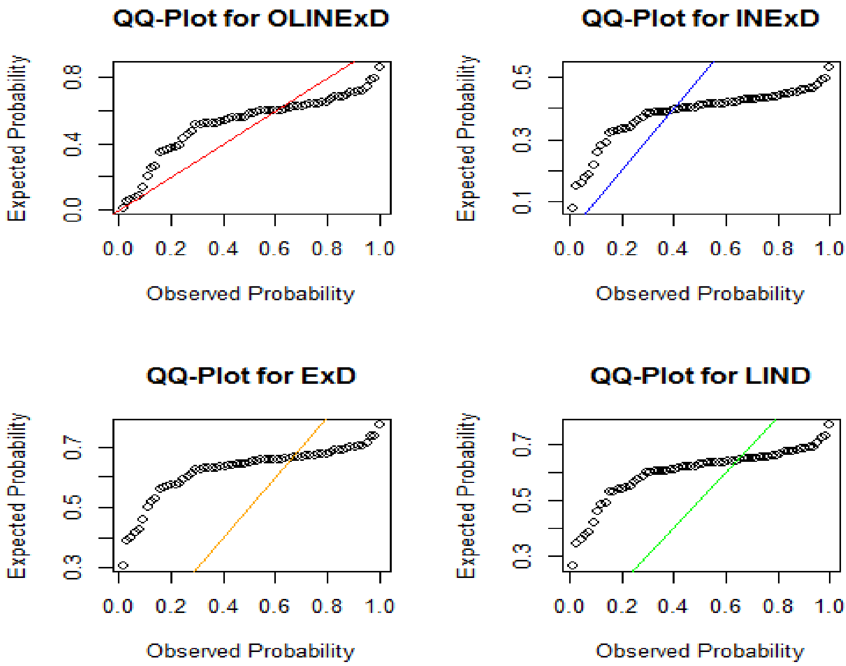


Figure 8: Probability Plots for the Fit of OLINExD, INExD, ExD & LIND Based on Dataset II.

Table 2 lists the MLEs of the parameters for the fitted models for both dataset I and II. The values of the statistics AIC, CAIC, BIC and HQIC are listed in Tables 3 for datasets I and II while those of the statistics A*,W* and K-S for datasets I and II are provided in Tables 4. For both two datasets (dataset I and II), the proposed OLINExD with two parameters provides the best fit compared to the LIND, INExD and ExD. Also, the estimated pdfs and cdfs displayed in Figure 6 as well as the P-P plots presented in Figures 7 and 8 clearly support and confirm the results in Tables 3 and 4.

7. SUMMARY AND CONCLUSION

In recent times, there has been a strong desire for developing more flexible statistical distributions in the area of statistical theory and applications. Most flexible compound classes of distributions have been introduced and used to describe various real life phenomena. This paper proposed a new lifetime model called the odd Lindley-Inverse Exponential distribution. The study provides some statistical properties of the distribution such as the ordinary moments, generating function, characteristics function, quantile function and related measures, survival and hazard functions as well as distribution of ordered statistics. The maximum likelihood method is used for estimating the model parameters and a simulation study is carried out to check the behavior of the maximum likelihood estimates under different values of the parameters and sample sizes. Based on the plots of the pdf of the distribution, it was discovered that the new model is skewed and flexible and its shape varies depending on the assumed parameters values. Also, the plots of the hazard function of the distribution are consistently decreasing irrespective of the assumed values of the model parameters which show that the new model would be useful for survival analysis of variables with decreasing failure rate. We also considered applying the new model to two real life datasets to demonstrate its practical importance and found that the proposed distribution provides consistently better fits compared to the other models fitted to these two datasets. Hence, it is our hope that the proposed distribution would be applied widely in areas such as reliability engineering as well as survival and lifetime data, among others.

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