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GENERALIZED EXPONENTIAL DISTRIBUTION FOR CONCOMITANTS OF ORDERED RANDOM VARIABLES FROM BAIRAMOV FAMILY

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ABSTRACT

This article mainly introduces the Bairamov Morgenstern family under generalized exponential distribution. We adopt the concomitants of generalized order statistics for this family to construct some distributional properties. In addition, recurrence relation between moments for the required model is obtained. Furthermore, the joint distribution of concomitants of ordinary order statistics is derived. Also, the asymptotic behavior of concomitants of ordinary order statistics is discussed.

KEYWORD

Bairamov family; Concomitants; Generalized exponential distribution; Generalized order statistics.

MSC2010: 62B10; 62G30.

1. INTRODUCTION

Bairamov et al. [2] considered a generalization of the well-known bivariate Farlie-Gumbel-Morgenstern (FGM) distribution by inserting extra parameters. We denote this model by BR(p_1, p_2, q_1, q_2). Therefore, in the current study, we transact with the distribution theory and applications of BR(p_1, p_2, q_1, q_2), which is specified by the cumulative distribution function (cdf) and probability density function (pdf), respectively, as follows

$$F_{X,Y}(x,y) = F_X(x)\, F_Y(y)[1 + \lambda\,(1 - F_X^{p_1}(x))^{q_1}(1 - F_Y^{p_2}(y))^{q_2}], \tag{1.1}$$

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)\,[1 + \lambda\,(1 - F_X^{p_1}(x))^{q_1-1}(1 - (1 + p_1 q_1)F_X^{p_1}(x))(1 - F_Y^{p_2}(y))^{q_2-1} \times (1 - (1 + p_2 q_2)F_Y^{p_2}(y))], \tag{1.2}$$

where $p_1, p_2 \geq 1$, $q_1, q_2 \in \mathbb{N}$, $F_X(x)$, $F_Y(y)$ and $f_X(x)$, $f_Y(y)$ are the marginal cdf's and pdf's of the random variables X and Y respectively. For BR(p_1, p_2, q_1, q_2), the parameter λ has the admissible range

$$-\min\left\{1, \frac{1}{p_1 p_2}\left(\frac{1 + p_1 q_1}{p_1(q_1 - 1)}\right)^{q_1-1}\left(\frac{1 + p_2 q_2}{p_2(q_2 - 1)}\right)^{q_2-1}\right\} \leq \lambda \leq \min\left\{\frac{1}{p_1}\left(\frac{1 + p_1 q_1}{p_1(q_1 - 1)}\right)^{q_1-1}, \frac{1}{p_2}\left(\frac{1 + p_2 q_2}{p_2(q_2 - 1)}\right)^{q_2-1}\right\}. \tag{1.3}$$

For BR$(1,1,q_1,q_2)$, we have $\left(\frac{1+q_1}{q_1-1}\right)^{q_1-1}\left(\frac{1+q_2}{q_2-1}\right)^{q_2-1}>1$, therefore, the admissible range of the parameter λ is:

$$-\left(\frac{1+q_1}{q_1-1}\right)^{q_1-1}\left(\frac{1+q_2}{q_2-1}\right)^{q_2-1}\leq\lambda\leq\min\left\{\left(\frac{1+q_1}{q_1-1}\right)^{q_1-1},\left(\frac{1+q_2}{q_2-1}\right)^{q_2-1}\right\}. \quad (1.4)$$

The partnership parameter λ is known as the dependence parameter of the random variables X and Y. If $\lambda=0$, then X and Y are independent.

The generalized exponential (GE) distribution was started by Gupta and Kundu [8]. It is so flexible and applied to many sections of reliability analysis. The random variable X has GE distribution, denoted by $X\sim GE(\theta;\alpha)$, if it has the cdf

$$F_X(x)=(1-\exp(-\theta x))^{\alpha}, x,\theta,\alpha>0. \quad (1.5)$$

For the kth moment of GE distribution, $GE(\theta;\alpha)$, Gupta and Kundu [8] showed that

$$\mu_k=\frac{\alpha\Gamma(k+1)}{\theta^k}\sum_{i=0}^{\chi(\alpha-1)}\frac{(-1)^i}{(i+1)^{k+1}}\binom{\alpha-1}{i}, \quad (1.6)$$

where $\chi(x)=x$, if x is integer and $\chi(x)=\infty$, if x is non-integer. Meanwhile, the moment generating function, mean and variance of $GE(\theta;\alpha)$ are given, respectively, by

$$M_X(t)=\alpha\beta\left(\alpha,1-\frac{t}{\theta}\right),\ \mu_1=E(X)=\frac{B(\alpha)}{\theta}, Var(X)=\frac{C(\alpha)}{\theta^2}, \quad (1.7)$$

where $\beta(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, $B(\alpha)=\psi(\alpha+1)-\psi(1)$, $C(\alpha)=\psi'(1)-\psi'(\alpha+1)$ and $\psi(.)$ is the digamma function, $\psi(1)=-\Gamma'(1)=0.57722$ is the Euler's constant, while $\psi'(.)$ is its derivation. For a quick preview of the previous literature, Tahmasebi and Jafari [11] launched some features of the classical FGM under bivariate GE distribution. Also, they present some distributional properties of concomitants of order statistics beside record values of this cdf. In contrast, they derived some recurrence relations between moments of concomitants of order statistics. Barakat et al. [[3], [4]] extended the results of Tahmasebi and Jafari [11] to Huang-Kotz Morgenstern and Bairamov-Kotz-Becki-Farlie-Gumble-Morgenstern type bivariate GE distribution, respectively. Furthermore, they studied some properties of concomitants of order statistics and record values for such a model.

The idea of generalized order statistics (gos) was first proposed by Kamps [9] that contains all forms of ordered random observations such as ordinary order statistics, sequential order statistics, kth record values and progressively Type-II censoring as special cases of gos. Let $n\in\mathbb{N}$, $k\geq1$, $m_1,\ldots,m_{n-1}\in\mathbb{R}$, $M_r=\sum_{j=r}^{n-1}m_j$, $1\leq r\leq n-1$, be parameters such that $\gamma_r=k+n-r+M_r\geq1$ for all $r\in\{1,2,\ldots,n-1\}$, and let $\widetilde{m}=(m_1,\ldots,m_{n-1})\in\mathbb{R}^{n-1}$. For a subclass of gos (called $m-gos$), where $m_1=m_2=\cdots=m_{n-1}=m$, the pdf of the rth $m-gos$, $X_{(r;k,m,n)}$, can be written as:

$$f_{(r;k,m,n)}(x)=\frac{h_{r-1}}{(r-1)!}(1-F(x))^{\gamma_r-1}f(x)\,w_m^{r-1}(F(x)), \quad (1.8)$$

where

$$h_{r-1} = \prod_{j=1}^{r} \gamma_j, \quad w_m(z) = p_m(z) - p_m(0), 0 < z < 1,$$

$$p_m(z) = \begin{cases} \frac{-(1-z)^{m+1}}{m+1}, & m \neq -1, \\ -\ln(1-z), & m = -1. \end{cases}$$

David et al. [5] illustrated the concomitants of ordinary order statistics. Under some bivariate distributions with cdf $F(x,y)$, let (X_i, Y_i), $i = 1,2,\ldots,n$, be n pairs of independent random variables. Let $X_{(r;n)}$ be the rth order statistics, then Y related with $X_{(r;n)}$ is called the concomitant of rth order statistics and is indicated by $Y_{[r;n]}$. The pdf and cdf of $Y_{[r;n]}$ are given by

$$g_{[r;n]}(y) = g_{Y_{[r;n]}}(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_{(r;n)}(x)dx, \tag{1.9}$$

$$G_{[r;n]}(y) \quad = \int_{-\infty}^{\infty} F_{Y|X}(y|x)\, f_{(r;n)}(x)\, dx, \tag{1.10}$$

where $f_{(r;n)}(x)$ is the pdf of the rth ordinary order statistics $X_{(r;n)}$.

Throughout this article, we deal with the Bairamov family defined in (1.1) when $p_1 = p_2 = 1$ (i.e. BR$(1,1,q_1,q_2)$) and under the distribution GE defined in (1.5) and we refer to it BR-GE$(\theta_1,\theta_2;\alpha_1,\alpha_2)$. Furthermore, all results of Tahmasebi and Jafari [11] and Barakat et al. [[3], [4]] are derived, generalized or extended to BR-GE$(\theta_1,\theta_2;\alpha_1,\alpha_2)$ for $m-gos$ and order statistics as a special case. The organization of the paper is as follows: Section 2 discuss the moments and the correlation with its admissible range for BR-GE$(\theta_1,\theta_2;\alpha_1,\alpha_2)$. Section 3 contains the recurrence relation for the single moments and the variance of bivariate concomitants of $m-gos$. Moreover, the asymptotic behavior of the concomitants of ordinary order statistics is given. Finally, the joint distribution of concomitants of paired order statistics and their moments are derived in Section 4.

2. MOMENTS AND CORRELATION OF BR-GE$(\boldsymbol{\theta_1},\boldsymbol{\theta_2};\boldsymbol{\alpha_1},\boldsymbol{\alpha_2})$

In this section, we obtain the $(n,m)th$ joint moments and correlation of BR-GE$(\theta_1,\theta_2;\alpha_1,\alpha_2)$. Before we go in details, the following remark is used to illustrate how to derive the moments.

Remark 2.1:

From (1.5), let the random variables $Z \sim GE(\theta;\alpha)$ and $U \sim GE(\theta,\alpha(i+1))$, then $F_U(z) = (1-e^{-\theta z})^{\alpha(i+1)}$ and the pdf of U is formulated as $f_U(z) = (i+1) f_Z(z) F_Z^i(z)$. Therefore, the expectation of U^n is $E(U^n) = \int_{-\infty}^{\infty}(i+1)z^n f_Z(z)F_Z^i(z)dx$. We can also note that

$$\int_{-\infty}^{\infty} x^n f_X(x)(1-F_X(x)\,)^q dx = \sum_{i=0}^{q} \binom{q}{i} (-1)^i \int_{-\infty}^{\infty} x^n f_X(x) F_X^i(x)dx, \tag{2.1}$$

$$\begin{aligned}\int_{-\infty}^{\infty} x^n f_X(x) F_X(x)(1-F_X(x)\,)^q dx \\ = \sum_{i=0}^{q} \binom{q}{i} (-1)^i \int_{-\infty}^{\infty} x^n f_X(x) F_X^{i+1}(x)dx.\end{aligned} \tag{2.2}$$

Now to obtain the moments, from the previous remark and the bivariate distribution BR$(1,1,q_1,q_2)$, where $X \sim GE(\theta_1;\alpha_1)$ and $Y \sim GE(\theta_2;\alpha_2)$. Then the $(n,m)th$ joint moments of BR-GE$(\theta_1,\theta_2;\alpha_1,\alpha_2)$ is given by

$$\begin{aligned} E(X^nY^m) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^m f_X(x) f_Y(y)[1+\lambda(1-F_X(x))^{q_1-1}(1-(1+ \\ &\quad q_1)F_X(x)) \times \big(1-F_Y(y)\big)^{q_2-1}\big(1-(1+q_2)F_Y(y)\big)\big] \\ &= E(x^n)E(y^m) + \lambda\Big[\sum_{i_1=0}^{q_1-1} I_1 E(U_1^n) - (1+q_1)\sum_{i_1=0}^{q_1-1} I_3 E(U_2^n)\Big] \\ &\quad \times \Big[\sum_{i_2=0}^{q_2-1} I_2 E(V_1^m) - (1+q_2)\sum_{i_2=0}^{q_2-1} I_4 E(V_2^m)\Big], \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} I_1 &= \frac{\binom{q_1-1}{i_1}(-1)^{i_1}}{i_1+1}, I_2 = \frac{\binom{q_2-1}{i_2}(-1)^{i_2}}{i_2+1}, \\ I_3 &= \frac{\binom{q_1-1}{i_1}(-1)^{i_1}}{i_1+2}, I_4 = \frac{\binom{q_2-1}{i_2}(-1)^{i_2}}{i_2+2}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &U_1 \sim GE(\theta_1,\alpha_1(i_1+1)), U_2 \sim GE(\theta_1,\alpha_1(i_1+2)), \\ &V_1 \sim GE(\theta_2,\alpha_2(i_1+1)), V_2 \sim GE(\theta_2,\alpha_2(i_1+2)), \end{aligned} \tag{2.5}$$

$i_1 = 0,\dots,q_1-1$, $i_2 = 0,\dots,q_2-1$. Therefore, from (2.3) and (1.7), we get

$$\begin{aligned} E(XY) &= \frac{B(\alpha_1)}{\theta_1}\frac{B(\alpha_2)}{\theta_2} \\ &+\lambda\left[\sum_{i_1=0}^{q_1-1} I_1 \frac{B\big(\alpha_1(i_1+1)\big)}{\theta_1} - (1+q_1)\sum_{i_1=0}^{q_1-1} I_3 \frac{B\big(\alpha_1(i_1+2)\big)}{\theta_1}\right] \\ &\times\left[\sum_{i_2=0}^{q_2-1} I_2 \frac{B(\alpha_2(i_2+1))}{\theta_2} - (1+q_2)\sum_{i_2=0}^{q_2-1} I_4 \frac{B(\alpha_2(i_2+2))}{\theta_2}\right]. \end{aligned} \tag{2.6}$$

Subsequently, the correlation of X and Y is

$$\begin{aligned} \rho_{X,Y} &= \frac{\lambda}{\sqrt{C(\alpha_1)C(\alpha_2)}} \\ &\Big[\sum_{i_1=0}^{q_1-1} I_1 B(\alpha_1(i_1+1)) - (1+q_1)\sum_{i_1=0}^{q_1-1} I_3 B(\alpha_1(i_1+2))\Big] \\ &\times\Big[\sum_{i_2=0}^{q_2-1} I_2 B\big(\alpha_2(i_2+1)\big) - (1+q_2)\sum_{i_2=0}^{q_2-1} I_4 B\big(\alpha_2(i_2+2)\big)\Big] \\ &= \lambda g(\alpha_1,\alpha_2,q_1,q_2), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} g(\alpha_1,\alpha_2,q_1,q_2) &= \frac{1}{\sqrt{C(\alpha_1)C(\alpha_2)}} \\ &\Big[\sum_{i_1=0}^{q_1-1} I_1 B\big(\alpha_1(i_1+1)\big) - (1+q_1)\sum_{i_1=0}^{q_1-1} I_3 B\big(\alpha_1(i_1+2)\big)\Big] \\ &\times\Big[\sum_{i_2=0}^{q_2-1} I_2 B\big(\alpha_2(i_2+1)\big) - (1+q_2)\sum_{i_2=0}^{q_2-1} I_4 B\big(\alpha_2(i_2+2)\big)\Big]. \end{aligned} \tag{2.8}$$

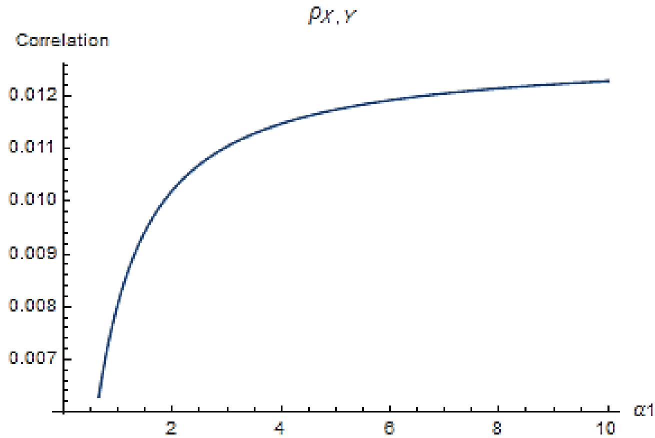

Figure 1: BR-GE$(3,3;\alpha_1,2)$, $\lambda = 0.5, q_1 = 3, q_2 = 2$

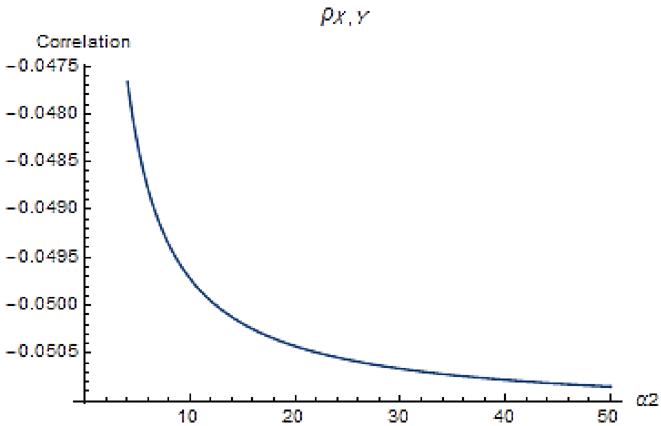

Figure 2: BR-GE$(2,0.9;3,\alpha_2)$, $\lambda = -2, q_1 = 3, q_2 = 2$

Clearly, for any $q_1, q_2 \in \mathbb{N}$, the function $g(\alpha_1, \alpha_2, q_1, q_2)$ is positive and increasing function with respect to each of α_1 and α_2. Thus, $\rho_{X,Y}$ is positive and increasing function, if $\lambda > 0$, and $\rho_{X,Y}$ is negative and decreasing function, if $\lambda < 0$, with respect to each of α_1 and α_2, see Figure (1) and (2). Meanwhile, from Barakat et al. [3] Page (4), we have

$$\lim_{\alpha\to\infty} \frac{B(\alpha(1+p))-B(\alpha)}{\sqrt{C(\alpha)}} = \frac{\sqrt{6}}{\pi}\log(1+p), \tag{2.9}$$

thus, we can show that

$$\begin{aligned} \lim_{\alpha_1,\alpha_2\to\infty} g(\alpha_1,\alpha_2,q_1,q_2) &= \frac{6}{\pi^2}\left[\sum_{i_1=0}^{q_1-1}\binom{q_1-1}{i_1}(-1)^{i_1}\log(1+i_1)\right] \\ &\left[\sum_{i_2=0}^{q_2-1}\binom{q_2-1}{i_2}(-1)^{i_2}\log(1+i_2)\right], \\ \lim_{\alpha_1,\alpha_2\to 0^+} g(\alpha_1,\alpha_2,q_1,q_2) &= 0. \end{aligned} \tag{2.10}$$

Therefore, the admissible range of $\rho_{X,Y}$ is

$$-\left(\frac{1+q_1}{q_1-1}\right)^{q_1-1}\left(\frac{1+q_2}{q_2-1}\right)^{q_2-1} g^*(q_1,q_2) \le \rho_{X,Y} \le \min\left\{\left(\frac{1+q_1}{q_1-1}\right)^{q_1-1} g^*(q_1,q_2), \left(\frac{1+q_2}{q_2-1}\right)^{q_2-1} g^*(q_1,q_2)\right\}, \quad (2.11)$$

where

$$g^*(q_1,q_2) = \frac{6}{\pi^2}\left[\sum_{i_1=0}^{q_1-1}\binom{q_1-1}{i_1}(-1)^{i_1}\log(1+i_1)\right]\left[\sum_{i_2=0}^{q_2-1}\binom{q_2-1}{i_2}(-1)^{i_2}\log(1+i_2)\right]. \quad (2.12)$$

Remark 2.2:

For $q_1, q_2 = 1$, all the previous results are reduced to those obtained by Tahmasebi and Jafari [11].

3. CONCOMITANTS OF $m-gos$ UNDER BR-GE$(\theta_1,\theta_2;\alpha_1,\alpha_2)$

From (1.2), the conditional pdf of Y given $X=x$ of BR$(1,1,q_1,q_2)$ is given by

$$f_{Y|X}(y|x) = f_Y(y)[1+\lambda(1-F_X(x))^{q_1-1}(1-(1+q_1)F_X(x))(1-F_Y(y))^{q_2-1} \times (1-(1+q_2)F_Y(y))]. \quad (3.1)$$

Mohamed [10] have introduced the pdf and cdf of BR$(1,1,q_1,q_2)$ of the concomitant of $m-gos$ by the following theorem.

Theorem 3.1:

Based on BR(p_1,p_2,q_1,q_2) with pdf given by (1.2) and cdf given by (1.1) (with $p_1=p_2=1$), utilizing (1.8) and (3.1), the pdf and cdf of the concomitant of rth $m-gos$, $Y_{[r;k,m,n]}$, are given by, $1\le r\le n$, respectively:

$$g_{[r;k,m,n]}(y) = f_Y(y)[1+\lambda\, R^*(r;k,m,n)(1-(1+q_2)F_Y(y))(1-F_Y(y))^{q_2-1}], \quad (3.2)$$

$$G_{[r;k,m,n]}(y) = f_Y(y)[1+\lambda\, R^*(r;k,m,n)(1-F_Y(y))^{q_2}], \quad (3.3)$$

where

$$R^*(r;k,m,n) = c_{r-1}\left\{\frac{(1+q_1)}{\prod_{i=1}^{r}(q_1+\gamma_i)} - \frac{q_1}{\prod_{i=1}^{r}(q_1+\gamma_i-1)}\right\}, \quad (3.4)$$

$\gamma_r = k+(n-r)(m+1)$, $n\in\mathbb{N}$, $k\ge 1$, $m_1 = \ldots = m_{n-1} = m \in \mathbb{R}$, $c_{r-1} = \prod_{i=1}^{r}\gamma_i$.

Let $Y\sim GE(\theta_2;\alpha_2)$ and $X\sim GE(\theta_1;\alpha_1)$, then based on BR-GE$(\theta_1,\theta_2;\alpha_1,\alpha_2)$, using (3.2), the pdf of $Y_{[r;k,m,n]}$ is as follows

$$g_{[r;k,m,n]}(y) = f_Y(y)\left[1 + \lambda R^*(r;k,m,n)\big(1 - (1+q_2)F_Y(y)\big)\big(1 - F_Y(y)\big)^{q_2-1}\right]$$
$$= f_Y(y)\Big[1 + \lambda R^*(r;k,m,n)\big(1 - (1+ q_2)F_Y(y)\big)\sum_{i_2=0}^{q_2-1}\binom{q_2-1}{i_2}(-1)^{i_2}F_y^{i_2}(y)\Big]$$
$$= f_Y(y) + \lambda R^*(r;k,m,n)\sum_{i_2=0}^{q_2-1} I_2(i_2+1)f_Y(y)F_Y^{i_2}(y) - (1+q_2)\lambda R^*(r;k,m,n)\sum_{i_2=0}^{q_2-1} I_4(i_2+2)f_Y(y)F_Y^{i_2+1}(y)$$
$$= f_Y(y) + \lambda R^*(r;k,m,n)\sum_{i_2=0}^{q_2-1} I_2 f_{V_1}(y) - (1+ q_2)\lambda R^*(r;k,m,n)\sum_{i_2=0}^{q_2-1} I_4 f_{V_2}(y), \tag{3.5}$$

where I_2, I_4, V_1, V_2 are defined in (2.4) and (2.5). Thus, using (3.5), the generating moment function of $Y_{[r;k,m,n]}$ is given by

$$M_{[r;k,m,n]}(t) = \alpha_2\beta\left(\alpha_2, 1-\frac{t}{\theta_2}\right) + \alpha_2\lambda R^*(r;k,m,n)\Big[\sum_{i_2=0}^{q_2-1} I_2\beta(\alpha_2(i_2+1), 1-\frac{t}{\theta_2}) -(1+q_2)\sum_{i_2=0}^{q_2-1} I_4\beta(\alpha_2(i_2+2), 1-\frac{t}{\theta_2})\Big]. \tag{3.6}$$

Accordingly, from (3.5) (or (3.6)), the lth moment of $Y_{[r;k,m,n]}$ is given by

$$\mu^l_{[r;k,m,n]}(t) = E\big(Y^l_{[r;k,m,n]}\big) = E(Y^l) + \lambda R^*(r;k,m,n)\Big[\sum_{i_2=0}^{q_2-1} I_2E(V_1^l) - (1+q_2)\sum_{i_2=0}^{q_2-1} I_4E(V_2^l)\Big]$$
$$= \frac{\alpha_2\Gamma(l+1)}{\theta_2^l}\sum_{i=0}^{\chi(\alpha_2-1)}\frac{(-1)^i}{(i+1)^{l+1}}\binom{\alpha_2-1}{i} + \lambda R^*(r;k,m,n)\Big[\sum_{i_2=0}^{q_2-1} I_2\frac{\alpha_2(i_2+1)\Gamma(l+1)}{\theta_2^l}\sum_{i=0}^{\chi(\alpha_2(i_2+1)-1)}\frac{(-1)^i}{(i+1)^{l+1}}\binom{\alpha_2(i_2+1)-1}{i}$$
$$-(1+q_2)\sum_{i_2=0}^{q_2-1} I_4\frac{\alpha_2(i_2+2)\Gamma(l+1)}{\theta_2^l}\sum_{i=0}^{\chi(\alpha_2(i_2+2)-1)}\frac{(-1)^i}{(i+1)^{l+1}}\binom{\alpha_2(i_2+2)-1}{i}\Big]. \tag{3.7}$$

In the sequel, for integer values of θ_2 and α_2, all the moments exist. Furthermore, we get the mean of $Y_{[r;k,m,n]}$ by putting $l = 1$ as follows

$$\mu_{[r;k,m,n]}(t) = \frac{B(\alpha_2)}{\theta_2} + \frac{\lambda R^*(r;k,m,n)}{\theta_2}\Big[\sum_{i_2=0}^{q_2-1} I_2B(\alpha_2(i_2+1)) - (1+q_2)\sum_{i_2=0}^{q_2-1} I_4B(\alpha_2(i_2+2))\Big]. \tag{3.8}$$

Therefore, the difference of the means between Y and $Y_{[r;k,m,n]}$ is

$$h(r;\lambda,\alpha_2,q_2) = \mu_{[r;k,m,n]}(t) - \frac{B(\alpha_2)}{\theta_2} = \frac{\lambda R^*(r;k,m,n)}{\theta_2}\Big[\sum_{i_2=0}^{q_2-1} I_2B(\alpha_2(i_2+1)) - (1+q_2)\sum_{i_2=0}^{q_2-1} I_4B(\alpha_2(i_2+2))\Big], \tag{3.9}$$

thus, we can see that $h(r;\lambda,\alpha_2,q_2) = 0$ if $\lambda = 0$.

Theorem 3.2:

For any $1 \le r \le n$*, by using (3.8), we get*

$$\mu_{[r+2;k,m,n]}-\frac{B(\alpha_2)}{\theta_2}=W_3\left\{\begin{matrix}\frac{B(\alpha_2)}{\theta_2}(\gamma_{r+2}+2q_1-1)+\gamma_{r+1}\mu_{[r;k,m,n]}\\ +(1-\gamma_{r+2}-\gamma_{r+1}-2q_1)\mu_{[r+1;k,m,n]}\end{matrix}\right\}, \tag{3.10}$$

where

$$W_3=\frac{-\gamma_{r+2}}{(q_1+\gamma_{r+2})(q_1+\gamma_{r+2}-1)}. \tag{3.11}$$

***Proof*:**

Since

$$\begin{aligned}R^*(r+1;k,m,n)&=c_r\left\{\frac{(1+q_1)}{\prod_{i=1}^{r+1}(q_1+\gamma_i)}-\frac{q_1}{\prod_{i=1}^{r+1}(q_1+\gamma_i-1)}\right\}\\&=\frac{\gamma_{r+1}}{q_1+\gamma_{r+1}}R^*(r;k,m,n)-\frac{\gamma_{r+1}W_2}{(q_1+\gamma_{r+1})(q_1+\gamma_{r+1}-1)},\end{aligned} \tag{3.12}$$

where $W_2=\frac{c_{r-1}q_1}{\prod_{i=1}^{r}(\gamma_i+q_1-1)}$. Therefore, from (3.12), we find that

$$W_2=\frac{(q_1+\gamma_{r+1})(q_1+\gamma_{r+1}-1)}{\gamma_{r+1}}\left[\frac{\gamma_{r+1}}{q_1+\gamma_{r+1}}R^*(r;k,m,n)-R^*(r+1;k,m,n)\right]. \tag{3.13}$$

Since

$$\begin{aligned}R^*(r+2;k,m,n)&=c_{r+1}\left\{\frac{(1+q_1)}{\prod_{i=1}^{r+2}(\gamma_i+q_1)}-\frac{q_1}{\prod_{i=1}^{r+2}(\gamma_i+q_1-1)}\right\}\\&=\gamma_{r+2}\gamma_{r+1}\left\{\frac{W_1}{(q_1+\gamma_{r+2})(q_1+\gamma_{r+1})}-\frac{W_2}{(q_1+\gamma_{r+2}-1)(q_1+\gamma_{r+1}-1)}\right\}\\&=\gamma_{r+2}\gamma_{r+1}\left\{\frac{R^*(r;k,m,n)}{(q_1+\gamma_{r+2})(q_1+\gamma_{r+1})}\right.\\&\qquad\left.+W_2\left(\frac{1}{(q_1+\gamma_{r+2})(q_1+\gamma_{r+1})}-\frac{1}{(q_1+\gamma_{r+2}-1)(q_1+\gamma_{r+1}-1)}\right)\right\}\\&=\frac{-\gamma_{r+2}}{(q_1+\gamma_{r+2})(q_1+\gamma_{r+2}-1)}[\gamma_{r+1}R^*(r;k,m,n)]\\&\qquad+R^*(r+1;k,m,n)(1-2q_1-\gamma_{r+1}-\gamma_{r+2})],\end{aligned} \tag{3.14}$$

where $W_1=\frac{c_{r-1}(1+q_1)}{\prod_{i=1}^{r}(\gamma_i+q_1)}=R^*(r;k,m,n)+W_2$.

Remark 3.1:

Based on order statistics (with $m=0$ and $k=1$), then, the formula (3.4) is reduced to the following

$$\begin{aligned}R^*(r;1,0,n)&=\left(\prod_{i=1}^{r}(n+1-i)\right)\left\{\frac{2}{\prod_{i=1}^{r}(n+2-i)}-\frac{1}{\prod_{i=1}^{r}(n+1-i)}\right\}\\&=\frac{2\prod_{i=1}^{r}(n+1-i)}{\prod_{i=1}^{r}(n+2-i)}-1\\&=\frac{2(n+1-r)}{n+1}-1\\&=\frac{n+1-2r}{n+1}.\end{aligned} \tag{3.15}$$

Moreover, the recurrence relation (3.10) is reduced to $\mu_{[r+2;n,1,0,n]}=\mu_{r+2:n}=2\mu_{r+1:n}-\mu_{r:n}$, of $Y_{[r;n,1,0,n]}=Y_{[r:n]}$, which is obtained by Tahmasebi and Jafari [11].

To obtain the variance of $Y_{[r;k,m,n]}$, multiplying the both sides of (3.5) by $(y - \mu_{[r;k,m,n]})^2$ and integrating, we get

$$\sigma^2_{[r;k,m,n]}(y) = \int_{-\infty}^{\infty} (y - \mu_{[r;k,m,n]})^2 f_Y(y) dy$$
$$+\lambda R^*(r;k,m,n) \sum_{i_2=0}^{q_2-1} I_2 \int_{-\infty}^{\infty} (y - \mu_{[r;k,m,n]})^2 f_{V_1}(y) dy$$
$$-(1+q_2)\lambda R^*(r;k,m,n) \sum_{i_2=0}^{q_2-1} I_4 \int_{-\infty}^{\infty} (y - \mu_{[r;k,m,n]})^2 f_{V_2}(y) dy$$
$$= \sigma_y^2 + (\mu_{[r;k,m,n]} - \mu_y)^2 + \lambda R^*(r;k,m,n)\left[\sum_{i_2=0}^{q_2-1} I_2 A_1 - (1+q_2)\sum_{i_2=0}^{q_2-1} I_4 A_2\right], \tag{3.16}$$

where

$$A_1 = \frac{C(\alpha_2(i_2+1))}{\theta_2^2} + \frac{(B(\alpha_2) - B(\alpha_2(i_2+1)))^2}{\theta_2^2}$$
$$+ \frac{1}{\theta_2^2}[\lambda R^*(r;k,m,n)[\sum_{j_2=0}^{q_2-1} I_2 B(\alpha_2(j_2+1))$$
$$-(1+q_2)\sum_{j_2=0}^{q_2-1} I_4 B(\alpha_2(j_2+2))]]^2$$
$$-2\frac{1}{\theta_2}[\lambda R^*(r;k,m,n)[\sum_{j_2=0}^{q_2-1} I_2 B(\alpha_2(j_2+1))$$
$$-(1+q_2)\sum_{j_2=0}^{q_2-1} I_4 B(\alpha_2(j_2+2))]]$$
$$\times [B(\alpha_2(i_2+1)) - B(\alpha_2)] \tag{3.17}$$

$$A_2 = \frac{C(\alpha_2(i_2+2))}{\theta_2^2} + \frac{(B(\alpha_2) - B(\alpha_2(i_2+2)))^2}{\theta_2^2}$$
$$+ \frac{1}{\theta_2^2}[\lambda R^*(r;k,m,n)[\sum_{j_2=0}^{q_2-1} I_2 B(\alpha_2(j_2+1))$$
$$-(1+q_2)\sum_{j_2=0}^{q_2-1} I_4 B(\alpha_2(j_2+2))]]^2$$
$$-2\frac{1}{\theta_2}[\lambda R^*(r;k,m,n)[\sum_{j_2=0}^{q_2-1} I_2 B(\alpha_2(j_2+1))$$
$$-(1+q_2)\sum_{j_2=0}^{q_2-1} I_4 B(\alpha_2(j_2+2))]]$$
$$\times [B(\alpha_2(i_2+2)) - B(\alpha_2)], \tag{3.18}$$

and from (3.8)

$$\mu_{[r;k,m,n]} - \mu_y = \frac{\lambda R^*(r;k,m,n)}{\theta_2}$$
$$\left[\sum_{i_2=0}^{q_2-1} I_2 B(\alpha_2(i_2+1)) - (1+q_2)\sum_{i_2=0}^{q_2-1} I_4 B(\alpha_2(i_2+2))\right]. \tag{3.19}$$

Clearly, from Remark (3.1), we get

$$\sigma^2_{[r:n]} = \frac{1}{\theta_2^2}[C(\alpha_2) + \delta_r(C(\alpha_2) - C(2\alpha_2)) - \delta_r(1+\delta_r)D^2(\alpha_2)], \tag{3.20}$$

where $\delta_r = \frac{\lambda(n-2r+1)}{n+1}$ and $D(\alpha_2) = B(2\alpha_2) - B(\alpha_2)$,
which is obtained by Barakat et al. [3].

For order statistics as a special case of $m - gos$, the limiting distribution of the nth concomitants order statistic of size n, $Y_{[n:n]}$, depends on the marginal distribution of X and the conditional distribution of Y given $X = x$, which is given by

$$\begin{aligned} F_{(Y|X)}(y|x) &= \int_0^y f_{Y|X}(v|x)dv \\ &= \int_0^y f_Y(v)[1+\lambda(1-F_X(x))^{q_1-1}(1-(1+q_1)F_X(x))(1- \\ &\quad F_Y(v))^{q_2-1} \times (1-(1+q_2)F_Y(v))]dv \\ &= F_Y(y)[1-\lambda(1-F_X(x))^{q_1-1}((1+q_1)F_X(x)-1)(1-F_Y(y))^{q_2}], \end{aligned} \quad (3.21)$$

and the following theorem gives the limiting distribution of $Y_{[n:n]}$.

Theorem 3.3:

Let X and Y be two random variables having GE distribution defined in (1.5), $X \sim GE(\theta_1; \alpha_1)$ *and* $Y \sim GE(\theta_2; \alpha_2)$, *then*

$$F_{[n:n]}(A_n\, y) \underset{n,w}{\longrightarrow} F_Y(y), \quad (3.22)$$

where $A_n = \frac{1}{\theta_2}$, $\underset{n,w}{\longrightarrow}$ refer to the weak convergence, as $n \to \infty$ (see Galambos [6]) and $F_{[n:n]}(.)$ is the cdf of the nth concomitants order statistic of size n.

***Proof*:**

From Barakat et al. [3], we get

$$P(X_{(n:n)} \leq a_n\, x + b_n) = F_X^n(a_n\, x + b_n) \underset{n,w}{\longrightarrow} e^{-e^{-x}}, \forall x, \quad (3.23)$$

where $a_n = \frac{1}{\theta_1}$, $b_n = -\log[\alpha_1 n]$ and the integer part of x denoted by [x]. Moreover, from (3.21), we get

$$F_{Y|X}(A_n\, y|X = a_n\, x + b_n) \underset{n,w,F_X(x)=1}{\longrightarrow} T(x,y) = F_Y(y). \quad (3.24)$$

On the other hand, we can easily check that the cdf $F_X(x)$ satisfies the von Mises condition, see Barakat et al. [3]. Thus, Theorem (5.5.1), in Galambos [6], (3.23) and (3.24) are the sufficient conditions for the relation

$$F_{[n:n]}(A_n\, y) \underset{n,w}{\longrightarrow} \int_{-\infty}^{\infty} T(x,y) e^{-e^{-x}} e^{-x} dx = F_Y(y), \quad (3.25)$$

which terminates the proof.

4. JOINT DISTRIBUTION OF CONCOMITANTS OF ORDER STATISTICS UNDER BR-GE($\boldsymbol{\theta_1, \theta_2; \alpha_1, \alpha_2}$)

The joint pdf of the order statistics $X_{(r:n)}$ and $X_{(s:n)}$, $1 \leq r < s \leq n$, $x_1 < x_2$, is given by:

$$\begin{aligned} f_{(r,s:n)}(x_1,x_2) &= \frac{1}{\beta(r,s-r,n-s+1)} F_X^{r-1}(x_1)(F_X(x_2) \\ &\quad -F_X(x_1))^{s-r-1}(1-F_X(x_2))^{n-s} \times f_X(x_1) f_X(x_2), \end{aligned} \quad (4.1)$$

where $\beta(a,b,c) = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}$, for more details see Arnold et al. [1]. Therefore, from (3.1) and (4.1), the joint pdf of concomitants $Y_{[r:n]}$ and $Y_{[s:n]}$, $r < s$, is

$$f_{(r,s:n)}(y_1,y_2)=\int_0^\infty\int_0^{x_2} f_{Y|X}(y_1|x_1)f_{Y|X}(y_2|x_2)f_{(r,s:n)}(x_1,x_2)dx_1dx_2$$
$$=f_Y(y_1)f_Y(y_2)\int_0^\infty\int_0^{x_2}\{1+\lambda^2A^*_{Y_1}A^*_{Y_2}[(1-F_X(x_1))^{q_1-1}(1-F_X(x_2))^{q_2-1}$$
$$-(1+q_1)F_X(x_1)(1-F_X(x_1))^{q_1-1}(1-F_X(x_2))^{q_2-1}$$
$$-(1+q_1)F_X(x_2)(1-F_X(x_1))^{q_1-1}(1-F_X(x_2))^{q_2-1}$$
$$+(1+q_1)^2F_X(x_1)F_X(x_2)(1-F_X(x_1))^{q_1-1}(1-F_X(x_2))^{q_2-1}]$$
$$+\lambda A^*_{Y_1}[(1-F_X(x_1))^{q_1-1}-(1+q_1)F_X(x_1)(1-F_X(x_1))^{q_1-1}]$$
$$+\lambda A^*_{Y_2}[(1-F_X(x_2))^{q_1-1}-(1+q_1)F_X(x_2)(1-F_X(x_2))^{q_1-1}]\}$$
$$\times\frac{1}{\beta(r,s-r,n-s+1)}F_X^{r-1}(x_1)(F_X(x_2)-F_X(x_1))^{s-r-1}(1-F_X(x_2))^{n-s}$$
$$\times f_X(x_1)f_X(x_2)dx_1dx_2, \quad (4.2)$$

where

$$A^*_{Y_1}=(1-F_Y(y_1))^{q_2-1}[1-(1+q_2)F_Y(y_1)],$$
$$A^*_{Y_2}=(1-F_Y(y_2))^{q_2-1}[1-(1+q_2)F_Y(y_2)]. \quad (4.3)$$

Remark 4.1:

To find the integration of (4.2) we use the following general forms of integrations, see Gradshteyn and Ryzhik [7],

1. $\int_a^b (x-a)^{\mu-1}(b-x)^{\nu-1}dx=(b-a)^{\mu+\nu-1}\beta(\mu,\nu),$ $b>a, Re(\mu), Re(\nu)>0.$ (4.4)

2. $\int_a^b x(x-a)^{\nu-1}(b-x)^{\mu-1}dx=(b-a)^{\mu+\nu-1}(a\mu+b\nu)\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu+1)},$ $b>a, Re(\mu), Re(\nu)>0.$ (4.5)

On the other hand, we have

$$J_1=\int_0^\infty\int_0^{x_2}(1-F_X(x_1))^{q_1-1}(1-F_X(x_2))^{q_1-1}dx_1dx_2$$
$$\times\frac{1}{\beta(r,s-r,n-s+1)}F_X^{r-1}(x_1)(F_X(x_2)-F_X(x_1))^{s-r-1}(1-F_X(x_2))^{n-s} \quad (4.6)$$
$$\times f_X(x_1)f_X(x_2)dx_1dx_2,$$

by substituting $u=F_X(x_1)$ and $v=F_X(x_2)$, $0<u,v<1$, we get

$$J_1=\frac{1}{\beta(r,s-r,n-s+1}\int_0^1\int_0^v u^{r-1}(1-u)^{q_1-1}(1-v)^{n+q_1-s-1}(v-u)^{s-r-1}dudv$$
$$=\frac{1}{\beta(r,s-r,n-s+1)}\int_0^1\int_u^1 u^{r-1}(1-u)^{q_1-1}(1-v)^{n+q_1-s-1}(v-u)^{s-r-1}dvdu \quad (4.7)$$
$$=\frac{\Gamma(n+1)\Gamma(n+2q_1-r-1)\Gamma(n+q_1-s)}{\Gamma(n-s+1)\Gamma(n+q_1-r)\Gamma(n+2q_1-1)},$$

$$J_2 = \int_0^\infty \int_0^{x_2} F_X(x_2)(1-F_X(x_1))^{q_1-1}(1-F_X(x_2))^{q_1-1}dx_1dx_2$$
$$\times \frac{1}{\beta(r,s-r,n-s+1)} F_X^{r-1}(x_1)(F_X(x_2)-F_X(x_1))^{s-r-1}(1-F_X(x_2))^{n-s}$$
$$\times f_X(x_1)f_X(x_2)dx_1dx_2$$
$$= \frac{1}{\beta(r,s-r,n-s+1)} \int_0^1 \int_u^1 vu^{r-1}(1-u)^{q_1-1} \quad (4.8)$$
$$(1-v)^{n+q_1-s-1}(v-u)^{s-r-1}dvdu$$
$$= \frac{\Gamma(n+1)\Gamma(n+2q_1-r-1)\Gamma(n+q_1-s)}{\Gamma(n-s+1)\Gamma(n+q_1-r+1)\Gamma(n+2q_1)}[(n+2q_1-1)s-r(s+q_1-1)],$$

$$J_3 = \frac{1}{\beta(r,s-r,n-s+1)}$$
$$\int_0^1 \int_u^1 u^r(1-u)^{q_1-1}(1-v)^{n+q_1-s-1}(v-u)^{s-r-1}dvdu \quad (4.9)$$
$$= \frac{r\Gamma(n+1)\Gamma(n+2q_1-r-1)\Gamma(n+q_1-s)}{\Gamma(n-s+1)\Gamma(n+q_1-r)\Gamma(n+2q_1)},$$

$$J_4 = \frac{1}{\beta(r,s-r,n-s+1)}$$
$$\int_0^1 \int_u^1 uvu^{r-1}(1-u)^{q_1-1}(1-v)^{n+q_1-s-1}(v-u)^{s-r-1}dvdu \quad (4.10)$$
$$= \frac{\Gamma(n+1)\Gamma(n+2q_1-r-1)\Gamma(n+q_1-s)}{\Gamma(n-s+1)\Gamma(n+q_1-r+1)\Gamma(n+2q_1+1)}$$
$$\left[r\left(n+q_1(r-1)+s(n+2q_1-r-1)\right)\right],$$

$$J_5 = \frac{1}{\beta(r,s-r,n-s+1)}$$
$$\int_0^1 \int_u^1 u^{r-1}(1-u)^{q_1-1}(1-v)^{n-s}(v-u)^{s-r-1}dvdu \quad (4.11)$$
$$= \frac{\Gamma(n+1)\Gamma(n+q_1-r)}{\Gamma(n-r+1)\Gamma(n+q_1)},$$

$$J_6 = \frac{1}{\beta(r,s-r,n-s+1)} \int_0^1 \int_u^1 u^r(1-u)^{q_1-1}(1-v)^{n-s}(v-u)^{s-r-1}dvdu$$
$$= \frac{r\Gamma(n+1)\Gamma(n+q_1-r)}{\Gamma(n-r+1)\Gamma(n+q_1+1)}, \quad (4.12)$$

$$J_7 = \frac{1}{\beta(r,s-r,n-s+1)} \int_0^1 \int_u^1 u^{r-1}(1-v)^{n+q_1-s-1}(v-u)^{s-r-1}dvdu$$
$$= \frac{\Gamma(n+1)\Gamma(n+q_1-s)}{\Gamma(n-s+1)\Gamma(n+q_1)}, \quad (4.13)$$

$$J_8 = \frac{1}{\beta(r,s-r,n-s+1)}$$
$$\int_0^1 \int_u^1 vu^{r-1}(1-u)^{q_1-1}(1-v)^{n+q_1-s-1}(v-u)^{s-r-1}dvdu \quad (4.14)$$
$$= \frac{s\Gamma(n+1)\Gamma(n+q_1-s)}{\Gamma(n-s+1)\Gamma(n+q_1+1)}.$$

Now, combining (4.7)-(4.14), with (4.2), we get

$$f_{(r,s:n)}(y_1,y_2) = f_Y(y_1)f_Y(y_2)\{1+\lambda^2 A^*_{Y_1}A^*_{Y_2}$$
$$[J_1-(1+q_1)J_2-(1+q_1)J_3+(1+q_1)^2J_4]$$
$$+\lambda A^*_{Y_1}[J_5-(1+q_1)J_6]+\alpha A^*_{Y_2}[J_7-(1+q_1)J_8]\} \quad (4.15)$$
$$= f_Y(y_1)f_Y(y_2)+T_1T_2(\lambda^2[J_1-(1+q_1)(J_2+J_3)+(1+q_1)^2J_4])$$
$$= f_Y(y_1)f_Y(y_2)+T_1T_2(\lambda^2[J_1-(1+q_1)(J_2+J_3)+(1+q_1)^2J_4])$$
$$+\lambda T_1 f_Y(y_2)[J_5-(1+q_1)J_6]+\lambda T_2 f_Y(y_1)[J_7-(1+q_1)J_8],$$

where

$$T_1 = \sum_{i_2=0}^{q_2-1} I_2 f_{V_1}(y_1) - (1+q_2)\sum_{i_2=0}^{q_2-1} I_4 f_{V_2}(y_1),$$
$$T_2 = \sum_{i_2=0}^{q_2-1} I_2 f_{V_1}(y_2) - (1+q_2)\sum_{i_2=0}^{q_2-1} I_4 f_{V_2}(y_2), \quad (4.16)$$

V_1, V_2 are defined in (2.5). Thus, we obtain the product moment $E[Y_{[r:n]}Y_{[s:n]}] = \mu_{[r,s:n]}$ directly from (4.15) as follows

$$\mu_{[r,s:n]} = \frac{B^2(\alpha_2)}{\theta_2^2} + \frac{K_1}{\theta_2^2}\{(\lambda^2[J_1 - (1+q_1)(J_2+J_3) + (1+q_1)^2 J_4]) \\ K_1 + \lambda B(\alpha_2)[J_5 + J_7 - (1+q_1)(J_6+J_8)]\}, \quad (4.17)$$

where

$$K_1 = \sum_{i_2=0}^{q_2-1} I_2 B(\alpha_2)(i_2+1) - (1+q_2)\sum_{i_2=0}^{q_2-1} I_4 B(\alpha_2)(i_2+2). \quad (4.18)$$

Therefore, by using (3.8), (3.16), with $m = 0$ and $k = 1$, and (4.17), the covariance and correlation between $Y_{[r:n]}$ and $Y_{[s:n]}$, $\sigma_{[r,s:n]} = \mu_{[r,s:n]} - \mu_{[r:n]}\mu_{[s:n]}$ and $\rho_{[r,s:n]} = \frac{\sigma_{[r,s:n]}}{\sqrt{\sigma^2_{[r:n]}\sigma^2_{[s:n]}}}$, respectively, can be obtained after simple calculations.

5. CONCLUSION

We derived the Bairamov Morgenstern family type bivariate GE distribution based on concomitant of $m-gos$. We also provided the correlation and its admissible range for such distribution. Moreover, we purposed the moment generating function, recurrence relation and variance of the concomitant of rth $m-gos$, $Y_{[r;k,m,n]}$, of BR-GE$(\theta_1,\theta_2;\alpha_1,\alpha_2)$. Finally, for BR-GE$(\theta_1,\theta_2;\alpha_1,\alpha_2)$ distribution of concomitant of order statistics, we studied its limiting distribution and the joint pdf of concomitants $Y_{[r:n]}$ and $Y_{[s:n]}$, $r < s$.

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REFERENCES

1. Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. (1992). *A First Course in Order Statistics*. John Wiley & Sons, New York.
2. Bairamov, I., Kotz, S. and Bekci, M. (2001). New generalized Farlie-Gumbel-Morgenstern distributions and concomitants of order statistics. *J. of App. Stat.*, 28(5), 521-536.
3. Barakat H.M., Nigm, E.M. and Syam, A.H. (2017). Concomitants of Ordered from Huang–Kotz FGM Type Bivariate Generalized Exponential Distribution. *Bull. Malays Math. Soc.*, 42(1), 337-353.
4. Barakat, H.M., Nigm, E.M. and Syam, A.H. (2018). Concomitants of Statistics and Record Values from General FGM Type Bivariate-Generalized Exponential Distribution. *FILOMAT*, 32(9), 3313-3324.

5. David, H.A., O'Connell, M.J. and Yang, S.S. (1977). Distribution and expected value of the rank of a concomitant of an order statistic. *Ann. of Stat.*, 5, 216-223.
6. Galambos, J. (1987). *The Asymptotic Theory of Extreme Order Statistics*. 2nd edn. Krieger, Malabar.
7. Gradshteyn, I.S. and Ryzhik, I.M. (2007). Table of Integrals, Series, and Products. Academic Press, 7th ed. Elsevier.
8. Gupta, R.D. and Kundu, D. (1999). Generalized exponential distributions. *Aust. N Z J Stat*., 41, 173-188.
9. Kamps, U. (1995). A concept of generalized order statistics. *J. of Stat. Plann Inf.*, 48, 1-23.
10. Mohamed, M.S. (2019). On concomitants of ordered random variables under general forms of morgenstern family. *FILOMAT*, 33(9), 2771-2780, doi.org/10.2298/FIL1909771M.
11. Mohamed, M.S. (2019). A measure of inaccuracy in concomitants of ordered random variables under Farlie-Gumbel-Morgenstern family. FILOMAT, 33(15), 4931-4942, doi.org/10.2298/FIL1915931M.
12. Mohamed, M.S. (2020). On Cumulative Tsallis Entropy and its Dynamic Past Version. Ind. J. of Pure. and Appl. Math., In press.
13. Tahmasebi, S. and Jafari, A.A. (2015). Concomitants of order statistics and record values from Morgenstern type bivariate-generalized exponential distribution. *Bull. Malays Math. Sci. Soc*., 38, 1411-1423.