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A NEW FAMILY OF DISTRIBUTIONS FOR GENERATING SKEWED MODELS: PROPERTIES AND APPLICATIONS

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ABSTRACT

We propose a new family of continuous distributions with two extra parameters named Transmuted Exponential- G family of distributions. We provide a special member for the new family of distributions. An explicit expression for some of its mathematical and structural properties such as reliability function, failure rate, ordinary moments, incomplete moments, generating function, Renyi entropy and order statistics were derived and presented. The method of maximum likelihood is used to estimate the parameters of the developed family of distributions. A simulation study is carried out to assess the performance of the maximum likelihood estimators in terms of biases and mean squared errors. Real-life data are used to validate the robustness of the developed family of distribution.

KEYWORDS

Transmuted Exponential, Reliability function, Maximum Likelihood, Order statistics.

1. INTRODUCTION

Real-life phenomenon are well or better described using statistical models (distributions). In reality, standard (classical) distributions may not be sufficient in describing real datasets. For example, normal distribution will not be a better choice for modeling asymmetric dataset.

The effectiveness of a probability distribution in modeling depends on its ability to properly capture the sensitive parts of a given data set. As data drawn from a population in numerous areas continue to exhibit complexity and changes in shape, which makes it difficult to easily choose a distribution that will provide a better fit to the datasets. Furthermore, the data generated is often characterized with the problems of elongation and asymmetry, which makes it difficult for the classical distributions to provide adequate fit to the real data. In this case, extending the existing models are more appropriate than transposing the data sets. The main motivation for using the Transmuted Exponential- G family is to construct heavy-tailed distributions with all types of hazard rates for modeling dataset which are characterized with the problems of elongation and asymmetry.

The addition of a shape parameter(s) in baseline (parent) model to improve the goodness-of-fits fall under generalization approach. Some generalized classes which are familiar to most researchers can be found in Marshall and Olkin (1997), Gupta et al. (1998), Eugene et al. (2002), Shaw and Buckley (2007), Pescim et al. (2012), Cordeiro et al. (2012), Cordeiro et al. (2013). The new approach for developing a family of distributions started with the work of Alzaatreh et al. (2013) who proposed a new method for generating different families of distributions called T–X family of distributions. Weibull- G by Bourguignon et al. (2014), A study on Mathematical properties of Marshall- Olkin- G family was carried out by (Cordeiro et al., 2014). A new family of generalized Cauchy distributions by Alzaatreh et al. (2015), Exponentiated Marshal-Olkin family of distributions by Cícero et al. (2015), A study on compounding of distribution by Tahir et al. (2015), Transmuted Exponentiated Generalized- G by Yousof et al. (2015), The Zografos-Balakrishnan- G family by Nadarajah et al. (2015), Owoloko et al. (2015) investigated the performance rating of the Transmuted Exponential (TE) distribution with to some other competing models, The Kumaraswamy Transmuted-G Family by Afify et al. (2016). Odd Lindley-G family by Gomes-Silva et al. (2017), The Beta Weibull-G Family by Yousof et al. (2017), A study on the general properties of Gompertz-G family by Alizadeh et al. (2017), Topp–Leone generated family by Rezaei et al. (2017), On generalized classes of exponential distribution by (Zubair et al., 2018), On Generating a New Family of Distributions Usingthe Tangent Function by Al-Mofleh (2018), The Alpha PowerTransformation Family: Properties and Applications by Mead et al. (2019), The quasixgamma-geometric distribution with application in medicine by (Sen 2019).

In this article, we derive and define a new tractable family based on T-X method using Transmuted Exponential Distributions as generator that hold for any parent distribution. The TE- G family of distributions has two parameter including scale and transmuted parameters which plays a vital role in stretching and adding skewness to a distribution respectively. An explicit expression for some of its structural properties comprising the reliability function, hazard function, incomplete and ordinary moments, generating function, order statistics and Renyi entropy were derived and presented.

The rest of the paper is outlined in sections as follows. In Section 2, we defined the TE-G family of distributions. Linear representation for the TE- G density function are presented in section 3. Some general mathematical properties of the proposed family including Survival and hazard function, ordinary and incomplete moments, generating functions, Renyi entropy and order statistics were derived and presented in Section 4. Maximum likelihood estimation (MLE) of the model parameters is investigated in Section 5. One special model of this family is presented and discussed in Section 6. Simulation study is carried out in section 7. In Section 8, we illustrated the potentiality of the new family using real and simulated datasets.

2. THE TRANSMUTED EXPONENTIAL- G FAMILY

Let T represent a continuous random variable with probability density function $r(t)$ defined on the interval [a, b]. The cumulative distribution function of a new class of distributions is defined as;

$$F(x) = \int_{a}^{w[G(x)]} r(t)dt \tag{1}$$

where $w[G(x)]$ Satisfies the conditions below

$w[G(x)] \in [a,b]$, $w[G(x)]$ is differentiable and monotonically non- decreasing

$$w[G(x)] \rightarrow a \, as \, x \rightarrow -\infty \ \& \ w[G(x)] \rightarrow b \, as \, x \rightarrow \infty .$$

The cdf of TE- G family of distributions is defined as;

$$F(x;\lambda,\theta,\xi) = \left(1-\left(1-G(x,\xi)\right)^{\lambda}\right)\left(1+\theta\left(1-G(x,\xi)\right)^{\lambda}\right) \quad (2)$$

The pdf corresponding to (2) is defined by:

$$f(x;\lambda,\theta,\xi) = \frac{g(x,\xi)}{1-G(x,\xi)}\lambda\left(1-G(x,\xi)\right)^{\lambda}\left(1-\theta+2\theta\left(1-G(x,\xi)\right)^{\lambda}\right) \quad (3)$$

where, $G(x,\xi)$ and $g(x,\xi)$ stand for baseline cdf and pdf respectively depending on a parameter vector ξ and $\lambda > 0$, $-1 \le \theta \le 1$ are two additional parameters i.e scale and transmuted parameter respectively.

Table 1
Some Special Cases of TE- G Family

λ	θ	$G(x)$	**Reduced Distribution**	**Author(s)**
1	-	$G(x)$	Transmuted-G family of distribution	Shaw & Buckley (2007)
1	0	$G(x)$	$G(x)$	-
1	-	Exponential	Transmuted Exponential distribution	Owoloko et al. (2015)

3. LINEAR REPRESENTATION

Here, we give the expansions for the cdf and pdf of TE- G family.

Recall that;

$$\left(1-z\right)^{b} = \sum_{j=0}^{\infty}(-1)^{j}\binom{b}{j}z^{j}$$

The cdf in (2) can be expressed as;

$$F(x;\lambda,\theta,\xi) = 1+(\theta-1)\sum_{j=0}^{\infty}(-1)^{j}\binom{\lambda}{j}G(x;\xi)^{j} - \theta\sum_{j=0}^{\infty}(-1)^{j}\binom{2\lambda}{j}G(x;\xi)^{j}$$

$$F(x;\lambda,\theta,\xi) = 1+\sum_{j=0}^{\infty}(-1)^{j}\left[(\theta-1)\binom{\lambda}{j}-\theta\binom{2\lambda}{j}\right]G(x;\xi)^{j}$$

$$F(x;\lambda,\theta,\xi) = 1+\sum_{j=0}^{\infty}t_{j}\Pi(x)^{j} \quad (4)$$

where, $t_{j} = (-1)^{j}\left[(\theta-1)\binom{\lambda}{j}-\theta\binom{2\lambda}{j}\right]$ and $\Pi(x)^{j} = G(x;\xi)^{j}$ is the cdf of Exp- G family with parameter j.

The corresponding TE- G density function is obtain by differentiating equation (4)

$$f(x;\lambda,\theta,\xi) = \sum_{j=0}^{\infty} jt_j g(x;\xi)G(x;\xi)^{j-1}$$

$$f(x;\lambda,\theta,\xi) = \sum_{j=0}^{\infty} t_j \pi(x)^j \tag{5}$$

where, $\pi(x)^j = jg(x;\xi)G(x;\xi)^{j-1}$ is the pdf of Exp- G family with power parameter j .

4. MATHEMATICAL PROPERTIES

Here, we explore the mathematical properties of TE- G family.

4.1 The Survival Function

The survival function of a Transmuted Exponential-G family is given by;

$$S(x) = 1 - F(x;\lambda,\theta,\xi)$$

$$S(x) = 1 - \left(1-\left(1-G(x,\xi)\right)^{\lambda}\right)\left(1+\theta\left(1-G(x,\xi)\right)^{\lambda}\right). \tag{6}$$

4.2 The Hazard Function

The hazard function of a Transmuted Exponential-G family is given by;

$$h(x) = \frac{f(x;\lambda,\theta,\xi)}{1-F(x;\lambda,\theta,\xi)}$$

$$h(x) = \frac{\lambda g(x,\xi)\left(1-G(x,\xi)\right)^{\lambda-1}(1-\theta) + 2\lambda\theta\left(1-G(x,\xi)\right)^{2\lambda-1} g(x,\xi)}{1-\left(1-\left(1-G(x,\xi)\right)^{\lambda}\right)\left(1+\theta\left(1-G(x,\xi)\right)^{\lambda}\right)}. \tag{7}$$

4.3 The Quantile Function

The quantile function of a Transmuted Exponential-G family is given by;

$$x_p = G^{-1}\left(1-\left(\frac{(\theta-1)\pm\sqrt{(\theta-1)^2+4\theta(1-p)}}{2\theta}\right)^{\frac{1}{\lambda}}\right) \tag{8}$$

where $0 < p < 1$ and G^{-1} is the quantile function of the baseline distribution.

When we substitute for p= 0.5 in (8) the result gives the median as;

$$x_{0.5} = G^{-1}\left(1-\left(\frac{(\theta-1)\pm\sqrt{(\theta-1)^2+2\theta}}{2\theta}\right)^{\frac{1}{\lambda}}\right). \tag{9}$$

4.3 Moments of TE- G Family of Distribution

Let Y be a random variable from a baseline distribution having Exponentiated- G distribution and also let X be a random variable from TE- G family, using (5) the rth ordinary moment of X is given as;

$$\mu_r' = \sum_{j=0}^{\infty} t_j E(Y_j^r) \tag{10}$$

where Y_j denotes the Exp- G distribution with power parameter j. Expressions for moments of several exp-G distributions are given in Nadarajah and Kotz (2006), which can be used to obtained $E(X^r)$.

4.4 Moment Generating Function of TE- G Family of Distribution

The Moment Generating Function of X say $M_x(t) = E(e^{tx})$ is given as;

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r' = \sum_{j,r=0}^{\infty} \frac{t^r t_j}{r!} E(Y_j^r) \tag{11}$$

4.5 Incomplete Moments of TE- G Family of Distribution

Bonferoni and Lorenz curves are refer to the first incomplete moment which are found to have a wide range of applications in many discipline such as Insurance, Economics and Medicine. The sth incomplete moments say;

$$\psi_s(t) = \int_{-\infty}^{t} x^s f(x;\lambda,\theta,\xi) dx$$

Using (5), we have

$$\psi_s(t) = \sum_{j=0}^{\infty} t_j \int_{-\infty}^{t} x^s \pi(x)^j dx \tag{12}$$

where, $\pi(x)^j = jg(x;\xi)G(x;\xi)^{j-1}$ is the pdf of Exp- G family with power parameter j and $t_j = (-1)^j \left[(\theta-1)\binom{\lambda}{j} - \theta\binom{2\lambda}{j} \right]$.

4.6 Entropy

The Renyi entropy is mathematical defined by;

$$I_\gamma(X) = \frac{1}{1-\gamma} \log\left(\int_{-\infty}^{\infty} f(x)^\gamma dx \right) \qquad ,\gamma > 0 \text{ and } \gamma \neq 1.$$

Using the pdf;

$$f(x;\lambda,\theta,\xi)=\lambda g(x;\lambda,\theta,\xi)\left(1-G(x;\lambda,\theta,\xi)\right)^{\lambda-1}\left[1-\theta\left(1-2\left(1-G(x;\lambda,\theta,\xi)\right)^{\lambda}\right)\right]$$

$$I_{\gamma}(X)=\frac{1}{1-\gamma}\log\left(\sum_{j,m=0}^{\infty}w_{j,m}\int_{-\infty}^{\infty}g(x;\lambda,\theta,\xi)^{\gamma}G(x;\lambda,\theta,\xi)^{j+m}dx\right) \qquad (13)$$

where

$$\sum_{j,m=0}^{\infty}w_{j,m}=(-1)^{j+m}\sum_{k,l=0}^{\infty}(-1)^{k+l}\binom{\gamma(\lambda-1)}{j}\binom{\gamma}{k}\binom{k}{l}\binom{\lambda l}{m}\theta^{k}2^{l}\lambda^{\gamma}.$$

4.7 Order Statistics of TE- G Family of Distribution

Suppose $X_1,...,X_n$ is a random sample from the TE- G family of distributions and let $X_{1:n},...,X_{n:n}$ be the corresponding order statistics.

The density of the i^{th} order statistics, $X_{1:n}$ can be written as;

$$f_{i:n}(x)=\frac{f(x)}{B(i,n-i+1)}F(x)^{i-1}\left(1-F(x)\right)^{n-i} \qquad (14)$$

where $B(.,.)$ stand for beta function.

$$f_{i:n}(x)=\frac{n!}{\left[(i-1)!(n-i)!\right]}\sum_{j=0}^{n-i}(-1)^{j}\binom{n-i}{j}f(x)F(x)^{j+i-1} \qquad (15)$$

The pdf of $X_{i:n}$ can be expressed as;

$$f_{i:n}(x)=\sum_{j=0}^{\infty}\left(\frac{(-1)^{j}n!\binom{n-i}{j}}{\left[(i-1)!(n-i)!\right]}\right)\sum_{k,m,w=0}^{\infty}d_{k,m,w}\pi_{k+m+w}(x)$$

where,

$$d_{k,m,w}=(-1)^{k+m+w}\sum_{h=0}^{\infty}(-1)^{h}\binom{j+i-1}{m}\binom{\lambda m}{w}\binom{j+i-1}{h}\binom{\lambda h}{k}\left((1-\theta)\binom{\lambda-1}{m}+2\theta\binom{2\lambda-1}{m}\right)$$

and $\pi_{k+m+w}(x)$ is the Exp- G.

5. ESTIMATION OF PARAMETERS OF THE TRANSMUTED EXPONENTIAL – G FAMILY OF DISTRIBUTIONS

The estimation of the parameters of the *TE-G* distribution is done by using the method of maximum likelihood estimation. Let $x_1, x_2, ..., x_n$ be a random sample from the *TE-G* distribution with unknown vector of parameters $\psi = (\lambda, \theta, \xi)^T$. The log-likelihood function is given by;

$$L(\psi) = n\log\lambda + \sum_{i=1}^{n}\log g(x_i;\xi) + (\lambda-1)\sum_{i=1}^{n}\log\left(1-G(x_i;\xi)\right) + \sum_{i=1}^{n}\log\left(1-\theta+2\theta\left(1-G(x_i;\xi)\right)^{\lambda}\right)$$

$$U_\lambda = \frac{\delta L(\psi)}{\delta\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\log\left(1-G(x_i;\xi)\right) + 2\theta\sum_{i=1}^{n}\frac{\left(1-G(x_i;\xi)\right)^{\lambda}\ln\left(1-G(x_i;\xi)\right)}{\left(1-\theta+2\theta\left(1-G(x_i;\xi)\right)^{\lambda}\right)}$$

$$U_\theta = \frac{\delta L(\psi)}{\delta\theta} = \sum_{i=1}^{n}\frac{2\left(1-G(x_i;\xi)\right)^{\lambda}-1}{\left(1-\theta+2\theta\left(1-G(x_i;\xi)\right)^{\lambda}\right)}$$

$$U_\xi = \frac{\delta L(\psi)}{\delta\xi} = \sum_{i=1}^{n}\frac{g^{'}(x_i;\xi)}{g(x_i;\xi)} - (\lambda-1)\sum_{i=1}^{n}\frac{G^{'}(x_i;\xi)}{\left(1-G(x_i;\xi)\right)} - 2\theta\lambda\sum_{i=1}^{n}\frac{\left(1-G(x_i;\xi)\right)^{\lambda-1}G^{'}(x_i;\xi)}{\left(1-\theta+2\theta\left(1-G(x_i;\xi)\right)^{\lambda}\right)}$$

where $G^{'}(\cdot)$ represent the derivatives of $G(x_i;\xi)$ with respect to ξ. Set the equations $U_\lambda = U_\theta = U_\xi = 0$ and solve them simultaneously to give the estimate $\widehat{\psi} = (\widehat{\lambda}, \widehat{\theta}, \widehat{\xi})^T$. In order to numerically maximize L, Newton- Rapson algorithm for nonlinear optimization is considered to be more appropriate approach for solving these equations. For an interval estimation of the parameters, we obtain the $p \times p$ observed information matrix $J(\psi) = \left\{\frac{\delta L}{\delta r\delta s}\right\}\{for\ r,s = \lambda,\theta,\xi\}$ whose elements can be computed numerically. Under standard regular conditions when $n \to \infty$, the distribution of $\hat{\psi}$ can be approximated by a multivariate normal $N_p\left(0, J(\hat{\psi})^{-1}\right)$ distribution to construct confidence intervals for the parameters.

6. SPECIAL MODELS OF THE TE- G FAMILY

The probability density function (3) will be very easy to handle when the parent or baseline $g(x;\xi)$ and $G(x;\xi)$ have simple analytical forms. These sub- distributions generalize some familiar classical models in the context of statistical literature. Here, we provide two sub- distributions of the TE- G family of distributions corresponding to the baseline Weibull (W) Distributions.

6.1 The TE- Weibull (TE-W) Distribution

Consider the pdf $g(x;k,\gamma)=\frac{k}{\gamma}\left(\frac{x}{\gamma}\right)^{k-1}e^{-\left(\frac{x}{\gamma}\right)^k}$ and cdf $G(x;k,\gamma)=1-e^{-\left(\frac{x}{\gamma}\right)^k}$ of the Weibull distribution with scale $\gamma>0$ and shape $k>0$ parameters. Inducing these functions in (2) and (3), the cdf and pdf of the Transmuted Exponential Weibull Distribution (for x>0) are respectively given as;

$$F(x;\lambda,\theta,k,\gamma)=\left(1-e^{-\lambda\left(\frac{x}{\gamma}\right)^k}\right)\left(1+\theta e^{-\lambda\left(\frac{x}{\gamma}\right)^k}\right) \tag{16}$$

and

$$f(x;\lambda,\theta,k,\gamma)=\frac{\lambda k}{\gamma}\left(\frac{x}{\gamma}\right)^{k-1}e^{-\left(\frac{x}{\gamma}\right)^k}\left(e^{-\left(\frac{x}{\gamma}\right)^k}\right)^{\lambda-1}(1-\theta)$$

$$+2\lambda\theta\left(e^{-\left(\frac{x}{\gamma}\right)^k}\right)^{2\lambda-1}\frac{k}{\gamma}\left(\frac{x}{\gamma}\right)^{k-1}e^{-\left(\frac{x}{\gamma}\right)^k}$$

$$f(x;\lambda,\theta,k,\gamma)=\frac{\lambda k(1-\theta)}{\gamma}\left(\frac{x}{\gamma}\right)^{k-1}e^{-\lambda\left(\frac{x}{\gamma}\right)^k}+\frac{2\lambda\theta k}{\gamma}\left(\frac{x}{\gamma}\right)^{k-1}e^{-2\lambda\left(\frac{x}{\gamma}\right)^k} \tag{17}$$

The plots of the cdf and pdf of the TE-WD are respectively displayed in figure 1 and 2 for some selected values $\lambda=a,\gamma=b,\theta=c,$ and $k=d$.

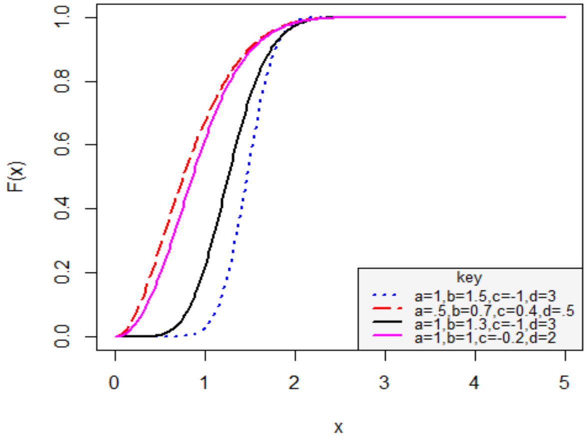

Figure 1: The cdf of TE-WD

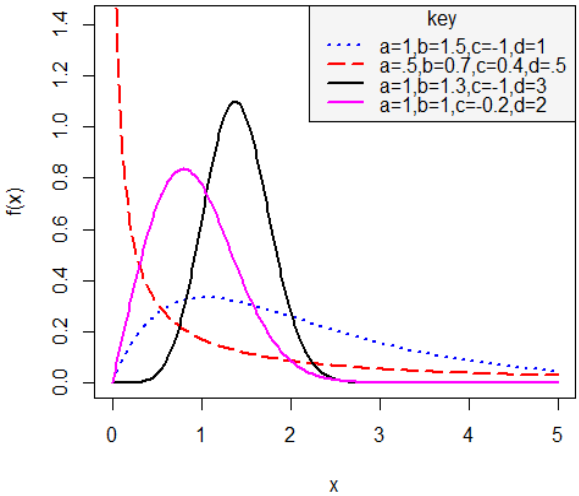

Figure 2: The pdf of TE-WD

6.1.1 Model Validity Check

To check the validity of the pdf in (17) the equation below is used;

$$\int_0^\infty f(x;\lambda,\theta,k,\gamma)dx = 1 \quad (18)$$

$$\int_0^\infty \left\{ \frac{\lambda k(1-\theta)}{\gamma}\left(\frac{x}{\gamma}\right)^{k-1} e^{-\lambda\left(\frac{x}{\gamma}\right)^k} + \frac{2\lambda\theta k}{\gamma}\left(\frac{x}{\gamma}\right)^{k-1} e^{-2\lambda\left(\frac{x}{\gamma}\right)^k} \right\} dx = 1$$

$$\int_0^\infty \frac{\lambda k(1-\theta)}{\gamma}\left(\frac{x}{\gamma}\right)^{k-1} e^{-\lambda\left(\frac{x}{\gamma}\right)^k} dx + \int_0^\infty \frac{2\lambda\theta k}{\gamma}\left(\frac{x}{\gamma}\right)^{k-1} e^{-2\lambda\left(\frac{x}{\gamma}\right)^k} dx$$

$$\frac{\lambda k(1-\theta)}{\gamma^k}\int_0^\infty x^{k-1} e^{-\lambda\left(\frac{x}{\gamma}\right)^k} dx + \frac{2\lambda\theta k}{\gamma^k}\int_0^\infty x^{k-1} e^{-2\lambda\left(\frac{x}{\gamma}\right)^k} dx$$

Recall that

$$\int_0^\infty x^n e^{-ax^b} dx = \frac{1}{b} a^{-\frac{(n+1)}{b}} \Gamma\left(\frac{n+1}{b}\right)$$

$$\therefore \int_0^\infty x^{k-1} e^{-\frac{\lambda}{\gamma^k}x^k} dx = \frac{1}{k}\left(\frac{\lambda}{\gamma^k}\right)^{-\frac{k}{k}} \Gamma\left(\frac{k}{k}\right) = \frac{\gamma^k}{\lambda k} \times 1 = \frac{\gamma^k}{\lambda k}$$

and

$$\int_0^\infty x^{k-1} e^{-\frac{2\lambda}{\gamma^k}x^k} dx = \frac{1}{k}\left(\frac{2\lambda}{\gamma^k}\right)^{-\frac{k}{k}} \Gamma\left(\frac{k}{k}\right) = \frac{\gamma^k}{2\lambda k} \times 1 = \frac{\gamma^k}{2\lambda k}$$

$$\Rightarrow \frac{\lambda k(1-\theta)}{\gamma^k} \times \frac{\gamma^k}{\lambda k} + \frac{2\theta\lambda k}{\gamma^k} \times \frac{\gamma^k}{2\lambda k} = 1-\theta+\theta = 1$$

Hence, the model in equation (17) is a valid probability density function.

6.1.2 Survival Function of TE-WD

$$S(x;\lambda,\theta,k,\gamma) = 1 - F(x;\lambda,\theta,k,\gamma)$$

$$S(x;\lambda,\theta,k,\gamma) = 1 - \left(1 - e^{-\lambda\left(\frac{x}{\gamma}\right)^k}\right)\left(1 + \theta e^{-\lambda\left(\frac{x}{\gamma}\right)^k}\right) \tag{19}$$

We can further simplify (19) as:

$$S(x;\lambda,\theta,k,\gamma) = 1 - \left(1 + \theta e^{-\lambda\left(\frac{x}{\gamma}\right)^k} - e^{-\lambda\left(\frac{x}{\gamma}\right)^k} - \theta e^{-2\lambda\left(\frac{x}{\gamma}\right)^k}\right)$$

$$= -\theta e^{-\lambda\left(\frac{x}{\gamma}\right)^k} + e^{-\lambda\left(\frac{x}{\gamma}\right)^k} + \theta e^{-2\lambda\left(\frac{x}{\gamma}\right)^k}$$

$$S(x;\lambda,\theta,k,\gamma)=\left(1-\theta+\theta e^{-\lambda\left(\frac{x}{\gamma}\right)^{k}}\right)e^{-\lambda\left(\frac{x}{\gamma}\right)^{k}}. \tag{20}$$

6.1.3 Hazard Function of TE-WD

$$h(x;\lambda,\theta,k,\gamma)=\frac{f(x;\lambda,\theta,k,\gamma)}{S(x;\lambda,\theta,k,\gamma)}$$

$$h(x;\lambda,\theta,k,\gamma)=\frac{\left(\frac{\lambda k(1-\theta)}{\gamma}\left(\frac{x}{\gamma}\right)^{k-1}+\frac{2\lambda\theta k}{\gamma}\left(\frac{x}{\gamma}\right)^{k-1}e^{-\lambda\left(\frac{x}{\gamma}\right)^{k}}\right)e^{-\lambda\left(\frac{x}{\gamma}\right)^{k}}}{\left(1-\theta+\theta e^{-\lambda\left(\frac{x}{\gamma}\right)^{k}}\right)e^{-\lambda\left(\frac{x}{\gamma}\right)^{k}}}$$

$$h(x;\lambda,\theta,k,\gamma)=\frac{\left(\frac{\lambda k(1-\theta)}{\gamma}\left(\frac{x}{\gamma}\right)^{k-1}+\frac{2\lambda\theta k}{\gamma}\left(\frac{x}{\gamma}\right)^{k-1}e^{-\lambda\left(\frac{x}{\gamma}\right)^{k}}\right)}{\left(1-\theta+\theta e^{-\lambda\left(\frac{x}{\gamma}\right)^{k}}\right)}. \tag{21}$$

The plots of the survival and hazard function of the TE-WD are respectively displayed in Figure 3 and 4 for selected values $\lambda=a,\gamma=b,\theta=c,\ and\ k=d$.

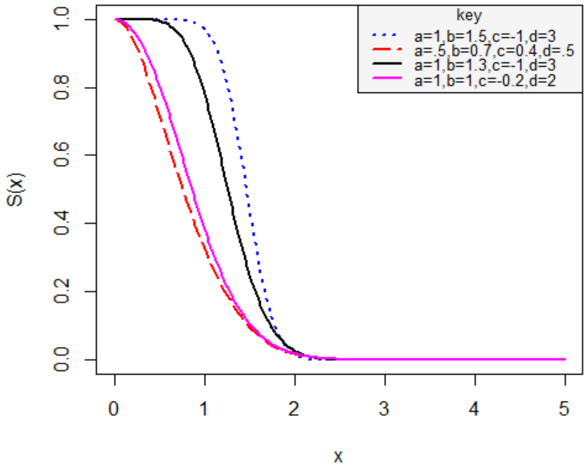

Figure 3: Survival Function of TE-WD

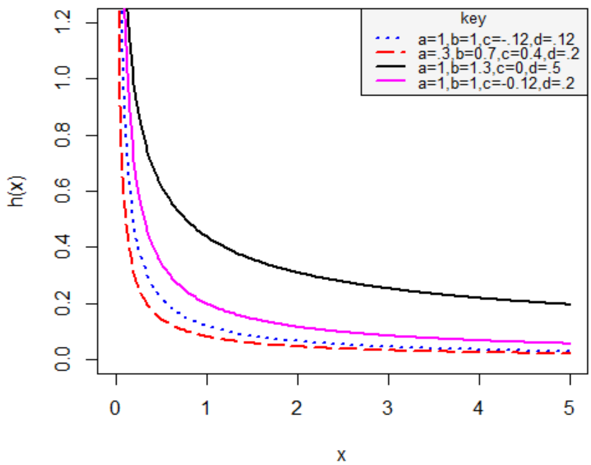

Figure 4: Hazard Function of TE-WD

6.1.4 Shapes of pdf and Hazard Function

In this subsection, we seek to investigate the behavior of the pdf and hazard function of the random variable X follows the distribution in (17) and (21) respectively as $x \to 0$ and *as* $x \to \infty$.

This involves considering $\lim_{x \to 0} f(x;\lambda,\theta,k,\gamma)$ *and* $\lim_{x \to \infty} f(x;\lambda,\theta,k,\gamma)$

As $x \to 0$

$$\lim_{x \to 0} f(x;\lambda,\theta,k,\gamma) = \lim_{x \to 0}\left(\frac{\lambda k(1-\theta)}{\gamma}\left(\frac{x}{\gamma}\right)^{k-1} e^{-\lambda\left(\frac{x}{\gamma}\right)^k} + \frac{2\lambda\theta k}{\gamma}\left(\frac{x}{\gamma}\right)^{k-1} e^{-2\lambda\left(\frac{x}{\gamma}\right)^k}\right) = 0$$

and

$$\lim_{x\to\infty} f(x;\lambda,\theta,k,\gamma) = \lim_{x\to\infty}\left(\frac{\lambda k(1-\theta)}{\gamma}\left(\frac{x}{\gamma}\right)^{k-1} e^{-\lambda\left(\frac{x}{\gamma}\right)^{k}} + \frac{2\lambda\theta k}{\gamma}\left(\frac{x}{\gamma}\right)^{k-1} e^{-2\lambda\left(\frac{x}{\gamma}\right)^{k}}\right) = 0$$

It can be deduced that, since $\lim_{x\to 0} f(x;\lambda,\theta,k,\gamma) = \lim_{x\to\infty} f(x;\lambda,\theta,k,\gamma) = 0$, the results shows that the TE-WD has only one mode.

The hazard function in (21) has the following limits 0 *and* $+\infty$

$$\lim_{x\to 0} h(x;\lambda,\theta,k,\gamma) = 0$$

and

$$\lim_{x\to\infty} h(x;\lambda,\theta,k,\gamma) = \lim_{x\to\infty}\left(\frac{\lambda k(1-\theta)}{\gamma}\left(\frac{x}{\gamma}\right)^{k-1} + \frac{2\lambda\theta k}{\gamma}\left(\frac{x}{\gamma}\right)^{k-1} e^{-\lambda\left(\frac{x}{\gamma}\right)^{k}}\right)$$

$$\times \lim_{x\to\infty}\left(1-\theta+\theta e^{-\lambda\left(\frac{x}{\gamma}\right)^{k}}\right)^{-1}$$

$$\lim_{x\to\infty} h(x;\lambda,\theta,k,\gamma) = \frac{\lambda k}{\gamma^{k}}.$$

6.1.5 Identifiablity of the Distribution

In this subsection, we discuss about the identifiability of TE-WD.

Let $\vartheta_1 = \lambda_1, k_1, \theta_1, \gamma_1$ and $\vartheta_2 = \lambda_2, k_2, \theta_2, \gamma_2$ be two set of parameters and $f_1(x;\vartheta_1)$ and $f_2(x;\vartheta_2)$ be the corresponding pdfs. By the definition of identifiability, we have, $f_1(x;\vartheta_1) = f_2(x;\vartheta_2)$ i.e.

$$x^{k_1-1}\left(\frac{\lambda_1 k_1(1-\theta_1)}{\gamma_1^{k_1}} e^{-\lambda_1\left(\frac{x}{\gamma_1}\right)^{k_1}} + \frac{2\lambda_1\theta_1 k_1}{\gamma_1^{k_1}} e^{-2\lambda_1\left(\frac{x}{\gamma_1}\right)^{k_1}}\right)$$

$$-x^{k_2-1}\left(\frac{\lambda_2 k_2(1-\theta_2)}{\gamma_2^{k_2}} e^{-\lambda_2\left(\frac{x}{\gamma_2}\right)^{k_2}} + \frac{2\lambda_2\theta_2 k_2}{\gamma_2^{k_2}} e^{-2\lambda_2\left(\frac{x}{\gamma_2}\right)^{k_2}}\right) = 0.$$

This expression is equal to zero for almost all x only when all its coefficients are zero, which is only possible when $\vartheta_1 = \vartheta_2$. Since the parameters $\lambda, k, \gamma > 0$, we conclude that the model is identifiable.

6.1.6 r^{th} Moments of TE-WD

The r^{th} Moments of TE-WD is given by

$$\mu_r' = \frac{\gamma^r \Gamma\left(1+\frac{r}{k}\right)}{\lambda^{\frac{r}{k}}}\left(1-\theta+\frac{\theta}{2^{\frac{r}{k}}}\right). \tag{22}$$

6.1.7 The r^{th} Moment about the Origin of TE-WD

The r^{th} moment about the origin of TE-WD is given by;

$$E(x-\mu)^r = \sum_{m=0}^{\infty}(-1)^m \binom{r}{m}\mu^m \left\{\frac{\lambda(1-\theta)}{\gamma^k}\left(\frac{\lambda}{\gamma^k}\right)^{-\left(\frac{r+k-m}{k}\right)} + \frac{2\lambda\theta}{\gamma^k}\left(\frac{2\lambda}{\gamma^k}\right)^{-\left(\frac{r+k-m}{k}\right)}\right\}\Gamma\left(\frac{r+k-m}{k}\right). \tag{23}$$

6.1.8 Quantile Function of TE-WD

The Quantile function of TE-WD is given by;

$$x_p = \gamma\left(\ln\left(\frac{(\theta-1)\pm\sqrt{(\theta-1)^2+4\theta(1-p)}}{2\theta}\right)^{-\frac{1}{\lambda}}\right)^{\frac{1}{k}}. \tag{24}$$

6.1.9 Coefficient of Skewness and Kurtosis

The Coefficient of Skewness is given by;

$$CS = \frac{E(x-\mu)^3}{\left(E(x-\mu)^2\right)^{\frac{3}{2}}}$$

$$CS = \frac{\sum_{m=0}^{\infty}(-1)^m\binom{3}{m}\mu^m\left\{\frac{\lambda(1-\theta)}{\gamma^k}\left(\frac{\lambda}{\gamma^k}\right)^{-\left(\frac{3+k-m}{k}\right)}+\frac{2\lambda\theta}{\gamma^k}\left(\frac{2\lambda}{\gamma^k}\right)^{-\left(\frac{3+k-m}{k}\right)}\right\}\Gamma\left(\frac{3+k-m}{k}\right)}{\left(\sum_{m=0}^{\infty}(-1)^m\binom{2}{m}\mu^m\left\{\frac{\lambda(1-\theta)}{\gamma^k}\left(\frac{\lambda}{\gamma^k}\right)^{-\left(\frac{2+k-m}{k}\right)}+\frac{2\lambda\theta}{\gamma^k}\left(\frac{2\lambda}{\gamma^k}\right)^{-\left(\frac{2+k-m}{k}\right)}\right\}\Gamma\left(\frac{2+k-m}{k}\right)\right)^{\frac{3}{2}}}. \tag{25}$$

The Coefficient of Kurtosis

$$CK = \frac{E\left(x-\mu\right)^4}{\left(E\left(x-\mu\right)^2\right)^2}$$

$$CK = \frac{\sum_{m=0}^{\infty}(-1)^m \binom{4}{m}\mu^m \left\{\frac{\lambda(1-\theta)}{\gamma^k}\left(\frac{\lambda}{\gamma^k}\right)^{-\left(\frac{4+k-m}{k}\right)} + \frac{2\lambda\theta}{\gamma^k}\left(\frac{2\lambda}{\gamma^k}\right)^{-\left(\frac{4+k-m}{k}\right)}\right\}\Gamma\left(\frac{4+k-m}{k}\right)}{\left(\sum_{m=0}^{\infty}(-1)^m \binom{2}{m}\mu^m \left\{\frac{\lambda(1-\theta)}{\gamma^k}\left(\frac{\lambda}{\gamma^k}\right)^{-\left(\frac{2+k-m}{k}\right)} + \frac{2\lambda\theta}{\gamma^k}\left(\frac{2\lambda}{\gamma^k}\right)^{-\left(\frac{2+k-m}{k}\right)}\right\}\Gamma\left(\frac{2+k-m}{k}\right)\right)^2}. \tag{26}$$

A simulation study was used to assess the behavior of skewness, kurtosis, mean and variance of the TE-W model. The results of these summary statistics are given in Table 2 and Table 3 for some arbitrary values of the model parameters. The results in Table 2 has shown that the skewness, kurtosis, mean and variance decreases as the values of the parameter k increase while for Table 3 the skewness and kurtosis increases as the values of the parameter θ increase while the mean and variance decreases as the values of the parameter θ increase.

Table 2: Skewness, Kurtosis, Mean and Variance for Some Arbitrary Choices of the Parameter Values

Parameters $\lambda = 1, \gamma = 1, \theta = 0.2$				
$k \downarrow$	Skewness	Kurtosis	Mean	Variance
0.4	2.2073	10.9856	6.7915	26.6025
0.6	0.8049	2.6580	3.4511	2.1405
0.8	0.5678	2.2156	2.4994	0.5452
1.0	0.4026	2.0051	2.0687	0.2206
1.6	0.2447	1.8764	1.5669	0.0676

Table 3: Skewness, Kurtosis, Mean and Variance for Some Arbitrary Choices of the Parameter Values

Parameters $\lambda = 3, k = 0.8, \gamma = 1$				
$\theta \downarrow$	Skewness	Kurtosis	Mean	Variance
0.2	2.3718	12.5862	1.5056	0.1978
0.3	2.5410	14.0946	1.1281	0.1669
0.4	2.7276	15.9047	0.8686	0.1407
0.6	3.1734	20.9007	0.5180	0.0959
0.8	3.6794	27.9628	0.2852	0.0574

7. SIMULATION STUDY

In this section, a simulation study is carried out to evaluate the performance of the MLEs of the TE-WD parameters. We generate 1000 samples of size, *n*=20, 50, 75, 300 and 500 of the TE-WD for fixed choice of parameters for $\lambda = 1, k = 0.2, \gamma = 0.4, \theta = 1$ and for$\lambda = 0.5, k = 0.5, \gamma = 0.4, \theta = 1$. The evaluation of estimates is based on the mean of the MLEs of the model parameters, bias and the mean squared error (MSE) of the MLEs. The empirical study was performed using the R programming language and the results are presented in Table 4 and Table 5. The values in Table 4 and Table 5 show that the estimates are quite stable and more importantly the values of the estimates are close to the true values for these sample sizes. Moreover, from Table 4 and Table 5 that the biases and MSEs decrease as n increases. Furthermore, from this simulation study we conclude that the maximum likelihood method is appropriate for estimating the TE-WD parameters. In fact, the MLEs of the parameters tend to be closer to the true parameter values when n increases.

Table 4: Average Values of the MLEs, Biases and MSEs of the TE-WD for $\boldsymbol{\lambda = 1, k = 0.2, \gamma = 0.4, \theta = 1}$

N	Parameter	Estimate	Bias	MSE
n=20	λ	1.0868	0.0868	0.0456
	k	0.2196	0.0196	0.0021
	γ	0.3184	-0.0816	0.0231
	θ	0.9130	-0.0870	0.0363
n=50	λ	1.0710	0.0710	0.0284
	k	0.2093	0.0093	0.0007
	γ	0.3422	-0.0578	0.0131
	θ	0.9202	-0.0798	0.0301
n=75	λ	1.0660	0.0660	0.0245
	k	0.2068	0.0068	0.0005
	γ	0.3503	-0.0497	0.0106
	θ	0.9204	-0.0796	0.0289
n=300	λ	1.0622	0.0622	0.0248
	k	0.2040	0.0040	0.0001
	γ	0.3750	-0.0250	0.0052
	θ	0.9267	-0.0733	0.0275
n=500	λ	1.0528	0.0528	0.0222
	k	0.2034	0.0034	6.8596
	γ	0.3808	-0.0192	0.0054
	θ	0.9383	-0.0617	0.0224

Table 5: Average values of the MLEs, Biases and MSEs of the TE-WD for $\boldsymbol{\lambda = 0.5, k = 0.5, \gamma = 0.4, \theta = 1}$

N	Parameter	Estimate	Bias	MSE
n=20	λ	0.5319	0.0319	0.0107
	k	0.5484	0.0484	0.0130
	γ	0.3841	-0.0159	0.0051
	θ	0.9232	-0.0768	0.0307
n=50	λ	0.5310	0.0310	0.0068
	k	0.5229	0.0229	0.0044
	γ	0.3878	-0.0122	0.0027
	θ	0.9247	-0.0752	0.0275
n=75	λ	0.5303	0.0303	0.0058
	k	0.5170	0.0170	0.0028

	γ	0.3885	-0.0115	0.0020
	θ	0.9222	-0.0778	0.0278
n=300	λ	0.5271	0.0271	0.0057
	k	0.5100	0.0100	0.0008
	γ	0.3941	-0.0059	0.0015
	θ	0.9283	-0.0717	0.0267
n=500	λ	0.5243	0.0243	0.0060
	k	0.5085	0.0085	0.0005
	γ	0.3971	-0.0029	0.0023
	θ	0.9383	-0.0617	0.0224

8. APPLICATIONS

Here, in order to demonstrate the potentiality of the TE-W model, we used a real-life dataset. The MLEs and the performance of the models are computed via R software.

8.1 Real Dataset

The data represents the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed, so we have complete data with the exact times of failure. For previous studies with the data sets see (Owoloko et al., 2015). We equally used this dataset to compare the TE-W model with Beta Weibull (BW), Kumaraswamy Weibull (KwW) and Exponentiated Generalized Weibull (EGW) distribution.

In order to determine the best out of the competing models, we will make use of some criteria including *AIC* (Akaike Information Criterion), *CAIC* (Consistent Akaike Information Criterion), *HQIC* (Hannan-Quinn Information Criteria) and *BIC* (Bayesian Information Criterion). These criteria are mathematically expressed as:

$$AIC = -2L + 2k,\ CAIC = -2L + \frac{2kn}{(n-k-1)},\ HQIC = -2L + 2k\log(\log(n))$$

and $BIC = -2L + klog(n)$

where L stand for log-likelihood function, k is the number of model parameters and n stand for the size of the sample. Furthermore, we equally compute other measures such as Anderson- Darling (A*), Cramer- Von Mises (W*) Statistic Kolmogorov Smirnov K-S and P- value.

Note: The model with the smallest value of these measures is consider to be the best among the competing models.

Table 6
Gives the Summary Statistics of the Dataset

n	Min.	Median	Mean	Variance	Max.	Skewness	Kurtosis
76	0.0251	1.7360	1.9590	2.4774	9.0960	1.9796	8.160792

Table 7
Estimated Parameters for the Dataset

Model	$\hat{\lambda}$	$\hat{\theta}$	$\hat{a}$	$\hat{b}$	$\hat{\gamma}$	$\hat{k}$
TE-W	0.6933	-0.8315	-	-	1.0552	1.0504
KwW	-	-	2.3383	12.2207	0.6790	7.7073
BW	-	-	1.4137	0.74	1.1224	1.3104
EGW	-	-	0.5056	1.4427	1.1013	0.8833

Table 6 provide the descriptive statistics of the real dataset while in Table 7, we provide the estimates of the parameters for the TE-WD and the competing distributions.

Table 8
Goodness-of-Fit Statistics for the Dataset

Model	-LL	AIC	CAIC	HQIC	BIC
TE-W	121.6115	251.223	251.7864	254.9489	260.5460
KwW	122.0722	252.1444	252.7078	255.8703	261.4674
BW	122.1564	252.3128	252.8762	256.0387	261.6357
EGW	122.1636	252.3272	252.8906	256.0531	261.6502

Table 9
Goodness-of-Fit Statistics for the Dataset

Model	W*	A*	K-S	(Bootstrapped PV)
TE-W	0.0973	0.5697	0.01526	(0.5175)
KwW	0.1139	0.6749	0.09756	(0.4369)
Bw	0.1166	0.6904	0.09880	(0.4211)
EGW	0.1167	0.6912	0.09876	(0.4217)

From Table 6, the skewness of the dataset is 1.9796 which shown clearly that the dataset is asymmetric and positively skewed. The results of Table 8 shows that the Transmuted Exponential Weibull (TE- W) distribution has the smallest values of the goodness-of-fit statistics and Table 9 shows that the Transmuted Exponential Weibull (TE- W) distribution has the smallest values of the W*, A* and K-S values. Therefore, the TE- W model considered as the best among the competing distributions.

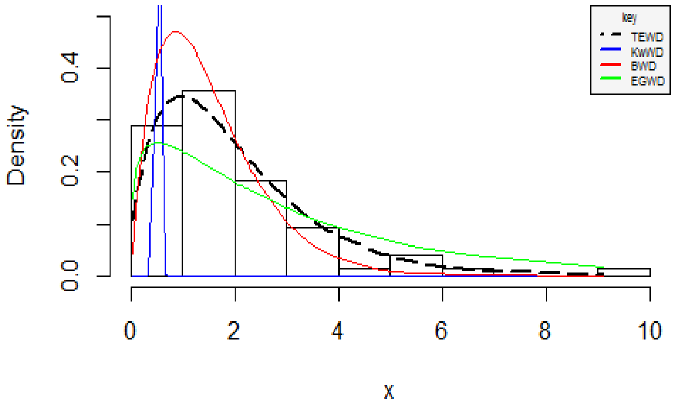

Figure 5: Plot of the Estimated Densities for the Fitted Models

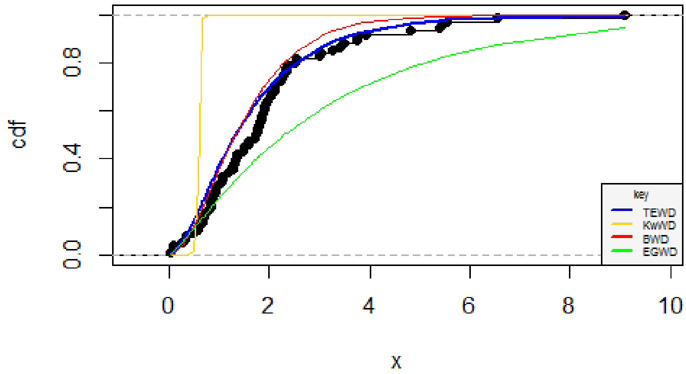

Figure 6: Plots of The estimated Distribution Functions of the Fitted Model

9. CONCLUSION

We define the Transmuted Exponential- G family in order to provide great flexibility to any continuous distribution by adding two extra parameters i.e transmuted and scale parameter. Some special members (models) for the new family of distributions can easily be gotten when the transmuted and scale parameter assume some values. An explicit expression for some of its mathematical and structural properties were derived and presented. The model parameters are estimated by the used of maximum likelihood approach. The effectiveness and potentiality of the TE- G family are exemplified by means of applications to a real and simulated datasets. The TE- W model gives a consistently better fit than the BW, KwW and EGW distribution when it comes to modelling asymmetric datasets.

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REFERENCES

1. Afify, A.Z., Cordeiro, G.M., Yousof, H.M., Alzaatreh, A. and Nofal, Z.M. (2016). The Kumaraswamy Transmuted-G Family of Distributions: Properties and Applications. *Journal of Data Science*, 14, 245-270.
2. Alizadeh, M., Cordeiro, G.M., Pinho, L.G.B. and Ghosh, I. (2017). The Gompertz-G family of distributions. *Journal of Statistical Theory and Practice*, 11(1), 179-207.
3. Al-Mofleh, H. (2018). On Generating a New Family of Distributions Using the Tangent Function. *Pakistan Journal of Statistics and Operation Research*, 14(3), 471-499.
4. Alzaatreh, A., Lee, C. and Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, 71(1), 63-79.
5. Alzaatreh, A., Lee, C., Famoye, F. and Ghosh, I. (2015). The generalized Cauchy family of distributions with applications. *Journal of Statistical Distributions and Applications*, 3(1). https://doi.org/10.1186/s40488-016-0050-3.
6. Bourguignon, M., Silva, R.B. and Cordeiro, G.M. (2014). The Weibull- G Family of Probability Distributions. *Journal of Data Science*, 12, 53-68.
7. Cícero, C.R., Cordeiro, G.M., Alizadeh, M., Diniz Marinho, P.R. and Campos Coêlho, H.F. (2015). Exponentiated Marshall-Olkin family of distributions. *Journal of Statistical Distributions and Applications*, 3(1), 1-21.
8. Cordeiro, G.M., Ortega, E.M.M. and Cunha, D.C.C. da. (2013). The Exponentiated Generalized Class of Distributions. *Journal of Data Science*, 11, 1-27.
9. Cordeiro, G.M., Ortega, E.M.M. and Silva, G. (2012). The Beta Extended Weibull family. *Journal of Probability and Statistical Science*, 10(10), 15-40.
10. Cordeiro, Gauss M., Lemonte, A.J. and Ortega, E.M.M. (2014). The Marshall-Olkin family of distributions: Mathematical properties and new models. *Journal of Statistical Theory and Practice*, 8(2), 343-366.
11. Eugene, N., Lee, C. and Famoye, F. (2002). Beta-Normal Distribution and its Applications. *Communications in Statistics - Theory and Methods*, 31(4), 497-512.
12. Gomes-Silva, F., Percontini, A., de Brito, E., Ramos, M.W., Venâncio, R. and Cordeiro, G.M. (2017). The odd Lindley-G family of distributions. *Austrian Journal of Statistics*, 46(1), 65-87.
13. Gupta, R.C., Gupta, P.L. and Gupta, R.D. (1998). Modeling failure time data by Lehman alternatives. *Communications in Statistics - Theory and Methods*, 27(4), 887-904.
14. Lee, E.T. (1992). *Statistical Methods for Survival Data Analysis*. New York: John Wiley.
15. Marshall, A.W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, 84(3), 641-652.
16. Mead, M.E., Cordeiro, G.M., Afify, A., Al-Mofleh, H. (2019). The Alpha Power Transformation Family: Properties and Applications. *Pakistan Journal of Statistics and Operation Research*, 15(3), 525-545.
17. Nadarajah, S., Cordeiro, G.M. and Ortega, E.M.M. (2015). The Zografos-Balakrishnan-G family of distributions: Mathematical properties and applications. *Communications in Statistics - Theory and Methods*, 44(1), 186-215.

18. Nadarajah, S. and Kotz, S. (2006). The Exponentiated Type Distributions. *Acta Applicandae Mathematica*, 92(2), 97-111.
19. Owoloko, E.A., Oguntunde, P.E. and Adejumo, A.O. (2015). Performance rating of the transmuted exponential distribution: An analytical approach. *SpringerPlus*, 4(1), 1-15.
20. Pescim, R.R., Cordeiro, G.M., Demétrio, C.G.B., Ortega, E.M.M. and Nadarajah, S. (2012). The new class of Kummer beta generalized distributions. *Sort*, 36(2), 153-180.
21. Rezaei, S., Sadr, B.B., Alizadeh, M. and Nadarajah, S. (2017). Topp-Leone generated family of distributions: Properties and applications. *Communications in Statistics - Theory and Methods*, *46*(6), 2893-2909.
22. Sen, S., Afify, A., Al-Mofleh, H. and Ahsanullah, M. (2019). The quasixgamma-geometric distribution with application in medicine. *FILOMAT*, 33(16), 5293-5332.
23. Shaw, W.T. and Buckley, I.R.C. (2007). *The Alchemy of Probability Distributions: Beyond Gram-Charlier & amp; Cornish-Fisher Expansions, and Skew-Normal or Kurtotic-Normal Distributions*. Retrieved from https://pdfs.semanticscholar.org/6585/a2d9ad4e2356e79b0c5603daea7af882edcf.pdf
24. Tahir, M.H., Mansoor, M., Zubair, M. and Hamedani, G.G. (2014). McDonald log-logistic distribution with an application to breast cancer data. *Journal of Statistical Theory and Applications*, 13(1), 65-82.
25. Tahir, M.H. and Cordeiro, G.M. (2015). Compounding of distributions: A survey and new generalized classes. In *Journal of Statistical Distributions and Applications* (Vol. 3). https://doi.org/10.1186/s40488-016-0052-1.
26. Yousof, H.M., Afify, A.Z., Alizadeh, M., Butt, N.S., Hamedani, G.G. and Ali, M.M. (2015). The transmuted exponentiated generalized-G family of distributions. *Pakistan Journal of Statistics and Operation Research*, 11(4), 441-464.
27. Yousof, H.M., Rasekhi, M., Afify, A.Z., Ghosh, I., Alizadeh, M. and Hamedani, G.G. (2017). The Beta Weibull-G Family of Distributions: Theory, Characterizations and Applications. *Pak. J. Statist.*, 33(2), 95-116.
28. Zubair, M., Alzaatreh, A., Cordeiro, G.M., Tahir, M.H. and Mansoor, M. (2018). On generalized classes of exponential distribution using T-X family framework. *Filomat*, 32(4), 1259-1272.