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FRÉCHET-WEIBULL DISTRIBUTION WITH APPLICATIONS TO EARTHQUAKES DATA SETS

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ABSTRACT

In this paper, we introduce a relatively new distribution called Fréchet-Weibull distribution which derived by using T-X method for generating families of continuous distributions. Many of its statistical properties were derived and presented in explicit form such as moments, mode, quantile function, Renyi, Tsallis and Shannon entropies, Lorenz and Bonferroni curves, and order statistics. The parameters estimation through the maximum likelihood estimation method was used and the results are applied to randomly generated data sets for studying behavior of distribution parameters. We applied the same procedure on two real-world data sets to show the flexibility of the proposed distribution.

KEYWORDS

Fréchet distribution, Weibull distribution, Quantile function, Moment Generating function, Order statistics, Entropy, Earthquakes.

1. INTRODUCTION

Extreme point distributions have developed as one of the most important statistical fields for the applied sciences. Analyzes of extreme point usually require estimation of the probability of events that are more extreme than any previously observed. For example, suppose a sea wall is required as part of its coastal defense design criteria to protect against all sea levels through one hundred years. Local sea level data may be available for a shorter time period, 10 years to tell. The goal is to estimate what might happen at sea level over the next one hundred years. Extreme point distributions provide a framework that allows for this type of study.

Weibull distribution is widely used in various applications especially in extreme value theory. Many papers have been introduced on the study of Weibull distribution due to its ability to fit data from many different fields such as engineering, physics, quality control, medicine,... etc. Lai et al. [9] introduced Weibull distributions and their applications. Ahmad et al. [1] introduced generalized new extended Weibull distribution, while Pobokov et al. [13] introduced transmuted Weibull distribution and along with some applications. Various other generalizations of Weibull distribution were introduced in Cordeiro et al. [5], [6] and Al-Aqtash et al. [2].

The probability density function (PDF) and the cumulative distribution function (CDF) of Weibull distribution, respectively, are given by (for $x \geq 0$)

$$d(x) = \frac{k}{\lambda}(\frac{x}{\lambda})^{k-1}\exp(-(\frac{x}{\lambda})^{k}), \tag{1}$$

$$D(x) = 1-\exp(-(\frac{x}{\lambda})^{k}), \tag{2}$$

where $\lambda > 0$ is the scale parameter and $k > 0$ is the shape parameter.

Fréchet distribution is also one of the most important distributions in extreme value theory. It has a large ability to be applied on extreme events such as earthquakes and floods. Da silva et al. [7] introduced a New lifetime model called the gamma extended Fréchet distribution. Nadarajah and Kotz [12] introduced the exponentiated Fréchet distribution, while Mead et al. [11] introduced the beta exponential Fréchet distribution. Mahmoud and Mandouh [10] introduced on the transmuted Fréchet distribution. Yousof et al. [16] showed some properties and applications on a six parameters version of Fréchet distribution.

PDF and CDF of Fréchet distribution, respectively, are given by (for $x > 0$)

$$m(x) = \frac{\alpha}{\beta}(\frac{\beta}{x})^{\alpha+1}\exp(-(\frac{\beta}{x})^{\alpha}), \tag{3}$$

$$M(x) = \exp(-(\frac{\beta}{x})^{\alpha}), \tag{4}$$

where $\beta > 0$ is the scale parameter and $\alpha > 0$ is the shape parameter.

In this paper we introduced a generalized distribution called Fréchet-Weibull distribution by using a relatively new method for generating new distributions, namely the T-X family which introduced by A. Alzaatreh et al. [3]. Its cumulative distribution function is defined as

$$G(x) = \int_{a}^{W(F(x))} r(t)dt = R[W(F(x))] \tag{5}$$

where X is a random variable with PDF $f(x)$ and CDF $F(x)$, and T is a continuous random variable defined on $[a,b]$ with PDF $r(t)$ and CDF $R(t)$. We consider the random variable X follows Weibull distribution with CDF (2), $W[F(x)] = -\log(1-D(x)) = (\frac{x}{\lambda})^{k}$ and the random variable T follows Fréchet distribution with PDF (3). As a result of using equation (5), we have Fréchet-Weibull distribution introduced in section 2.

This paper is organized as follows. In section 2, we presented CDF and PDF of the proposed model with some related function such as survival, hazard, reverse hazard functions. Some of its statistical properties were derived and represented in section 3. In section 4, PDF and CDF of i^{th} order statistics were derived with studying the limit distribution of the maximum order statistics. Estimation of proposed model parameters was performed by maximum likelihood method in section 5. In section 6, firstly, Behavior of model parameters was study by randomly generated data sets, secondly, two real-world data sets was applied to explain the flexibility of the proposed model.

2. FRÉCHET-WEIBULL DISTRIBUTION (FWD)

A random variable X is said to have FWD with four parameters α, β, λ, k, where its CDF is defined as

$$F(x) = \exp(-\beta^{\alpha}(\frac{\lambda}{x})^{\alpha k}), x > 0, \alpha, \beta, \lambda, k > 0 \tag{6}$$

and its corresponding PDF is given by

$$f(x) = \alpha k \beta^{\alpha} \lambda^{\alpha k} x^{-1-\alpha k} \exp(-\beta^{\alpha}(\frac{\lambda}{x})^{\alpha k}) \tag{7}$$

where α and k are shape parameters, and λ and β are scale parameters.

The survival, hazard and reverse hazard functions, respectively, for FWD are given by

$$S(x) = 1 - \exp(-\beta^{\alpha}(\frac{\lambda}{x})^{\alpha k}), \tag{8}$$

$$h(x) = \frac{\alpha k \beta^{\alpha} \lambda^{\alpha k} x^{-1-\alpha k} \exp(-\beta^{\alpha}(\frac{\lambda}{x})^{\alpha k})}{1 - \exp(-\beta^{\alpha}(\frac{\lambda}{x})^{\alpha k})}, \tag{9}$$

$$r(x) = \alpha k \beta^{\alpha} \lambda^{\alpha k} x^{-1-\alpha k}. \tag{10}$$

2.1 Asymptotic Behavior

The limits of PDF for FWD at $x = 0$ and $x = \infty$ are given by

$$\lim_{x\to 0} f(x) = \alpha k \beta^{\alpha} \lambda^{\alpha k} \lim_{x\to 0} x^{-1-\alpha k} \exp(-\beta^{\alpha}(\frac{\lambda}{x})^{\alpha k}) = 0,$$

$$\lim_{x\to \infty} f(x) = \alpha k \beta^{\alpha} \lambda^{\alpha k} \lim_{x\to \infty} x^{-1-\alpha k} \exp(-\beta^{\alpha}(\frac{\lambda}{x})^{\alpha k}) = 0,$$

respectively, while the limits of survival function, respectively, at the same points are given by

$$\lim_{x\to 0} S(x) = 1, \lim_{x\to \infty} S(x) = 0.$$

2.2 The Effect of the Parameters on PDF, CDF, S(x) and h(x)

Plots of PDF (7), CDF (6), S(x) (8) and h(x) (9) of FWD were displayed in Figures 1, 2, 3 and 4 for different parameters values, respectively.

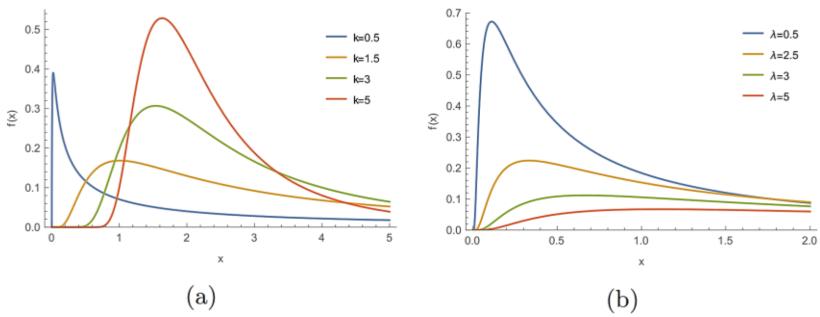

Figure 1: The Effect of *k* and *λ* Parameters on the PDF of FWD

Figure (1a) shows how PDF behave, affected by the change of parameter k, where $\alpha = 0.5$, $\beta = 3$ and $\lambda = 1.5$, while Figure (1b) shows the behavior of PDF with changing the parameter λ, where $\alpha = 0.5$, $\beta = 2$ and $k = 1$.

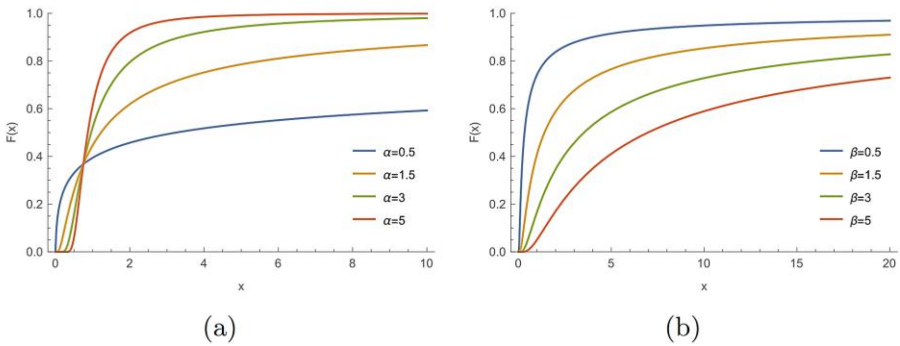

Figure 2: The Effect of *α* and *β* Parameters on the CDF of FWD

Figure (2a) shows how CDF behave, affected by the change of parameter α, where $\beta = 0.5$, $\lambda = 3$ and $k = 0.5$, while Figure (2b) shows the behavior of CDF with changing the parameter β, where $\alpha = 1$, $\lambda = 0.5$ and k=0.75.

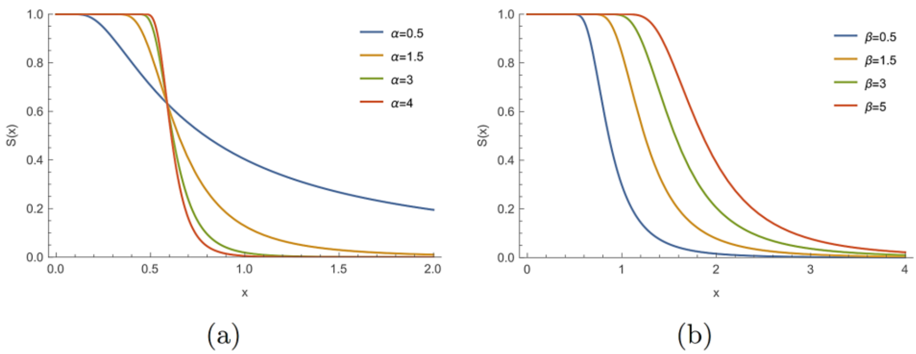

Figure 3: The Effect of *α* and *β* Parameters on the S(x) of FWD

Figure (3a) shows how S(x) behave, affected by the change of parameter k, where α = 0.5, β = 3 and λ = 1.5, while Figure (3b) shows the behavior of S(x) with changing the parameter λ, where α = 0.5, β = 2 and k = 1.

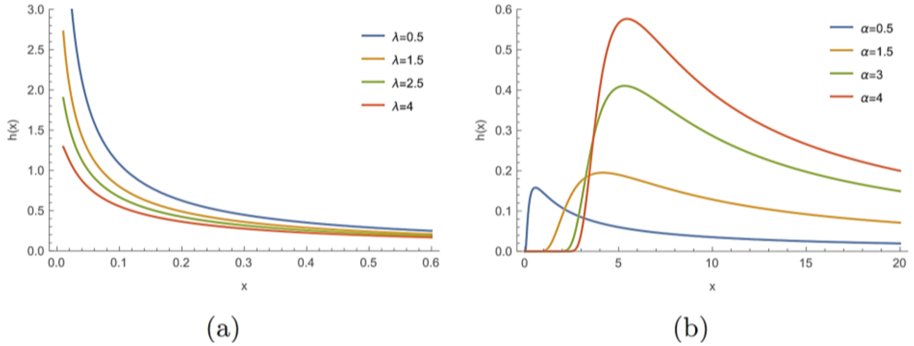

Figure 4: The Effect of λ and α Parameters on the h(x) of FWD

Figure (4a) shows how h(x) behave, affected by the change of parameter λ that h(x) is a decreasing function of λ, where α = 0.5, β = 1 and k = 0.5, while Figure (4b) shows the behavior of h(x) with changing the parameter α that h(x) is an upside down bathtub function of α, where β = 3, λ = 1.5 and k = 1.

3. STATISTICAL PROPERTIES

In this section, we introduce some important statistical properties of FWD such as moments about the origin along with they relations with first four central moments about the mean which used to determine coefficients of skewness, kurtosis, and variation. Also, quantile function, mode, moment generating function and mean residual life function, Renyi, Tsallis, and Shannon entropies along with Lorenz and Bonferroni curves were presented in this section.

3.1 Moments

The r^{th} moments μ'_r about the origin of FWD is given by

$$\mu'_r = E(x^r) = \int_{x=0}^{\infty} x^r f(x)dx = \lambda^r \beta^{\frac{r}{k}} \Gamma(1 - \frac{r}{\alpha k}), r < \alpha k,$$

by setting r = 1, 2, 3 and 4 we obtain the first four moments about the origin of FWD, respectively.

The mean and variance, respectively, of FWD are given by

$$\mu'_1 = \mu = \lambda \beta^{\frac{1}{k}} \Gamma(1 - \tfrac{1}{\alpha k}) \ , \ \sigma^2 = \lambda^2 \beta^{\frac{2}{k}} [\Gamma(1 - \tfrac{2}{\alpha k}) - (\Gamma(1 - \tfrac{1}{\alpha k}))^2].$$

First four central moments, respectively, about the mean for FWD can be obtained by the following relations

$$\begin{aligned}
\mu_1 &= \mu_1' - \mu = 0, \\
\mu_2 &= \mu_2' - (\mu_1')^2 = \lambda^2 \beta^{\frac{2}{k}} [\Gamma(1 - \frac{2}{\alpha k}) - (\Gamma(1 - \frac{1}{\alpha k}))^2], && (11) \\
\mu_3 &= \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3 = \lambda^3 \beta^{\frac{3}{k}} [\Gamma(1 - \frac{3}{\alpha k}) - 3\Gamma(1 - \frac{2}{\alpha k})\Gamma(1 - \frac{1}{\alpha k}) + 2(\Gamma(1 - \frac{1}{\alpha k}))^3], && (12) \\
\mu_4 &= \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 \\
&= \lambda^4 \beta^{\frac{4}{k}} [\Gamma(1 - \frac{4}{\alpha k}) + \Gamma(1 - \frac{1}{\alpha k})[-4\Gamma(1 - \frac{3}{\alpha k}) + 6\Gamma(1 - \frac{2}{\alpha k})\Gamma(1 - \frac{1}{\alpha k}) - 3(\Gamma(1 - \frac{1}{\alpha k}))^3]], && (13)
\end{aligned}$$

By using equations (11), (12) and (13), we can determining coefficients of skewness, kurtosis and variation, respectively, as the following

$$\begin{aligned}
\beta_1 &= \frac{(\mu_3)^2}{(\mu_2)^3} = \frac{(\Gamma(1 - \frac{3}{\alpha k}) - 3\Gamma(1 - \frac{2}{\alpha k})\Gamma(1 - \frac{1}{\alpha k}) + 2(\Gamma(1 - \frac{1}{\alpha k}))^3)^2}{(\Gamma(1 - \frac{2}{\alpha k}) - (\Gamma(1 - \frac{1}{\alpha k}))^2)^3}, \\
\beta_2 &= \frac{\mu_4}{(\mu_2)^2} = -3 + \frac{\Gamma(1 - \frac{4}{\alpha k}) + 3(\Gamma(1 - \frac{2}{\alpha k}))^2 - 4\Gamma(1 - \frac{3}{\alpha k})\Gamma(1 - \frac{1}{\alpha k})}{(\Gamma(1 - \frac{2}{\alpha k}) - (\Gamma(1 - \frac{1}{\alpha k}))^2)^2}, \\
CV &= \frac{\sigma}{\mu} \times 100 = \frac{\sqrt{\Gamma(1 - \frac{2}{\alpha k}) - (\Gamma(1 - \frac{1}{\alpha k}))^2}}{\Gamma(1 - \frac{1}{\alpha k})} \times 100.
\end{aligned}$$

3.2 Quantile Function and Mode

By determining the inverse function of CDF (6) of FWD, we obtain its quantile function $Q(p)$ as the following

$$x_p = Q(p) = \inf\{x \in R : F(x) \geq p\} = \lambda(-\beta^{-\alpha} \ln p)^{\frac{-1}{\alpha k}}, \tag{14}$$

by setting p = 0.25, 0.5 and 0.75 we obtain first, second and third quartiles of FWD, respectively. Let p follows uniform distribution then $Q(p)$ (14) can be used for generated data sets of size n for FWD as the following

$$x_i = \lambda(-\beta^{-\alpha} \ln p_i)^{\frac{-1}{\alpha k}}, \quad i = 1, 2, \ldots, n. \tag{15}$$

By taking the logarithm of PDF (7) and differentiating it with equality to zero, we have the mode of FWD as following

$$x_0 = \lambda(\frac{\alpha k \beta^\alpha}{1 + \alpha k})^{\frac{1}{\alpha k}}.$$

3.3 Moment Generating Function

The moment generating function of FWD is given by

$$M(t) = \sum_{m=0}^{\infty} \frac{t^m}{m!} \lambda^m \beta^{\frac{m}{k}} \Gamma(1 - \frac{m}{\alpha k}),$$

the characteristics function of FWD is given by

$$\phi(t) = \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \lambda^m \beta^{\frac{m}{k}} \Gamma(1 - \frac{m}{\alpha k}).$$

3.4 Mean Residual Life Function

The mean residual life function for FWD of is given by

$$\mu(t)=E(T-t|T>t)=\frac{1}{S(t)}\int_t^\infty uf(u)du-t=\frac{\beta^{\frac{1}{k}}\lambda}{S(t)}\Gamma(1-\frac{1}{\alpha k},\beta^{\alpha}(\frac{\lambda}{t})^{\alpha k})-t,$$

where $\Gamma\left(1-\frac{1}{\alpha k},\beta^{\alpha}\left(\frac{\lambda}{\mu}\right)^{\alpha k}\right)$ is lower incomplete gamma function. Similarly, the mean deviation about another measure of central tendency can be obtained by replacing μ in previous equation by this measure.

3.5 Entropy

Information theory has mathematical origin in entropy notion that is related to thermodynamic and statistical mechanics. In 1948, the definition of Shannon entropy was introduced. After 1948, variate extensions of the Shannon entropy has been introduced such as Renyi entropy (1961), and Tsallis entropy (1988) (for more information see Tabass et al. [15]).

Renyi, Tsallis and Shannon entropies of FWD are given by

$$\begin{aligned}
R_r(X) &= \frac{1}{1-r}\log\int_{x=0}^\infty f^r(x)dx=\frac{1}{1-r}\log[(\frac{\alpha k}{\lambda})^{r-1}\beta^{\frac{1}{k}(1-r)}r^{\frac{1}{\alpha k}-\frac{r}{\alpha k}-r}\Gamma(\frac{r}{\alpha k}+r-\frac{1}{\alpha k})],\\
T_r(X) &= \frac{1}{1-r}[\int_{x=0}^\infty f^r(x)dx-1]=\frac{1}{1-r}[(\frac{\alpha k}{\lambda})^{r-1}\beta^{\frac{1}{k}(1-r)}r^{\frac{1}{\alpha k}-\frac{r}{\alpha k}-r}\Gamma(\frac{r}{\alpha k}+r-\frac{1}{\alpha k})-1],\\
S_H(X) &= -\int_{x=0}^\infty f(x)\log f(x)dx=\lim_{r\to 1}R_r(X)=1+\log\lambda+\frac{1}{k}\log\beta-\log\alpha k+(1+\frac{1}{\alpha k})(\gamma),
\end{aligned}$$

respectively, where $r > 0$, $r \neq 1$ and $\gamma=-\int_{x=0}^\infty e^{-x}\ln(x)dx$ which is Euler Mascheroni constant.

3.6 Lorenz and Bonferroni Curves

Inequality is an important characteristic of non-negative distributions which can represented by inequality curves such as Lorenz and Bonferroni curves. The Lorenz curve is the oldest but also the most used and Bonferroni curve is another classical curve. Lorenz and Bonferroni curves, respectively, of FWD are given by

$$\begin{aligned}
L_X(p) &= \frac{1}{\mu}\int_{x=0}^{x_p} xf(x)dx=\frac{1}{\Gamma(1-\frac{1}{\alpha k})}\int_{-\ln p}^\infty y^{\frac{-1}{\alpha k}}e^{-y}dy,\\
B_X(p) &= \frac{1}{\mu F(x)}\int_0^{x_p} xf(x)dx=\frac{L_X(p)}{F(x)}.
\end{aligned}$$

4. ORDER STATISTICS

Let $X_1,X_2,...,X_n$ is a random sample from FWD. Let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ denote the corresponding order statistics. PDF and CDF of the i^{th} order statistic of FWD are given by

$$\begin{aligned}
f_{i:n}(x) &= \lim_{\delta x \to 0}\left(\frac{P(x < X_{i:n} \leqslant x + \delta x)}{\delta x}\right) = \frac{n!}{(i-1)!(n-i)!}(F(x))^{i-1}(1-F(x))^{n-i}f(x) \\
&= \frac{n!\alpha k \beta^{\alpha}\lambda^{\alpha k}}{(i-1)!(n-i)!}x^{-1-\alpha k}\exp(-i\beta^{\alpha}(\frac{\lambda}{x})^{\alpha k})(1-\exp(-\beta^{\alpha}(\frac{\lambda}{x})^{\alpha k}))^{n-i}, \\
F_{i:n}(x) &= \sum_{r=i}^{n}\binom{n}{r}(F(x))^{r}(1-F(x))^{n-r} = \sum_{r=i}^{n}\binom{n}{r}\exp(-r\beta^{\alpha}(\frac{\lambda}{x})^{\alpha k})(1-\exp(-\beta^{\alpha}(\frac{\lambda}{x})^{\alpha k}))^{n-r},
\end{aligned}$$

respectively, by setting $i = 1$ and n we obtain the distribution of minimum and maximum order statistics of FWD, respectively.

Suppose that $Z_n = X_{n:n}$ =maximum $(X_1,X_2,\ldots,X_n)$ from FWD, then the limiting distribution of Z_n can be obtained by the theorem (2.1.1) in Galambos [8] as the following

$$\lim_{n \to +\infty} P(Z_n < b_n x) = \exp(-x^{-\alpha k}) \qquad , x > 0, b_n = \lambda(-\beta^{-\alpha}\ln(1-\frac{1}{n}))^{\frac{-1}{\alpha k}}.$$

5. PARAMETERS ESTIMATION

For estimating the parameters of FWD we use maximum likelihood estimation. Let $X = (X_1,X_2,\ldots,X_n)$ be independent random sample having PDF (7), then the log-likelihood function is given by

$$\log L(x) = n\log\alpha + n\log k + n\alpha\log\beta + n\alpha k\log\lambda - \beta^{\alpha}\lambda^{\alpha k}\sum_{i=1}^{n}x_i^{-\alpha k} - (1+\alpha k)\sum_{i=1}^{n}\log x_i \tag{16}$$

From equation (16), we get

$$\begin{aligned}
\frac{\partial \log L(x)}{\partial \alpha} &= \frac{n}{\alpha} + n\log\beta + nk\log\lambda - k\sum_{i=1}^{n}\log x_i \\
&\quad - \beta^{\alpha}\log\beta\sum_{i=1}^{n}(\frac{\lambda}{x_i})^{\alpha k} - k\beta^{\alpha}\sum_{i=1}^{n}(\frac{\lambda}{x_i})^{\alpha k}\log\frac{\lambda}{x_i} \\
\frac{\partial \log L(x)}{\partial \beta} &= \frac{n\alpha}{\beta} - \alpha\beta^{\alpha-1}\lambda^{\alpha k}\sum_{i=1}^{n}x_i^{-\alpha k} \\
\frac{\partial \log L(x)}{\partial \lambda} &= \frac{n\alpha k}{\lambda} - \alpha k\beta^{\alpha}\lambda^{\alpha k-1}\sum_{i=1}^{n}x_i^{-\alpha k} \\
\frac{\partial \log L(x)}{\partial k} &= \frac{n}{k} + n\alpha\log\lambda - \alpha\beta^{\alpha}\sum_{i=1}^{n}(\frac{\lambda}{x_i})^{\alpha k}\log\frac{\lambda}{x_i} - \alpha\sum_{i=1}^{n}\log x_i
\end{aligned}$$

We can obtain the estimates of unknown parameters by setting the last four equations equal zero, but solving these equations simultaneously to get the unknown parameters $\hat{\alpha}$, $\hat{\beta}$, $\hat{\lambda}$ and $\hat{k}$ in explicit form is mathematically complicated, so these estimates will be obtained numerically.

6. APPLICATIONS

In this section, we study behavior of parameters of FWD by using randomly generated data sets and the proposed distribution superiority for fitting data sets was presented by applications on two real-world data sets.

6.1 Randomly Generated Data

A number of thousand random sample were generated for each samples size n = 50, 250 and 400 by using equation (15) as $x_i = \lambda(-\beta^{-\alpha} \ln u_i)^{\frac{-1}{\alpha k}}$ with parameters α = 1.5, β = 0.5, λ = 0.8 and k = 1, where u is uniformly distributed. Table (1) shows the estimates, biases and mean squared errors (MSEs) of the parameters for each sample size. It is easy to notice that estimates are close to their actual as the sample size increases. Also, Bias and MSE for estimated parameters are decrease and tends to zero as increasing of the sample size.

Table 1

Estimates, Biases and Mean Squared Errors of $\hat{\alpha}$, $\hat{\beta}$, $\hat{\lambda}$ and $\hat{k}$

n	Estimates	Bias	MSE
50	$\hat{\alpha} = 1.879298$	0.379298	0.143867
	$\hat{\beta} = 0.5068211$	0.006821073	4.652704×10^{-5}
	$\hat{\lambda} = 0.9330208$	0.1330208	0.01769454
	$\hat{k} = 0.8815254$	−0.1184746	0.01403623
250	$\hat{\alpha} = 1.672903$	0.1729031	0.02989548
	$\hat{\beta} = 0.4916992$	−0.008300754	6.890252×10^{-5}
	$\hat{\lambda} = 0.8868115$	0.086811541	0.007536243
	$\hat{k} = 0.9228719$	−0.07712806	0.005948737
400	$\hat{\alpha} = 1.665155$	0.165155	0.02727618
	$\hat{\beta} = 0.5025657$	0.002565699	6.582811×10^{-6}
	$\hat{\lambda} = 0.8634504$	0.06345036	0.004025949
	$\hat{k} = 0.9229422$	−0.07705782	0.005937908

6.2 Real-World Data

In this subsection we fitted the FWD to earthquakes data sets using maximum likelihood estimation along with compared the proposed FWD with generalized new extended Weibull distribution (GNEXWD) (see Ahmad et al. [1]), Lindley Weibull distribution (LWD) (see Cordeiro et al. [5]), the half-logistic generalized Weibull distribution (HLGWD) (see Anwar and Bibi [4]) and Weibull distribution (WD).

In order to compare the distributions we calculated the Akaikes information criterion (AIC), the corrected Akaikes information criterion (AICC), the Bayesian information criterion (BIC), Hannan Quinn information criterion (HQIC) and consistent Akaike information criteria (CAIC). The model with minimum AIC (or AICC, BIC, CAIC and HQIC) value is chosen as the best model to fit the data. The parameters are estimated by using the maximization of the log-likelihood function and the calculations are performed by using Wolfram Mathematica software version 10.

First earthquakes data set: The data represents the magnitudes earthquakes occurred between December 2017 and December 2018 which contains many variables and each variable contains 14509 observation. We will study the magnitude variable of earthquakes with minimum magnitude 3.9 across the world. Data was collected by an American federal agency called USGS which is responsible for recording and reporting earthquake activity in The U.S. and the world. It uses an advanced seismic system to collect the data. U. S. G. S. makes their collected data available on line for public through their governmental website. (see U. S. G. S. [14])

Second earthquakes data set: We will use the dataset earth quakes issued from the datasets R library. This locates the earthquakes off Fiji islands. It gives the locations of 1000 seismic events. The events occurred in a cube near Fiji islands since 1964. The dataset contains 1000 observations of 5 variables: the latitude (lat), longitude (long), Depth in km (depth), magnitude (mag) and the numeric number of stations reporting (stations), our study will be on magnitude variable. They in turn obtained it from Dr. John Wood house, Dept. of Geophysics, Harvard University

Table (2) gives the descriptive for the data sets, respectively. Tables (3) and (4) present the maximum likelihood estimates of the parameters together with the log-likelihood function, AIC, AICC, BIC, HQIC and CAIC values for the data sets, respectively. From Tables (3) and (4), we conclude that how good is our proposed distribution for fitting earth quakes data sets than other related and compared distributions.

Table 2
Descriptive Statistics for Earthquakes Data Sets

Data	Min	Max	Mean	Variance	First Quantile	Median	Third Quantile	Skewness	Kurtosis
I	3.9	8.2	4.53172	0.167572	4.3	4.5	4.7	1.6645	8.50133
II	4	6.4	4.6204	0.162226	4.3	4.6	4.9	0.7686	3.5103

Table 3
Estimates, Log L, AIC, AICC, BIC, HQIC and CAIC for First Earthquakes Data Set

Distribution	estimates	Log L	AIC	AICC	BIC	HQIC	CAIC
FWD	$\hat{\alpha} = 3.76549$ $\hat{\beta} = 0.620181$ $\hat{\lambda} = 4.90099$ $\hat{k} = 3.96361$	-5479.53	10967.1	10967.1	10997.4	10977.1	10967.1
GNEXWD	$\hat{\theta} = 5.8638 \times 10^{-12}$ $\hat{a} = 7.07203 \times 10^{-12}$ $\hat{b} = 5.20007$ $\hat{\sigma} = 113.865$	-8087.38	16182.8	16182.8	16213.1	16192.8	16182.8
LWD	$\hat{\beta} = 6.74561$ $\hat{\theta} = 0.023439$ $\hat{\alpha} = 0.413629$	-9598.68	19203.4	19203.4	19226.1	19210.9	19203.4
HLGWD	$\hat{w} = 0.692183$ $\hat{\eta} = 8.55473$ $\hat{\gamma} = 5.0839 \times 10^{-6}$	-10138.2	20282.4	20282.4	20305.1	20289.9	20282.4
WD	$\hat{k} = 8.58528$ $\hat{\lambda} = 4.72893$	-10960.4	21924.8	21924.8	21939.9	21929.8	21924.8

Table 4
Estimates, Log L, AIC, AICC, BIC, HQIC and CAIC for Second Earthquakes Data Set

Distribution	estimates	Log L	AIC	AICC	BIC	HQIC	CAIC
FWD	$\hat{\alpha} = 3.42366$ $\hat{\beta} = 0.89899$ $\hat{\lambda} = 4.53867$ $\hat{k} = 4.07125$	-454.301	916.603	916.643	936.234	924.064	916.643
GNEXWD	$\hat{\theta} = 1.02 \times 10^{-12}$ $\hat{a} = 2.08 \times 10^{-9}$ $\hat{b} = 5.67261$ $\hat{\sigma} = 129.199$	-536.016	1080.03	1080.07	1099.66	1087.49	1080.07
LWD	$\hat{\beta} = 7.75034$ $\hat{\theta} = 0.0117135$ $\hat{\alpha} = 0.408374$	-579.376	1164.75	1164.78	1179.47	1170.35	1164.78
HLGWD	$\hat{w} = 1.46506$ $\hat{\eta} = 3.88814$ $\hat{\gamma} = 0.002276$	-978.416	1962.83	1962.86	1977.56	1968.43	1962.86
WD	$\hat{k} = 10.6726$ $\hat{\lambda} = 4.81258$	-633.239	1270.48	1270.49	1280.29	1274.21	1270.49

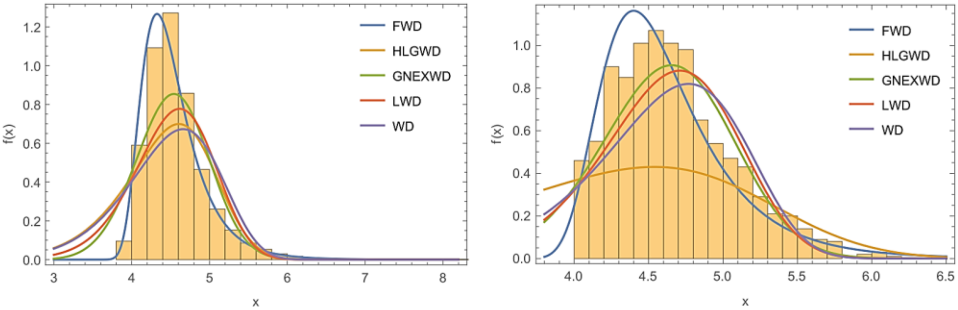

Figure 5: Histogram of Data Sets Compared to the Fitted PDFs

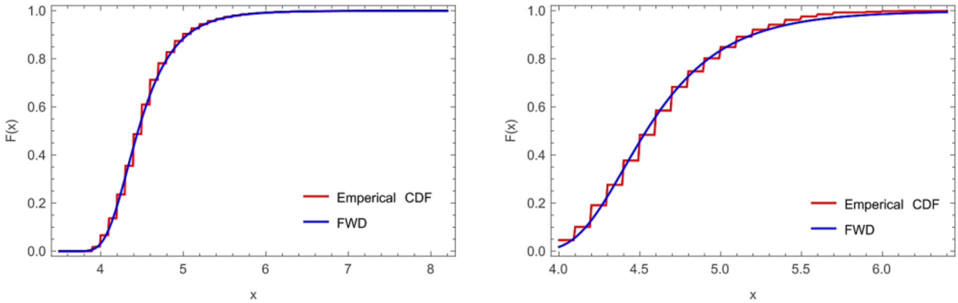

Figure 6: Empirical CDF of Data Sets and the Fitted CDF of FWD

Figure (5) illustrate the histograms and the fitted PDFs of FWD, HLGWD, GNEXWD and LWD for each data set, respectively. It also shown that the proposed model has the superiority to fit the used data sets than other compared models, while Figure (6) shows that how our choice for real-world data sets is suitable for FWD.

7. CONCLUSION

This paper introduced a relatively new generalized distribution called FWD which is an extend to Weibull distribution and is a member of T-X method for generating continuous distributions. Statistical properties of FWD were studied such as moments, mode, Quantile function, moment generating function and mean residual life function. Renyi, Tsallis and Shannon entropies, and inequality curves such as Lorenz and Bonferroni curves were also derived. PDF and CDF of the i^{th} order statistic were derived along with limiting distribution of maximum order statistics. The randomly generated data sets results showed that all estimators perform very well in terms of their bias, mean square errors as the sample size increase. Two real-world data applications were used to prove flexibility and potentiality of FWD. These applications showed that FWD gives better fits than other extensions of Weibull distribution. We expect that FWD will use in many fields such as geology, reliability, medicine, engineering, and life testing.

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