**Pak. J. Statist.**
**2020 Vol. 36(2), 117-133**

# ESTIMATION OF TWO PARAMETER POWERED INVERSE RAYLEIGH DISTRIBUTION

**Nashaat Jasim Mohammed Anber**
Information Technology Department, Technical College of Management, Baghdad
Middle Technical University, Baghdad, Iraq
Email: dr.nashaat@mtu.edu.iq

## ABSTRACT

In this article, we generalize the Inverse Rayleigh distribution using the power transformation. We provide a comprehensive description of the mathematical properties of the powered distribution. The maximum likelihood estimator of the parameters is derived. The Bayes estimation based on square error loss function is computed for general prior information by using Lindely and Tierney and Kadane's (T-K) approximation methods. A Comparisons among the suggested estimation methods have been made using the mean square error criteria and it is observed that The Lindely approximation method is more efficient than maximum likelihood method and T-K approximation method for all cases used. One real data analysis is performed for illustration.

## KEYWORDS

Inverse Rayleigh Distribution, powered transformation , Maximum Likelihood method, Bayes Method, lindely approximation, Tierney and Kadane's (T-K) Approximation.

## 1. INTRODUCTION

The Rayleigh distribution is introduced by Lord Rayleigh in (1880), it is special case from two parameter Weibull distribution and has a hazard function is an increasing function of time. It has many applications in relaliability, medical image analysis, signal analysis and survival analysis.

Voda (1972) introduced the inverse Rayleigh distribution and applied it in reliability and data analysis[16]. In recent years, several extensions of probability distributions have been made to increase its applicability In many applied sciences such as medicine, engineering and finance, amongst others.[3,4,6,7,8,10,11,13,14]

El-Helbawy and Abdel-Monem (2005) have obtained Bayesian estimators of the parameter of the inverse Rayleigh distribution under four loss functions[2]. In (2013) Sindhua, Aslama and Ferozeb considered Bayes estimation of the parameters of the inverse Rayleigh distribution for left censored data.[14]

In (2016) Khan and King introduced the transmuted modified inverse Rayleigh distribution by using quadratic rank transmutation map, which extends the modified Inverse Rayleigh distribution.[7]

In (2017) Manzoor and Memon used lower record values to estimate parameters of Inverse Rayleigh distribution by using three different methods.[10]

© 2020 Pakistan Journal of Statistics 

In (2018) Jan, Fatima and Ahmad introduced Transmuted Generalized Inverse Rayleigh Distribution and used to analyze data from Medical Science and Engineering.[5]

The main aim of this paper is to extend the one parameter inverse Rayleigh distribution to two parameter inverse Rayleigh distribution called powered inverse Rayleigh distribution (PIRD) through power transformation and describe the properties and estimation methods of its parameters.

The probability density Function of inverse Rayleigh distribution (IR) is

$$f(x;\theta) = \frac{2}{\theta x^3} e^{-\frac{1}{\theta x^2}} ; x, \theta > 0 \qquad (1)$$

where $\theta$ is scale parameter.

We will powered the inverse Rayleigh variate through the transformation as follow:

$t = x^{\frac{1}{\alpha}}$ , where $\alpha > 0$
$\Rightarrow x = t^{\alpha}$ , $dx = \alpha t^{\alpha-1} dt$

Then the two parameter powered inverse Rayleigh distribution (PIRD) will be

$$f(t;\alpha,\theta) = \frac{2\alpha}{\theta t^{2\alpha+1}} e^{-\frac{1}{\theta t^{2\alpha}}} ; t, \alpha, \theta > 0 \qquad (2)$$

where $\alpha$ is shape parameter and $\theta$ is scale parameter. Clearly when $\alpha = 1$, we get one parameter IRD, also the PIRD satisfy the transformation $\left(\frac{1}{t}\right) \sim powered\ Rayleigh(\alpha, \theta)$ and $(\frac{1}{t})^{\alpha} \sim Rayleigh(\theta)$.

We observe from plot of PDF for different values of parameters that PIRD is skewed from right and the peak increasing as shape parameter increase.

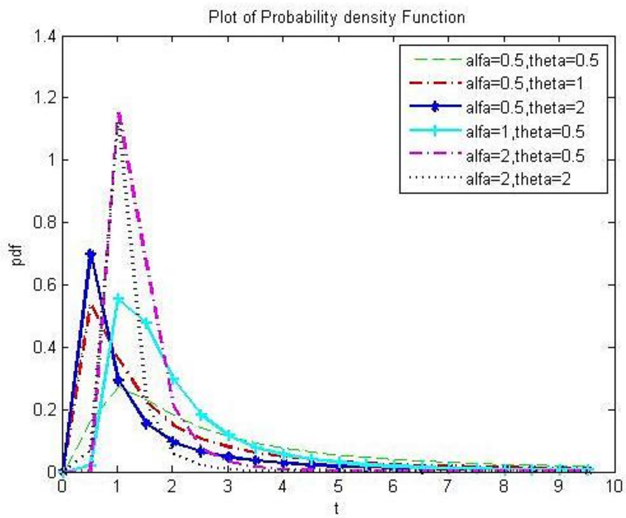


**Figure 1: Probability Density Function Plot of PIRD**

The cumulative distribution function will be

$$F(\mathrm{t}) = pr(T \leq t) = \int_0^t f(u)du = e^{-\frac{1}{\theta t^{2\alpha}}}\,;\ t, \alpha, \theta > 0 \qquad (3)$$

We observe from plot of cdf for different values of parameters curves increased more heavily as shape parameter increase.

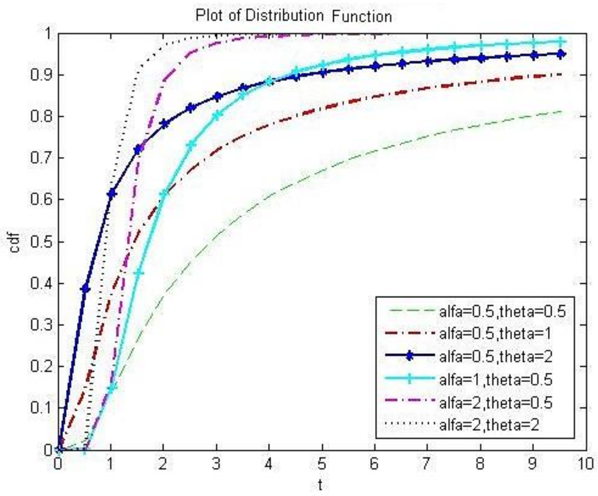


**Figure 2: Plot of Distribution Function of PIRD**

The reliability function will be

$$R(\mathrm{t}) = 1 - F(t) = 1 - e^{-\frac{1}{\theta t^{2\alpha}}} \qquad (4)$$

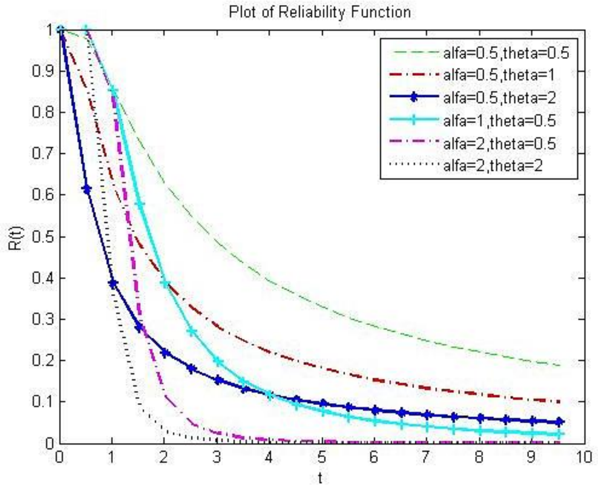


**Figure 3: Reliability Function Plot of PIRD**

The hazard function will be

$$h(\mathrm{t}) = \frac{f(t)}{R(t)} = \frac{\frac{2\alpha}{\theta t^{2\alpha+1}} e^{-\frac{1}{\theta t^{2\alpha}}}}{1-e^{-\frac{1}{\theta t^{2\alpha}}}} \quad (5)$$

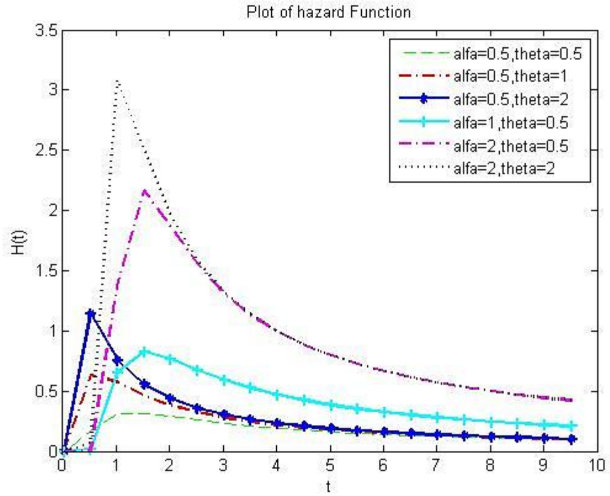


**Figure 4: Hazard Function Plot of PIRD**

The hazard function's behaviour starts increasing and then decreasing after peak point, and this behaviour become sharpness as values of parameters increases.

The General Formula of non-central moments will be

$$\mu_r = E(t^r) = \text{Gamma}\left[1 - \frac{r}{2\alpha}\right]\left(\frac{1}{\theta}\right)^{\frac{r}{2\alpha}}, \text{Provided } r < 2\alpha \quad (6)$$

The mode, median, skewness, kurtosis and coefficient of variation will be

$$Mode = \sqrt{\left(\frac{2\alpha}{\theta(2\alpha+1)}\right)^{\frac{1}{\alpha}}} \quad (7)$$

$$Median = \sqrt{\left(\frac{1}{\theta log(2)}\right)^{\frac{1}{\alpha}}} \quad (8)$$

$$Skewness = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} = \frac{Gamma\left[1-\frac{3}{2\alpha}\right]}{\sqrt{Gamma\left[1-\frac{1}{\alpha}\right]^3}}, \text{ Provided } 3 < 2\alpha \quad (9)$$

$$Kurtosis = \frac{\mu_4}{(\mu_2)^2} = \frac{Gamma\left[1-\frac{2}{\alpha}\right]}{Gamma\left[1-\frac{1}{\alpha}\right]^2}, \text{ Provided } 2 < \alpha \quad (10)$$

$$C.V = \frac{\sigma}{\mu_1} = \frac{Gamma\left[1-\frac{1}{\alpha}\right] - Gamma\left[1-\frac{1}{2\alpha}\right]^2}{Gamma\left[1-\frac{1}{2\alpha}\right]}\left(\frac{1}{\theta}\right)^{\frac{1}{2\alpha}}, \text{ Provided } 1 < 2\alpha \quad (11)$$

## 2. ESTIMATION METHODS

### 2.1 Maximum Likelihood Estimator (MLE) [1]

Assume that $t_{1,}t_{2,}\ldots,t_n$ be a complete random sample of PIRD$(\alpha,\theta)$ then, the likelihood function of the sample can be obtained below

$$\mathrm{L}(\alpha,\theta)=\prod_{i=1}^{n} f(t_i)=\frac{(2\alpha)^n}{\theta^n}\prod_{i=1}^{n} t_i^{-2\alpha-1}\, e^{-\sum_{i=1}^{n}\frac{1}{\theta t_i^{2\alpha}}} \quad (12)$$

To obtain MLE, firstly we differentiate the log-likelihood equation w.r.t. the parameters and equate it to zero. Thus, the logarithm of the likelihood function will be

$$\mathrm{LogL}(\alpha,\theta)=\mathrm{nlog}(2\alpha)-\mathrm{nlog}(\theta)-(2\alpha+1)\sum_{i=1}^{n}\log(t_i)-\sum_{i=1}^{n}\frac{1}{\theta t_i^{2\alpha}} \quad (13)$$

The derivative of equation (13) for $(\alpha,\theta)$ respectively will be;

$$\frac{\partial LogL}{\partial\alpha}=\frac{n}{\hat{\alpha}}-2\sum_{i=1}^{n}\log(t_i)+\frac{2}{\hat{\theta}}\sum_{i=1}^{n}\frac{1}{\hat{\theta} t_i^{2\hat{\alpha}}}\log(t_i)=0 \quad (14)$$

$$\frac{\partial LogL}{\partial\theta}=\frac{-n}{\hat{\theta}}+\frac{1}{\hat{\theta}^2}\sum_{i=1}^{n}\frac{1}{t_i^{2\hat{\alpha}}}=0 \quad (15)$$

From equation (15), we have

$$\hat{\theta}_{mle}=\frac{\sum_{i=1}^{n}\frac{1}{t_i^{2\hat{\alpha}}}}{n} \quad (16)$$

By substitute equation (16) by equation (14), we have

$$\frac{n}{\alpha}-2\sum_{i=1}^{n}\log(t_i)+\frac{2n}{\sum_{i=1}^{n}\frac{1}{t_i^{2\hat{\alpha}}}}\sum_{i=1}^{n}\frac{1}{t_i^{2\hat{\alpha}}}\log(t_i)=0 \quad (16)$$

Therefore the MLE of $\alpha$ is the solution of Equation (16), but that equation is nonlinear equation in $\alpha$ , so an analytical solution is not possible ,so we proposed the use of Newton-Raphson (N-R) algorithm to get the numerical solution that represent the value of maximum likelihood estimator of $\alpha$.

### 2.2 Bayesian Method [1]

In this section, we have obtained the Bayes estimates for the unknown parameters α and $\theta$ based on complete data sample. In Bayesian analysis, the parameter of interest is to be considered as a random variable and follows some prior distribution. Here, we assume that, parameters α ∼ gamma (a,b) density and $\theta$ ∼ gamma (c,d), where $\alpha,\theta$ are independent, i.e.

$$\pi_1(\alpha)=\alpha^{b-1}e^{-a\alpha}\ ;\ \alpha>0 \quad (17)$$

$$\pi_2(\theta)=\theta^{d-1}e^{-c\theta}\ ;\ \theta>0 \quad (18)$$

where a,b,c and d are the hyper-parameters which are assumed known. The above considered prior is more applicable in the sense that, it is more flexible and assumes different variety of prior which may be the reason behind its popularity.

Under square error loss function, which is more popular, The Bayesian estimator of any function of parameters $g(\alpha,\theta)$ will be the posterior expectation of that function, so we have

$$\hat{g}(\alpha,\theta) = E_{post}[g(\alpha,\theta)\backslash t_1, t_2, \dots, t_n] = \frac{\int_0^\infty \int_0^\infty g(\alpha,\theta)L(\alpha,\theta)\pi(\alpha,\theta)d\alpha d\theta}{\int_0^\infty \int_0^\infty L(\alpha,\theta)\pi(\alpha,\theta)d\alpha d\theta} \quad (19)$$

From equation (19) we observed that, the analytical solution of the integrals is not possible due to the implicit mathematical form and thus, we require some approximation techniques. Therefore, here the authors are proposing the use of two different approximation method namely Lindley's approximation method and T-K approximation method for evaluation the Bayes estimates of the parameters.

### 2.2.1 Lindley's Approximation Method [9]

We can easy to observed that the Bayesian estimation of the function $g(\alpha,\theta)$ as in equation (15) is in the form of ratio of two integrals for which simplified closed forms are not available. Thus, one can apply Lindley's (1980) approximation methods to evaluate these estimates. Using this method the Bayes estimate of the function $g(\alpha,\theta)$ is obtained by following equations

$$\hat{g}_{Lindley}(\alpha,\theta) = \hat{g}_{mle}(\alpha,\theta) + 0.5[l_{30}B_{12} + l_{03}B_{21} + l_{21}C_{12} + l_{12}C_{21}] + P_1A_{12} + P_2A_{21} \quad (20)$$

where

$$W_i = \frac{\partial g(\lambda_1,\lambda_2)}{\partial \lambda_i}, \text{ where } i = 1,2,\ \lambda_1 = \alpha,\ \lambda_1 = \theta$$

$$A = \sum_{i=1}^{2}\sum_{j=1}^{2} W_{ij}T_{ij}$$

$$W_{ij} = \frac{\partial^2 g(\lambda_1,\lambda_2)}{\partial \lambda_i \partial \lambda_j}, \text{ where } i = 1,2,\ \lambda_1 = \alpha,\ \lambda_1 = \theta$$

where $T$ is the inverse of Fisher Information Matrix with size (2×2).

Let

$$I_{11} = \frac{\partial^2 L(\alpha,\theta)}{\partial \alpha^2} = \frac{1}{\alpha^2} + \frac{4}{\theta}\sum_{i=1}^{n}\frac{1}{t_i^{2\alpha}}(\log(t_i))^2,$$

$$I_{12} = I_{21} = \frac{\partial^2 L(\alpha,\theta)}{\partial \alpha \partial \theta} = \frac{2}{\theta^2}\sum_{i=1}^{n}\frac{1}{t_i^{2\alpha}}(\log(t_i))$$

$$I_{22} = \frac{\partial^2 L(\alpha,\theta)}{\partial \theta^2} = \frac{2}{\theta^3}\sum_{i=1}^{n}\frac{1}{t_i^{2\alpha}}$$

$$T = I^{-1} = \frac{1}{(I_{11}I_{22} - I_{12}^2)}\begin{pmatrix} I_{22} & -I_{12} \\ -I_{21} & I_{11} \end{pmatrix}$$

$$B_{ij} = (W_iT_{ii} + W_jT_{ij})T_{ii}$$

$$C_{ij} = 3W_iT_{ii}T_{ij} + W_j(T_{ii}T_{jj} + 2T_{ij}^2)$$

$$A_{ij} = W_iT_{ii} + W_jT_{ji}$$

To estimate $\alpha$ we get: $g(\alpha,\theta) = \alpha$, then we have

$$\hat{\alpha}_{Lindley} = \hat{\alpha}_{mle} + 0.5[l_{30}B_{12} + l_{03}B_{21} + l_{21}C_{12} + l_{12}C_{21}] + P_1A_{12} + P_2A_{21} \quad (21)$$

where

$W_1 = 1, W_2 = 0$
$A = 0$
$W_{ij} = 0$
$B_{12} = (W_1T_{12} + W_2T_{12})T_{11} = T_{12}T_{11}$
$B_{21} = (W_2T_{22} + W_1T_{21})T_{22} = T_{21}T_{22}$
$C_{12} = 3W_1T_{11}T_{22} = 3T_{11}T_{22}$
$C_{21} = 3W_2T_{22}T_{21} + W_1(T_{22}T_{11} + 2T_{21}^2) = (T_{22}T_{11} + 2T_{21}^2)$
$A_{12} = W_1T_{11} = T_{11}$
$A_{21} = W_2T_{22} + W_1T_{12} = T_{12}$

To estimate $\theta$ we get: $g(\alpha,\theta) = \theta$, then we have

$$\hat{\theta}_{Lindley} = \hat{\theta}_{mle} + 0.5[l_{30}B_{12} + l_{03}B_{21} + l_{21}C_{12} + l_{12}C_{21}] + P_1A_{12} + P_2A_{21} \quad (22)$$

where

$W_1 = 0, W_2 = 1$
$A = 0$
$W_{ij} = 0$
$B_{12} = (W_1T_{12} + W_2T_{12})T_{11} = T_{12}T_{11}$
$B_{21} = (W_2T_{21} + W_1T_{21})T_{22} = T_{21}T_{22}$
$C_{12} = 3W_1T_{11}T_{22} + W_2(T_{11}T_{22} + 2T_{12}^2) = T_{11}T_{22} + 2T_{12}^2$
$C_{21} = 3W_2T_{22}T_{21} + W_1(T_{22}T_{11} + 2T_{21}^2) = 3T_{22}T_{21}$
$A_{12} = W_1T_{11} + W_2T_{21} = T_{21}$
$A_{21} = W_2T_{22} + W_1T_{12} = T_{22}$.

### 2.2.2 T-K Approximation Method [15]

Although Lindley's approximation plays an important role in the Bayesian analysis, however, this approximation requires the evaluation of third derivatives of the log-likelihood function which is very tedious as number of parameter increases. Tierney and Kadane (1986) proposed a procedure to approximate the ratio of two integrals such as (19). To apply the Tierney-Kadane's approximation, suppose that

$$Q(\alpha,\theta) = Log\big(L(\alpha,\theta)\big) + Log(\pi(\alpha,\theta)) \quad (23)$$

$$l(\alpha,\beta) = \frac{[Q(\alpha,\beta)]}{n} \quad (24)$$

$$l^*(\alpha,\beta) = \frac{[Q(\alpha,\beta)+\log(g(\alpha,\theta))]}{n}. \quad (25)$$

Then using the method of Tierney and Kadane the posterior expectation of the function $g(\alpha,\theta)$ is approximated as;

$$\hat{g}(\alpha,\theta)_{T-K} = \left|\frac{|\Sigma^*|}{|\Sigma|}\right|^{\frac{1}{2}} e^{n(l^*(\hat{\theta})-l(\hat{\theta}))} \tag{26}$$

where

$\hat{\theta}$: is maximum likelihood estimate of parameter.

$|\Sigma| = (D_{11}D_{22} - D_{12}^2)$

$D_{11} = \frac{\partial^2 l(\alpha,\theta)}{\partial\alpha^2} = \frac{-1}{\alpha^2} - \frac{4}{n\theta}\sum_{i=1}^{n}\frac{(\log(t_i))^2}{t_i^{2\alpha}} - \frac{b-1}{n\alpha^2}$

$D_{12} = D_{21} = \frac{\partial^2 l(\alpha,\theta)}{\partial\alpha\partial\theta} = \frac{-2}{n\theta^2}\sum_{i=1}^{n}\frac{\log(t_i)}{t_i^{2\alpha}}$

$D_{22} = \frac{\partial^2 l(\alpha,\theta)}{\partial\theta^2} = \frac{1}{\theta^2} - \frac{2}{n\theta^3}\sum_{i=1}^{n}\frac{1}{t_i^{2\alpha}} - \frac{d-1}{n\theta^2}$

To estimate $\alpha$, we have $g(\alpha,\theta) = \alpha$

$$l^*(\alpha,\beta) = \frac{[Q(\alpha,\beta)+\log(\alpha)]}{n} \tag{27}$$

$\frac{\partial l^*(\alpha,\beta)}{\partial\alpha} = \frac{\partial l(\alpha,\beta)}{\partial\alpha} + \frac{1}{n\alpha} = \frac{1}{\alpha} - \frac{2}{n}\sum_{i=1}^{n}\log(t_i) + \frac{2}{n\theta}\sum_{i=1}^{n}\frac{\log(t_i)}{t_i^{2\alpha}} + \frac{b-1}{n\alpha} - \frac{a}{n} + \frac{1}{n\alpha}$

$\frac{\partial l^*(\alpha,\beta)}{\partial\theta} = \frac{\partial l(\alpha,\beta)}{\partial\theta} = \frac{-1}{\theta} + \frac{1}{n\theta^2}\sum_{i=1}^{n}\frac{1}{t_i^{2\alpha}} + \frac{d-1}{n\theta} - \frac{c}{n}$

$D_{11}^* = \frac{\partial^2 l^*(\alpha,\beta)}{\partial\alpha^2} = D_{11} - \frac{1}{n\alpha^2}$

$D_{12}^* = D_{21}^* = \frac{\partial^2 l^*(\alpha,\beta)}{\partial\theta\partial\alpha} = D_{12}$

$D_{22}^* = \frac{\partial^2 l^*(\alpha,\beta)}{\partial\theta^2} = D_{22}$

$$\hat{\alpha}_{T-K} = \left|\frac{|\Sigma^*|}{|\Sigma|}\right|^{\frac{1}{2}} e^{n(l^*(\hat{\alpha})-l(\hat{\alpha}))} \tag{28}$$

To estimate $\theta$, we have $g(\alpha,\theta) = \theta$

$$l^*(\alpha,\beta) = \frac{[Q(\alpha,\beta)+\log(\theta)]}{n} \tag{29}$$

where

$\frac{\partial l^*(\alpha,\beta)}{\partial\alpha} = \frac{\partial l(\alpha,\beta)}{\partial\alpha} + \frac{1}{n\alpha} = \frac{1}{\alpha} - \frac{2}{n}\sum_{i=1}^{n}\log(t_i) + \frac{2}{n\theta}\sum_{i=1}^{n}\frac{\log(t_i)}{t_i^{2\alpha}} + \frac{b-1}{n\alpha} - \frac{a}{n}$

$\frac{\partial l^*(\alpha,\beta)}{\partial\theta} = \frac{\partial l(\alpha,\beta)}{\partial\theta} = \frac{-1}{\theta} + \frac{1}{n\theta^2}\sum_{i=1}^{n}\frac{1}{t_i^{2\alpha}} + \frac{d-1}{n\theta} - \frac{c}{n} + \frac{1}{n\theta}$

$D_{11}^* = \frac{\partial^2 l^*(\alpha,\beta)}{\partial\alpha^2} = D_{11}$

$D_{12}^* = D_{21}^* = \frac{\partial^2 l^*(\alpha,\beta)}{\partial\theta\partial\alpha} = D_{12}$

$D_{22}^* = \frac{\partial^2 l^*(\alpha,\beta)}{\partial\theta^2} = D_{22} - \frac{1}{n\theta^2}$

$$\hat{\theta}_{T-K} = \left|\frac{|\Sigma^*|}{|\Sigma|}\right|^{\frac{1}{2}} e^{n(l^*(\hat{\theta})-l(\hat{\theta}))}. \tag{30}$$

## 3. SIMULATION STUDY

In this section, Monte Carlo simulation method has been used and the obtained results for the three estimation methods are compared. The simulation process were done using sample sizes (15,25,50 and 100) that are represented small, moderate and large sample sizes and different combinations for hyper parameters (a, b, c and d) from (0.5, 1) and parameters ($\alpha, \theta = 0.5, 1, 2$). A random samples for each sample size is generated by using the following formula

$$x = [-\theta \log(U)]^{-\frac{1}{2\alpha}} \quad (31)$$

where

U: is a uniform random variate.

built on 1000 replications, we used mean square error (mse) measures to check the performance and The results are reported in tables 1 to 9.

**Table 1**
**Average of Parameter Estimates and its mse where**
$(\boldsymbol{\alpha = 0.5, \theta = 0.5, a = 0.5, b = 0.5, c = 0.5, d = 0.5})$

| $\alpha$ | $\theta$ | a | b | c | d |
|---|---|---|---|---|---|
| 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 |
| N | estimates | mle | lindely | T-K | best |
| 15 | Alfa | 0.766203 | 0.712329 | 0.782327 | |
| | mse_alfa | 0.544122 | 0.498374 | 0.578164 | lindely |
| | theta | 2.887157 | 2.882883 | 3.943798 | |
| | mse_theta | 60.56687 | 60.75637 | 122.6922 | mle |
| 25 | Alfa | 0.442948 | 0.420217 | 0.448162 | |
| | mse_alfa | 0.206861 | 0.204655 | 0.210854 | lindely |
| | theta | 1.222894 | 1.226746 | 1.285146 | |
| | mse_theta | 3.065853 | 3.133995 | 3.511804 | mle |
| 50 | Alfa | 0.569011 | 0.555032 | 0.572336 | |
| | mse_alfa | 0.134492 | 0.129694 | 0.136441 | lindely |
| | theta | 1.052829 | 1.052043 | 1.081826 | |
| | mse_theta | 3.256115 | 3.277351 | 3.541409 | mle |
| 100 | Alfa | 0.576373 | 0.570634 | 0.577986 | |
| | mse_alfa | 0.00647 | 0.005607 | 0.006724 | lindely |
| | theta | 0.464364 | 0.46286 | 0.466784 | |
| | mse_theta | 0.004854 | 0.004992 | 0.004717 | T-k |
| percentage of cases | | 38% | 50% | 12% | |

**Table 2**
**Average of Parameter Estimates and its mse where**
$(\alpha = 0.5, \theta = 1, a = 0.5, b = 0.5, c = 0.5, d = 0.5)$

| $\alpha$ | $\theta$ | a | b | c | d |
|---|---|---|---|---|---|
| **0.5** | **1** | **0.5** | **0.5** | **0.5** | **0.5** |
| **N** | **estimates** | **mle** | **lindely** | **T-K** | **best** |
| 15 | Alfa | 0.672507 | 0.664637 | 0.685975 | |
| | mse_alfa | 0.043907 | 0.041062 | 0.049383 | lindely |
| | Theta | 1.066122 | 1.039666 | 1.11317 | |
| | mse_theta | 0.077431 | 0.066801 | 0.094309 | lindely |
| 25 | alfa | 0.540895 | 0.533701 | 0.547194 | |
| | mse_alfa | 0.180215 | 0.181233 | 0.18522 | mle |
| | theta | 1.659292 | 1.650245 | 1.73825 | |
| | mse_theta | 1.98937 | 2.031872 | 2.400466 | mle |
| 50 | alfa | 0.368151 | 0.362513 | 0.370317 | |
| | mse_alfa | 0.206154 | 0.212726 | 0.207703 | mle |
| | theta | 1.359661 | 1.361213 | 1.382052 | |
| | mse_theta | 0.649517 | 0.666569 | 0.69663 | mle |
| 100 | alfa | 0.60693 | 0.605504 | 0.608658 | |
| | mse_alfa | 0.01508 | 0.014732 | 0.015476 | lindely |
| | theta | 1.02662 | 1.024438 | 1.032735 | |
| | mse_theta | 0.010296 | 0.00995 | 0.010836 | lindely |
| percentage of cases | | 50% | 50% | 0% | |

**Table 3**
**Average of Parameter Estimates and its mse where**
$(\alpha = 1, \theta = 0.5, a = 0.5, b = 0.5, c = 0.5, d = 0.5)$

| $\alpha$ | $\theta$ | a | b | c | d |
|---|---|---|---|---|---|
| **1** | **0.5** | **0.5** | **0.5** | **0.5** | **0.5** |
| **N** | **estimates** | **mle** | **Lindely** | **T-K** | **best** |
| 15 | alfa | 1.260562 | 1.152488 | 1.285027 | |
| | mse_alfa | 0.105378 | 0.051981 | 0.120436 | lindely |
| | theta | 0.484838 | 0.472383 | 0.502018 | |
| | mse_theta | 0.014443 | 0.016283 | 0.015077 | mle |
| 25 | alfa | 1.254727 | 1.196938 | 1.269331 | |
| | mse_alfa | 0.103609 | 0.073598 | 0.112342 | lindely |
| | theta | 0.532273 | 0.527487 | 0.544043 | |
| | mse_theta | 0.050076 | 0.052344 | 0.05311 | mle |
| 50 | alfa | 1.139343 | 1.110269 | 1.145683 | |
| | mse_alfa | 0.026413 | 0.018345 | 0.028319 | lindely |
| | theta | 0.488849 | 0.485478 | 0.493895 | |
| | mse_theta | 0.004945 | 0.00517 | 0.004946 | mle |
| 100 | alfa | 1.244819 | 1.227532 | 1.248322 | |
| | mse_alfa | 0.071129 | 0.062494 | 0.072928 | lindely |
| | theta | 0.478803 | 0.476845 | 0.48127 | |
| | mse_theta | 0.002243 | 0.002366 | 0.002159 | T-K |
| percentage of cases | | 38% | 50% | 12% | |

**Table 4**
**Average of Parameter Estimates and its mse where**
**$(\alpha = 1, \theta = 1, a = 0.5, b = 0.5, c = 0.5, d = 0.5)$**

| $\alpha$ | $\theta$ | a | b | c | d |
|---|---|---|---|---|---|
| **1** | **1** | **0.5** | **0.5** | **0.5** | **0.5** |
| **N** | **estimates** | **mle** | **lindely** | **T-K** | **best** |
| 15 | alfa | 1.262863 | 1.252916 | 1.287988 | |
| | mse_alfa | 0.206515 | 0.228444 | 0.226687 | mle |
| | theta | 1.218251 | 1.261722 | 1.279913 | |
| | mse_theta | 0.202497 | 0.308398 | 0.267065 | mle |
| 25 | alfa | 1.266251 | 1.249336 | 1.281013 | |
| | mse_alfa | 0.12874 | 0.122326 | 0.138474 | lindely |
| | theta | 1.213349 | 1.236174 | 1.248355 | |
| | mse_theta | 0.218534 | 0.27221 | 0.25366 | mle |
| 50 | alfa | 1.254555 | 1.23575 | 1.261673 | |
| | mse_alfa | 0.089453 | 0.079453 | 0.093434 | lindely |
| | theta | 1.020867 | 1.02134 | 1.033216 | |
| | mse_theta | 0.028445 | 0.028396 | 0.030107 | lindely |
| 100 | alfa | 1.155617 | 1.148819 | 1.158861 | |
| | mse_alfa | 0.034772 | 0.032455 | 0.03586 | lindely |
| | theta | 1.110221 | 1.109324 | 1.117067 | |
| | mse_theta | 0.020426 | 0.020017 | 0.022141 | lindely |
| percentage of cases | | 38% | 62% | 0% | |

**Table 5**
**Average of Parameter Estimates and its mse where**
**$(\alpha = 0.5, \theta = 1, a = 0.5, b = 0.5, c = 1, d = 0.5)$**

| $\alpha$ | $\theta$ | a | b | c | d |
|---|---|---|---|---|---|
| **0.5** | **0.5** | **1** | **0.5** | **1** | **0.5** |
| **N** | **estimates** | **mle** | **lindely** | **T-K** | **best** |
| 15 | alfa | 0.766203 | 0.6829 | 0.782327 | |
| | mse_alfa | 0.544122 | 0.474988 | 0.578164 | lindely |
| | theta | 2.887157 | 2.88229 | 3.943798 | |
| | mse_theta | 60.56687 | 60.88129 | 122.6922 | mle |
| 25 | alfa | 0.442948 | 0.411565 | 0.448162 | |
| | mse_alfa | 0.206861 | 0.204467 | 0.210854 | lindely |
| | theta | 1.222894 | 1.230219 | 1.285146 | |
| | mse_theta | 3.065853 | 3.171766 | 3.511804 | mle |
| 50 | alfa | 0.569011 | 0.549568 | 0.572336 | |
| | mse_alfa | 0.134492 | 0.128051 | 0.136441 | lindely |
| | theta | 1.052829 | 1.052132 | 1.081826 | |
| | mse_theta | 3.256115 | 3.289183 | 3.541409 | mle |
| 100 | alfa | 0.576373 | 0.568792 | 0.577986 | |
| | mse_alfa | 0.00647 | 0.005342 | 0.006724 | lindely |
| | theta | 0.464364 | 0.462387 | 0.466784 | |
| | mse_theta | 0.004854 | 0.005006 | 0.004717 | T-K |
| percentage of cases | | 38% | 50% | 12% | |

**Table 6**
**Average of Parameter Estimates and its mse where**
**$(\alpha = 0.5, \theta = 1, a = 1, b = 0.5, c = 1, d = 0.5)$**

| $\alpha$ | $\theta$ | a | b | c | d |
|---|---|---|---|---|---|
| **0.5** | **1** | **1** | **0.5** | **1** | **0.5** |
| **N** | **estimates** | **mle** | **lindely** | **T-K** | **best** |
| 15 | alfa | 0.672507 | 0.635894 | 0.685975 | |
| | mse_alfa | 0.043907 | 0.029756 | 0.049383 | lindely |
| | theta | 1.066122 | 0.999428 | 1.11317 | |
| | mse_theta | 0.077431 | 0.054994 | 0.094309 | lindely |
| 25 | alfa | 0.540895 | 0.521839 | 0.547194 | |
| | mse_alfa | 0.180215 | 0.180478 | 0.18522 | mle |
| | theta | 1.659292 | 1.636358 | 1.73825 | |
| | mse_theta | 1.98937 | 2.058624 | 2.400466 | mle |
| 50 | alfa | 0.368151 | 0.356991 | 0.370317 | |
| | mse_alfa | 0.206154 | 0.213639 | 0.207703 | mle |
| | theta | 1.359661 | 1.358 | 1.382052 | |
| | mse_theta | 0.649517 | 0.67807 | 0.69663 | mle |
| 100 | alfa | 0.60693 | 0.602433 | 0.608658 | |
| | mse_alfa | 0.01508 | 0.014025 | 0.015476 | lindely |
| | theta | 1.02662 | 1.020186 | 1.032735 | |
| | mse_theta | 0.010296 | 0.009524 | 0.010836 | lindely |
| percentage of cases | | 50% | 50% | 0% | |

**Table 7**
**Average of Parameter Estimates and its mse where**
**$(\alpha = 1, \theta = 0.5, a = 1, b = 0.5, c = 1, d = 0.5)$**

| $\alpha$ | $\theta$ | a | b | c | d |
|---|---|---|---|---|---|
| **1** | **0.5** | **1** | **0.5** | **1** | **0.5** |
| **N** | **estimates** | **mle** | **lindely** | **T-K** | **best** |
| 15 | alfa | 1.418622 | 1.237958 | 1.44698 | |
| | mse_alfa | 0.286458 | 0.133828 | 0.31643 | lindely |
| | theta | 0.553786 | 0.540476 | 0.576632 | |
| | mse_theta | 0.123719 | 0.123047 | 0.139779 | lindely |
| 25 | alfa | 1.132334 | 1.03177 | 1.145377 | |
| | mse_alfa | 0.524721 | 0.454023 | 0.540424 | lindely |
| | theta | 1.131606 | 1.128748 | 1.208168 | |
| | mse_theta | 4.637504 | 4.704789 | 5.605368 | mle |
| 50 | alfa | 1.185273 | 1.141245 | 1.191932 | |
| | mse_alfa | 0.064278 | 0.048058 | 0.067157 | lindely |
| | theta | 0.475454 | 0.471048 | 0.480455 | |
| | mse_theta | 0.011337 | 0.011692 | 0.011283 | T-K |
| 100 | alfa | 1.260114 | 1.23235 | 1.263691 | |
| | mse_alfa | 0.085883 | 0.070945 | 0.087871 | lindely |
| | theta | 0.491798 | 0.489476 | 0.494319 | |
| | mse_theta | 0.002891 | 0.002972 | 0.002879 | T-K |
| percentage of cases | | 12% | 63% | 25% | |

**Table 8**
**Average of Parameter Estimates and its mse where**
**$(\alpha = 1, \theta = 1, a = 1, b = 0.5, c = 1, d = 0.5)$**

| $\alpha$ | $\theta$ | a | b | c | d |
|---|---|---|---|---|---|
| **1** | **1** | **1** | **0.5** | **1** | **0.5** |
| **N** | **estimates** | **mle** | **lindely** | **T-K** | **best** |
| 15 | alfa | 1.316219 | 1.88113 | 1.342208 | |
| | mse_alfa | 0.158383 | 2.763559 | 0.178287 | mle |
| | theta | 1.377664 | 1.964404 | 1.455027 | |
| | mse_theta | 0.517511 | 2.935582 | 0.649001 | mle |
| 25 | alfa | 1.254641 | 1.172209 | 1.26904 | |
| | mse_alfa | 0.101196 | 0.058209 | 0.109755 | Lindely |
| | theta | 0.990958 | 0.970314 | 1.015229 | |
| | mse_theta | 0.044376 | 0.040626 | 0.047637 | Lindely |
| 50 | alfa | 1.203216 | 1.166183 | 1.210091 | |
| | mse_alfa | 0.074833 | 0.057011 | 0.078083 | Lindely |
| | theta | 1.084473 | 1.071021 | 1.097846 | |
| | mse_theta | 0.021141 | 0.017342 | 0.024149 | Lindely |
| 100 | alfa | 1.151077 | 1.135067 | 1.154268 | |
| | mse_alfa | 0.031858 | 0.026865 | 0.032892 | Lindely |
| | theta | 1.109978 | 1.102306 | 1.116833 | |
| | mse_theta | 0.026531 | 0.024103 | 0.028386 | Lindely |
| percentage of cases | | 25% | 75% | 0% | |

**Table 9**
**Average of Parameter Estimates and its mse where**
**$(\alpha = 1, \theta = 0.5, a = 1, b = 1, c = 1, d = 1)$**

| $\alpha$ | $\theta$ | a | b | c | d |
|---|---|---|---|---|---|
| **1** | **0.5** | **1** | **1** | **1** | **1** |
| **N** | **estimates** | **mle** | **lindely** | **T-K** | **Best** |
| 15 | alfa | 1.258233 | 1.138818 | 1.28183 | |
| | mse_alfa | 0.187369 | 0.098557 | 0.205166 | Lindely |
| | theta | 0.522269 | 0.518543 | 0.540295 | |
| | mse_theta | 0.025707 | 0.027353 | 0.028225 | Mle |
| 25 | alfa | 1.170055 | 1.106909 | 1.18278 | |
| | mse_alfa | 0.060356 | 0.034324 | 0.065714 | Lindely |
| | theta | 0.538081 | 0.535319 | 0.548995 | |
| | mse_theta | 0.013542 | 0.013889 | 0.014945 | Mle |
| 50 | alfa | 1.29708 | 1.259181 | 1.304437 | |
| | mse_alfa | 0.135663 | 0.109631 | 0.140724 | Lindely |
| | theta | 0.433502 | 0.43063 | 0.438129 | |
| | mse_theta | 0.014672 | 0.015325 | 0.014233 | T-k |
| 100 | alfa | 1.223492 | 1.204422 | 1.226995 | |
| | mse_alfa | 0.058991 | 0.05014 | 0.060629 | Lindely |
| | theta | 0.459862 | 0.458476 | 0.462248 | |
| | mse_theta | 0.00534 | 0.005499 | 0.005182 | T-k |
| percentage of cases | | 25% | 50% | 25% | |

## 4. NUMERICAL RESULTS

Some of the points are very clear from these results, and listed as follow:

1. It is observed that as the sample size increases, the biases, the MSEs of the estimators decrease.
2. The Lindely approximation method is more efficient than maximum likelihood method and T-K approximation method for all cases used.
3. It is observed that as the sample size increases, the Lindely approximation method's efficient increases.
4. Lindely approximation method will be more efficient as values of parameters increases.
5. It is observed that as the T-K method is the worst method among methods that used.

## 5. APPLICATION

In this section we present an analysis based on real data set to show that the PIRD can be a better model than the Inverse Rayleigh Distribution IRD, Rayleigh Distribution RD, Weibull Distribution WD. The data represent measurements of coating weights ($gm/m^2$) by chemical method on top center side (TCS), where that process are done in order to improve the quality of steel roofing, the data originally reported by Rao G.S. and Mbwambo S. (2019)[12]. The data set is presented in table 10. Descriptive data analysis of the sample data as shown in table 11, that indicates the data are positively skewed and a Platykurtic distribution.

**Table 10**
**Coating Weights by Chemical Method on TCS**

| TCS | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 36.8 | 47.2 | 35.6 | 36.7 | 55.8 | 58.7 | 42.3 | 37.8 | 55.4 |
| 45.2 | 31.8 | 48.3 | 45.3 | 48.5 | 52.8 | 45.4 | 49.8 | 48.2 |
| 54.5 | 50.1 | 48.4 | 44.2 | 41.2 | 47.2 | 39.1 | 40.7 | 40.3 |
| 41.2 | 30.4 | 42.8 | 38.9 | 34 | 33.2 | 56.8 | 52.6 | 40.5 |
| 40.6 | 45.8 | 58.9 | 28.7 | 37.3 | 36.8 | 40.2 | 58.2 | 59.2 |
| 42.8 | 46.3 | 61.2 | 58.4 | 38.5 | 34.2 | 41.3 | 42.6 | 43.1 |
| 42.3 | 54.2 | 44.9 | 42.8 | 47.1 | 38.9 | 42.8 | 29.4 | 32.7 |
| 40.1 | 33.2 | 31.6 | 36.2 | 33.6 | 32.9 | 34.5 | 33.7 | 39.9 |

**Table 11**
**Descriptive Statistics of Coating Weights**

| N | Min | Max | Mean | Median | Mode | Standard Deviation | Skewness | Kurtosis Excess |
|---|---|---|---|---|---|---|---|---|
| 72 | 28.7 | 61.2 | 43.0917 | 42.3 | 42.8 | 8.2514 | 0.4238 | -0.617 |

To verify that PIRD is suitable model for the data sets a minus two times of negative log-likelihood value, Akaike information criteria (AIC), Bayesian information criteria

(BIC), corrected Akaike information criterion (AICC), Hannan-Quinn information criterion (HQIC), consistent akaike information criterion (CAIC), Kolmogorov - smirnov K-S distance and p value are used.

where

$$\mathrm{AIC} = 2\mathrm{k} - 2\mathrm{logl} \tag{32}$$

$$\mathrm{BIC} = \mathrm{klog(n)} - 2\mathrm{logl} \tag{33}$$

$$\mathrm{AICC} = \mathrm{AIC} + \frac{2\mathrm{k}(\mathrm{k}+1)}{\mathrm{n}-\mathrm{k}-1} \tag{34}$$

$$\mathrm{HQIC} = -2\mathrm{logl} + 2\mathrm{klog(log(n))} \tag{35}$$

$$\mathrm{CAIC} = -2\mathrm{logl} + \frac{2\mathrm{kn}}{\mathrm{n}-\mathrm{k}-1} \tag{36}$$

where $logl$ denotes the log-likelihood at MLEs, $k$ is the number of parameters, and $n$ is a sample size. The table 12 show that the PIRD has lower values for $-2Logl$, (AIC), (BIC), (AICC), (HQIC), (CAIC), K-S distance and highest P-value than IRD, RD and WD. So, which indicate that the PIRD could be chosen as the best model than IRD, RD and WD models. The empirical and estimated cdf of PIRD, IRD, RD and WD are displayed in Figure 5.

**Table 12**
**Parameter Estimates and Different Criteria for Comparison**

| **Model** | **Parameter estimate** | $-2Logl$ | **AIC** | **BIC** | **AICC** | **HQIC** | **CAIC** | **K-S** | **P-value** |
|---|---|---|---|---|---|---|---|---|---|
| PIRD | $\hat{\alpha} = 3.1131$ $\hat{\theta} = 1.3926e-10$ | 510.129 | 514.129 | 518.682 | 514.303 | 515.942 | 514.303 | 0.115 | 0.280 |
| IRD | $\hat{\theta} = 5.9849e-04$ | 593.604 | 595.604 | 597.880 | 595.661 | 596.510 | 595.661 | 0.365 | 0.000 |
| RD | $\hat{\theta} = 962.0160$ | 593.789 | 595.789 | 598.066 | 595.846 | 596.695 | 595.846 | 0.363 | 0.000 |
| WD | $\hat{\alpha} = 46.5429$ $\hat{\beta} = 5.5669$ | 512.184 | 516.184 | 520.737 | 516.357 | 517.996 | 516.357 | 0.118 | 0.246 |

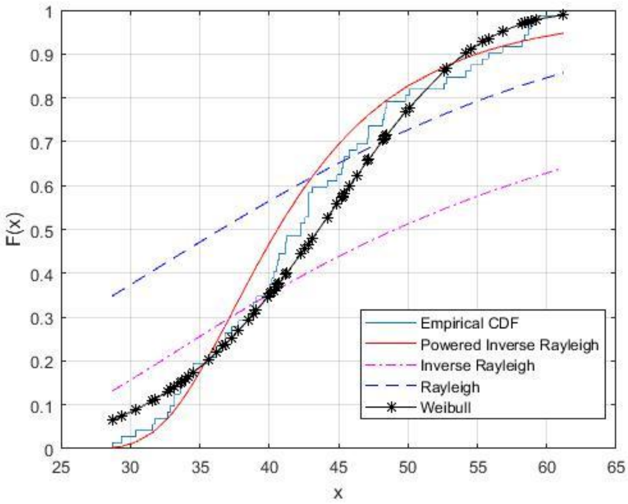


**Figure 4: Empirical, Fitted Powered Inverse Rayleigh, Inverse Rayleigh, Rayleigh and Weibull cdf of TCS Data**

## 5. CONCLUSION

We generalized the one parameter Inverse Rayleigh distribution through power transformation, the two parameter powered Inverse Rayleigh distribution has scale and shape parameters. We used classical and Bayesian estimation methods by assume general prior distribution. Since the posterior distribution is difficult to integral, we suggest use Lindely approximation method and T-K Method to find Bayesian estimate of parameters. A numerical comparison through simulation Study was conducted. It is observed that The Lindely approximation method is more efficient than maximum likelihood method and T-K approximation method for all cases used. We applied the two parameter powered inverse Rayleigh distribution to real data and indicated that could be chosen as the best model than inverse Rayleigh distribution, Rayleigh distribution and Weibull distribution.

## REFERENCES

[1] DasGupta, A. (2011). Finite sample theory of order statistics and extremes. In *Probability for Statistics and Machine Learning* (pp. 221-248). Springer, New York, NY.

[2] El-Helbawy, A.A. and Abd-El-Monem (2005). Bayesian Estimation and Prediction for the Inverse Rayleigh Lifetimd Distribution. In *Proceeding of the 40th Annual Conference of Statistics, Computer Sciences and Operation Research*, Cairo University (pp. 45-59).

[3] Haq, M.A. (2016). Transmuted Exponentiated Inverse Rayleigh Distribution. *J. Stat. Appl. Pro.*, 5(2), 337-343.

[4] Haq, M.A. (2016). Kumaraswamy Exponentiated Inverse Rayleigh Distribution. *Mathematical Theory and Modeling*, 6(3), 93-104.

[5] Jan, U., Fatima, K. and Ahmad, S.P. (2018). Transmuted Generalized Inverse Rayleigh Distribution and Its Applications to Medical Science and Engineering. *Appl. Math. Inf. Sci. Lett*., 6(3), 149-163.

[6] Khan, M.S. (2014). Modified inverse Rayleigh distribution. *International Journal of Computer Applications*, 87(13), 28-33.

[7] Khan, M.S. and Kinga, R. (2016). New generalized inverse Weibull distribution for lifetime modeling. *Communications for Statistical Applications and Methods*, 23(2), 147-161.

[8] Leao, J., Saulo, H., Bourguignon, M., Cintra, R., Rego, L. and Cordeiro, G. (2013). On some properties of the beta Inverse Rayleigh distribution. *Chilean Journal of Statistics*, 4(2), 111-131.

[9] Lindley, D.V. (1980). Approximate Bayesian Method. *Traabayos Estatistica*, 31, 223-237.

[10] Manzoor, S. and Memon, A.Z. (2017). Analysis of Inverse Rayleigh Distribution based on Lower Record Values. *International Journal of Advanced Science and Technology*, 102, 35-48.

[11] Merovci, F. (2013). Transmuted Rayleigh Distribution. *Austrian Journal of Statistics*, 42(1), 21-31.

[12] Rao G.S. and Mbwambo, S. (2019). Exponentiated Inverse Rayleigh Distribution and an Application to Coating Weights of Iron Sheets Data. *Journal of Probability and Statistics.* https://doi.org/10.1155/2019/7519429.

[13] Rodrigues, J.A., Silva, A.P.C.M. and Hamedani, G.G. (2016). The Exponentiated Kumaraswamy Inverse Weibull Distribution with Application in Survival Analysis. *Journal of Statistical Theory and Applications*, 15(1), 8-24.

[14] Sindhua, T.N., Aslama, M. and Ferozeb, N. (2013). Bayes estimation of the parameters of the inverse Rayleigh distribution for left censored data. *ProbStat Forum*, 6, 42-59.

[15] Tierney, L. and Kadane, J. (1986). Accurate approximations for posterior moments and marginal densities. *Journal of the American Statistical Association*, 81, 82-86.

[16] Voda, V.G. (1972). On the inverse Rayleigh distributed random variable. *Rep. Statist. App. Res*., JUSE, 19, 13-21.