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STATISTICAL INFERENCES UNDER MODIFIED WEIBULL DISTRIBUTION BASED ON PROGRESSIVE FIRST-FAILURE CENSORING SCHEME

Saieed F. Ateya[1,2] and **Elham A. Madhagi**[3,4]

[1] Mathematics & Statistics Department, Taif University, Hawia, Taif, Saudi Arabia.
[2] Mathematics Department, Faculty of Science, Assiut University, Egypt.
[3] Department of Statistics, College of Science, Qassim University, Qassim, Saudi Arabia.
[4] Mathematics Department, Faculty of Education, Hodeidah University, Hodeidah, Yemen.

ABSTRACT

In this paper, point and interval estimations under modified Weibull (MW) distribution have been studied based on progressive first-failure censored scheme. The Bayes estimates ($BE's$) have been computed based on squared error (SE) and (Linex) loss functions and using Markov Chain Monte Carlo ($MCMC$) algorithm. Also, based on this censoring scheme, the interval estimation problem of the parameters of MW distribution have been studied. A Monte Carlo simulation study has been carried out to compare the performances of the different methods by computing the mean squared errors ($MSE's$). Finally, point and interval estimates for all parameters have been studied based on a real data set as an illustrative example.

KEYWORDS

Modified Weibull distribution; Progressive first-failure censoring; Markov Chain Monte Carlo method.

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1. INTRODUCTION

The Weibull distribution is one of the most popular and widely used models of failure time in life testing and reliability theory. The Weibull distribution are shown to be useful for modeling and analysis of life time data in medical, biological and engineering sciences. Applications of the Weibull distribution in various fields are given in Zaharim et al. [23], Gotoh et al. [6], Shamilov et al. [16], Vicen-Bueno et al. [21], Niola et al. [15] and Green et al. [7]. A great deal of research is done on estimating the parameters of the Weibull distribution using both classical and Bayesian techniques, and a very good summary of this work can be found in Johnson et al. [10]. Hossain and Zimmer [8] have discussed some comparisons of estimation methods for Weibull parameters using complete and censored samples. Jaheen and Harbi[9] studied the Bayesian estimation of the exponentiated Weibull distribution using Markov chain Monte Carlo simulation. The modified Weibull distribution was proposed by Lai et al. [12] as a new lifetime distribution. They have shown the capability of the model for modeling a bathtub-shaped hazard-rate function. In addition, they characterized the model through the Weibull plot paper. Further, they have shown that the modified Weibull model compares well with other competing models to fit data that exhibit a bathtub-shaped hazard-rate function. Sultan [17] studied the record values from the modified Weibull distribution and studied its applications. Ateya and Alharthi [2,3] studied the estimation problem under a finite mixture of MW distribution using the traditional maximum likelihood method and using the EM algorithm. Vasile et al. [20] used the Bayes method to estimate the parameters of the modified Weibull distribution and Upadhyaya and Gupta [18] studied the Bayes analysis of the modified Weibull distribution using Markov chain Monte Carlo simulation. Ateya [1] study the estimation problem under a censored sample of generalized order statistics from MW distribution. Mohammed et al. [13] studied the estimation problem based on progressive first-failure censored scheme under exponentiated exponential distribution. Also, Kotb and Raqab [11] studied the statistical inference problem for modified Weibull distribution based on progressively type-II censored data.

A random variable X has a MW distribution with the parameters β, τ and λ if its probability density function (pdf) is given by

$$f(x|\beta,\tau,\lambda) = \tau(\beta+\lambda x)x^{\beta-1}\, exp(\lambda x)\, exp(-\tau x^{\beta} e^{\lambda x}),\quad x \geq 0, (\tau > 0, \beta \geq 0, \lambda \geq 0). \tag{1.1}$$

The cumulative distribution function (cdf) of this distribution can be written as

$$F(x|\beta,\tau,\lambda) = 1 - exp(-\tau x^{\beta} e^{\lambda x}). \tag{1.2}$$

2. A PROGRESSIVE FIRST-FAILURE CENSORING SCHEME

In this section, the first-failure censoring is combined with progressive censoring scheme as in Wu and Kus [22]. Suppose that n independent groups with k items within each group are put on life test. R_1 groups and the group in which the first failure is observed are randomly removed from the test as soon as the first failure $X^R_{1;m,n,k}$ has occurred, R_2 groups and the group in which the second failure is observed are randomly removed from the test as soon as the second failure $X^R_{2;m,n,k}$ has occurred, and finally when the mth failure $X^R_{m;m,n,k}$ is observed, the remaining groups R_m are removed from the test. Then $X^R_{1;m,n,k} < X^R_{2;m,n,k} < \ldots < X^R_{m;m,n,k}$ are called progressively first-failure censored order statistics with the progressive censored scheme $R = (R_1, R_2, \ldots, R_m)$. It is clear that $n = m + \sum_{i=1}^{m} R_i$. If the failure times of the $n \times k$ items originally in the test are from a continuous population with cdf $F(x)$ and pdf $f(x)$, the joint pdf for $X^R_{1;m,n,k}, X^R_{2;m,n,k}, \ldots, X^R_{m;m,n,k}$ is given by Wu and Kus [22] as follows:

$$\begin{aligned} &f_{1,2,\ldots,m}\left(X^R_{1;m,n,k}, X^R_{2;m,n,k}, \ldots, X^R_{m;m,n,k}\right) \\ &\quad = Ak^m \prod_{i=1}^{m} f(x^R_{i;m,n,k})[1 - F(x^R_{i;m,n,k})]^{k(R_i+1)-1}, \\ &\qquad 0 < x^R_{1;m,n,k} < x^R_{2;m,n,k} < \ldots < x^R_{m;m,n,k} < \infty, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} A = \; & n(n - R_1 - 1)(n - R_1 - R_2 - 2) \\ & \ldots (n - R_1 - R_2 - \ldots - R_{m-1} - m + 1) \end{aligned} \tag{2.2}$$

This censoring scheme has advantages in terms of reducing test time, in which more items are used but only m of $n \times k$ items are failures. Note that using the above notation, some censoring rules can be accommodated such as the first-failure censored order statistics when $R = (0,0,\dots,0)$, a progressive type-II censored order statistics when $k = 1$, a usual type-II censored order statistics when $k = 1$ and $R = (0,0,\dots,n-m)$, and a complete sample if $k = 1$ and $R = (0,0,\dots,0)$, with $n = m$. Also, it should be noted that the progressive first-failure censored sample $X^R_{1;m,n,k}, X^R_{2;m,n,k}, \dots, X^R_{m;m,n,k}$ with $cdf\ F(x)$, can be viewed as a progressive type-II censored sample from a population with cdf $1-(1-F(x))^k$.

3. POINT ESTIMATION

3.1 Maximum Likelihood Estimation

Let $X_i = X^R_{i;m,n,k}$, $i = 1,2,\dots,m$, be the progressive first-failure censored order statistics from MW distribution with censored scheme $R = (R_1, R_2, \dots, R_m)$ and its realization denoted by $x^R_{i;m,n,k}$, $i = 1,2,\dots,m$ which can be written for simplicity as $x = (x_1,\dots,x_m)$. The likelihood function of the parameters β, τ and λ given the vector of observations x can be obtained by substituting from (1.1) and (1.2) in (2.1) to be of the form

$$L\left(\beta,\tau,\lambda \middle| x\right)\ \alpha\ \tau^m \prod_{i=1}^{m} (\beta+\lambda x_i) x_i^{\beta-1} \\ exp(\lambda x_i)\ exp\left[-k(R_i+1)\ \tau x_i^{\beta} e^{\lambda x_i}\right],\ \ \beta>0,\ \tau>0,\ \lambda>0. \tag{3.1}$$

By taking the natural logarithm for the likelihood function (3.1), differentiating with respect to all parameters and then setting to zero, three nonlinear equations will be obtained. By solving these nonlinear equations numerically, the maximum likelihood estimates ($MLE's$) of all parameters have been obtained.

3.2 Bayes Estimation

Suppose that the prior belief of the experimenter is measured by the trivariate prior suggested by Ateya[1] which of the form

$$\pi(\beta,\tau,\lambda)\ \propto\ \frac{1}{\Gamma(\beta)} \beta^{c_1+c_3-1}\ \tau^{\beta+c_3-1}\ \lambda^{\beta-1}\, exp[-\beta\ (\tau+c_2)-\tau\ \lambda], \\ \beta>0,\ \tau>0,\ \lambda>0,\ (c_1>0,\ c_2>0,\ c_3>0), \tag{3.2}$$

where $\boldsymbol{c_1}, \boldsymbol{c_2}$ and $\boldsymbol{c_3}$ are the prior parameters (also known as hyperparameters).

Therefore, the joint posterior $\boldsymbol{pdf}$ of the parameters $\boldsymbol{\beta}, \boldsymbol{\tau}$ and $\boldsymbol{\lambda}$ can be obtained from (3.1) and (3.2) in the form

$$\boldsymbol{\pi^*(\beta,\tau,\lambda|x) = \frac{A}{\Gamma(\beta)} \beta^{c_1+c_3-1}\ \tau^{\beta+c_3+m-1}\ \lambda^{\beta-1}\, exp[-\beta\ (\tau+c_2)-\tau\ \lambda]} \\ \boldsymbol{\prod_{i=1}^{m}[(\beta+\lambda x_i) x_i^{\beta-1}\, exp(\lambda x_i)\, exp[-k(R_i+1)\tau x_i^{\beta} e^{\lambda x_i}]]}, \tag{3.3}$$

where $\boldsymbol{A}$ is a normalizing constant.

Using the $\boldsymbol{MCMC}$ method, the Bayes estimate ($\boldsymbol{BE}$) of any function $\boldsymbol{\eta(\beta,\tau,\lambda)}$ under $\boldsymbol{SE}$ and Linex loss functions are given, respectively, by

$$\hat{\eta}_{BS} = \frac{1}{N-M}\sum_{i=M+1}^{N}\eta(\beta_i,\tau_i,\lambda_i)\,, \tag{3.4}$$

and

$$\hat{\eta}_{BL} = -\frac{1}{a}ln[\frac{1}{N-M}\sum_{i=M+1}^{N}exp(-a\eta(\beta_i,\tau_i,\lambda_i))], \tag{3.5}$$

where β_i, τ_i and λ_i are generated from the posterior pdf, M is the burn-in period (that is, a number of iterations before the stationary distribution is achieved) and a is a constant.

For more details about $MCMC$ methods, see, for example, Upadhyaya and Gupta[18] and Upadhyaya et al.[19]. The Gibbs is an algorithm for simulating from the full conditional posterior distributions while the Metropolis-Hatings algorithm generate sampling from an (essentially) arbitrary proposal distribution (i.e., a Markov transition kernel).

4. INTERVAL ESTIMATION

In this section, the approximate confidence interval (ACI), bootstrap-p confidence interval (BCI),credibility confidence interval (CCI) and highest posterior density interval (HPD) for the parameters β, τ and λ have been studied.

4.1 Approximate Confidence Interval

Let $X^R_{1;m,n,k} < X^R_{2;m,n,k} <\ldots< X^R_{m;m,n,k}$ denote a progressive first-failure censored sample from MW distribution with parameters β, τ and λ. In this section, the approximate confidence intervals for the parameters of MW distribution have been obtained based on progressive first-failure censored using the Fisher information matrix $I(\beta,\tau,\lambda)$ which can be estimated by $I(\hat{\beta},\hat{\tau},\hat{\lambda})$ in the form

$$I(\hat{\beta},\hat{\tau},\hat{\lambda}) = \begin{bmatrix} -\frac{\partial^2\ell}{\partial\beta^2} & -\frac{\partial^2\ell}{\partial\beta\partial\tau} & -\frac{\partial^2\ell}{\partial\beta\partial\lambda} \\ -\frac{\partial^2\ell}{\partial\beta\partial\tau} & -\frac{\partial^2\ell}{\partial\tau^2} & -\frac{\partial^2\ell}{\partial\tau\partial\lambda} \\ -\frac{\partial^2\ell}{\partial\beta\partial\lambda} & -\frac{\partial^2\ell}{\partial\tau\partial\lambda} & -\frac{\partial^2\ell}{\partial\lambda^2} \end{bmatrix}_{(\hat{\beta},\hat{\tau},\hat{\lambda})}, \tag{4.1}$$

where ℓ is the log likelihood of the parameters β, τ and λ.

The Approximate confidence intervals for β, τ and λ can be obtained, respectively, by

$$\hat{\beta} \mp z_{\frac{\alpha}{2}}\sqrt{\nu_{11}} \quad \hat{\tau} \mp z_{\frac{\alpha}{2}}\sqrt{\nu_{22}} \quad and \quad \hat{\lambda} \mp z_{\frac{\alpha}{2}}\sqrt{\nu_{33}}, \tag{4.2}$$

where ν_{11}, ν_{22} and ν_{33} are the elements on the main diagonal of the covariance matrix $I^{-1}(\hat{\beta},\hat{\tau},\hat{\lambda})$ and $z_{\frac{\alpha}{2}}$ is the standard normal variate.

4.2 Bootstrap Confidence Interval

In this section, confidence intervals based on the parametric percentile bootstrap method ($\boldsymbol{Bootstrap-p}$) have been obtained based on the idea of Efron [5]. The algorithms for estimating the confidence intervals of the parameters using $\boldsymbol{Bootstrap-p}$ method are illustrated as the following:

1. From the original data $\boldsymbol{x}=(\boldsymbol{x_1},\boldsymbol{x_2},\ldots,\boldsymbol{x_n})$ compute the $\boldsymbol{MLE's}$ of the parameters $\boldsymbol{\beta},\boldsymbol{\tau}$ and $\boldsymbol{\lambda}$, say $\boldsymbol{\hat\beta},\boldsymbol{\hat\tau}$ and $\boldsymbol{\hat\lambda}$, respectively.
2. Using $\boldsymbol{\hat\beta},\boldsymbol{\hat\tau}$ and $\boldsymbol{\hat\lambda}$, a bootstrap sample of upper ordered values $\boldsymbol{x}^*$ is generated.
3. As in Step $\mathbf{1}$, based on $\boldsymbol{x}^*$, compute the bootstrap sample estimates of $\boldsymbol{\beta},\boldsymbol{\tau}$ and $\boldsymbol{\lambda}$ say $\boldsymbol{\hat\beta}^*,\boldsymbol{\hat\tau}^*$ and $\boldsymbol{\hat\lambda}^*$.
4. Repeat Steps 2 and 3 $\boldsymbol{N}$ times representing $\boldsymbol{N}$ bootstrap $\boldsymbol{MLEs}$ of $\boldsymbol{\beta},\boldsymbol{\tau}$ and $\boldsymbol{\lambda}$ based on $\boldsymbol{N}$ bootstrap samples.
5. Arrange all $\boldsymbol{\hat\beta}^*$'s, $\boldsymbol{\hat\tau}^*$'s and $\boldsymbol{\hat\lambda}^*$'s in an ascending order to obtain the bootstrap samples $(\boldsymbol{\hat\beta}^{*\mathbf{1}},\boldsymbol{\hat\beta}^{*\mathbf{2}},\ldots,\boldsymbol{\hat\beta}^{*\boldsymbol{N}})$, $(\boldsymbol{\hat\tau}^{*\mathbf{1}},\boldsymbol{\hat\tau}^{*\mathbf{2}},\ldots,\boldsymbol{\hat\tau}^{*\boldsymbol{N}})$ and $(\boldsymbol{\hat\lambda}^{*\mathbf{1}},\boldsymbol{\hat\lambda}^{*\mathbf{2}},\ldots,\boldsymbol{\hat\lambda}^{*\boldsymbol{N}})$.
6. A two-sided $(\mathbf{1}-\boldsymbol{\alpha})\times\mathbf{100}\%$ $\boldsymbol{BCI}$ of $\boldsymbol{\beta}$, say $[\boldsymbol{\beta}_L^*,\boldsymbol{\beta}_U^*]$ is then given by $[\boldsymbol{\hat\beta}^{*\boldsymbol{N}(\alpha/\mathbf{2})}, \boldsymbol{\hat\beta}^{*\boldsymbol{N}(\mathbf{1}-\alpha/\mathbf{2})}]$.
7. Also, a two-sided $(\mathbf{1}-\boldsymbol{\alpha})\times\mathbf{100}\%$ $\boldsymbol{BCI}$ of $\boldsymbol{\tau}$, say $[\boldsymbol{\tau}_L^*,\boldsymbol{\tau}_U^*]$ is then given by $[\boldsymbol{\hat\tau}^{*\boldsymbol{N}(\alpha/\mathbf{2})},\boldsymbol{\hat\tau}^{*\boldsymbol{N}(\mathbf{1}-\alpha/\mathbf{2})}]$.
8. Finally, a two-sided $(\mathbf{1}-\boldsymbol{\alpha})\times\mathbf{100}\%$ $\boldsymbol{BCI}$ of $\boldsymbol{\lambda}$, say $[\boldsymbol{\lambda}_L^*,\boldsymbol{\lambda}_U^*]$ is then given by $[\boldsymbol{\hat\lambda}^{*\boldsymbol{N}(\alpha/\mathbf{2})},\boldsymbol{\hat\lambda}^{*\boldsymbol{N}(\mathbf{1}-\alpha/\mathbf{2})}]$.

4.3 Credibility Confidence Interval

For a specified value of $\boldsymbol{\alpha}$, $(\mathbf{1}-\boldsymbol{\alpha})\times\mathbf{100}\%$ $\boldsymbol{CCI}$ $(\boldsymbol{L_\beta},\boldsymbol{U_\beta})$ for $\boldsymbol{\beta}$, $(\mathbf{1}-\boldsymbol{\alpha})\times\mathbf{100}\%$ $\boldsymbol{CCI}$ $(\boldsymbol{L_\tau},\boldsymbol{U_\tau})$ for $\boldsymbol{\tau}$ and $(\mathbf{1}-\boldsymbol{\alpha})\times\mathbf{100}\%$ $\boldsymbol{CCI}$ $(\boldsymbol{L_\lambda},\boldsymbol{U_\lambda})$ for $\boldsymbol{\lambda}$ have been defined, respectively by

$$\begin{aligned}
&\int_{L_\beta}^{\infty}\boldsymbol{\pi_1^*(\beta|x)d\beta}=\mathbf{1}-\frac{\boldsymbol{\alpha}}{\mathbf{2}}, \quad \int_{U_\beta}^{\infty}\boldsymbol{\pi_1^*(\beta|x)d\beta}=\frac{\boldsymbol{\alpha}}{\mathbf{2}},\\
&\int_{L_\tau}^{\infty}\boldsymbol{\pi_2^*(\tau|x)d\tau}=\mathbf{1}-\frac{\boldsymbol{\alpha}}{\mathbf{2}}, \quad \int_{U_\tau}^{\infty}\boldsymbol{\pi_2^*(\tau|x)d\tau}=\frac{\boldsymbol{\alpha}}{\mathbf{2}},\\
&\int_{L_\lambda}^{\infty}\boldsymbol{\pi_3^*(\lambda|x)d\lambda}=\mathbf{1}-\frac{\boldsymbol{\alpha}}{\mathbf{2}}, \quad \int_{U_\lambda}^{\infty}\boldsymbol{\pi_3^*(\lambda|x)d\lambda}=\frac{\boldsymbol{\alpha}}{\mathbf{2}},
\end{aligned} \tag{4.3}$$

where $\boldsymbol{\pi_1^*(\beta|x)}$, $\boldsymbol{\pi_2^*(\tau|x)}$ and $\boldsymbol{\pi_3^*(\lambda|x)}$ are the marginal density functions of $\boldsymbol{\beta}$, $\boldsymbol{\tau}$ and $\boldsymbol{\lambda}$, respectively. In many cases it will be very difficult to obtain the marginal $\boldsymbol{pdf}$ from the posterior $\boldsymbol{pdf}$. So, Gibbs sampler and Metropolis Hastings algorithms are used to generate $(\boldsymbol{\beta_1},\boldsymbol{\tau_1},\boldsymbol{\lambda_1}),(\boldsymbol{\beta_2},\boldsymbol{\tau_2},\boldsymbol{\lambda_2}),\ldots,(\boldsymbol{\beta_N},\boldsymbol{\tau_N},\boldsymbol{\lambda_N})$ from $\boldsymbol{\pi^*(\beta,\tau,\lambda|x)}$.

Using these generated values of $\boldsymbol{\beta},\boldsymbol{\tau}$ and $\boldsymbol{\lambda}$, the marginal posteriors $\boldsymbol{pdf's}$ can be written in the forms

$$\begin{aligned}\pi_1^*(\beta|x) &= \frac{1}{N}\sum_{i=1}^{N}\pi^*(\beta,\tau_i,\lambda_i|x),\\ \pi_2^*(\tau|x) &= \frac{1}{N}\sum_{i=1}^{N}\pi^*(\tau,\beta_i,\lambda_i|x),\\ \pi_3^*(\lambda|x) &= \frac{1}{N}\sum_{i=1}^{N}\pi^*(\lambda,\beta_i,\tau_i|x).\end{aligned} \tag{4.4}$$

Substituting from (4.4) in (4.3), simple formulas have been obtained to compute the credibility intervals for β, τ and λ in the following form

$$\begin{aligned}&\frac{1}{N}\sum_{i=1}^{N}\int_{L_\beta}^{\infty}\pi^*(\beta,\tau_i,\lambda_i|x)d\beta = 1-\frac{\alpha}{2}, \quad \frac{1}{N}\sum_{i=1}^{N}\int_{U_\beta}^{\infty}\pi^*(\beta,\tau_i,\lambda_i|x)d\beta = \frac{\alpha}{2}\\ &\frac{1}{N}\sum_{i=1}^{N}\int_{L_\tau}^{\infty}\pi^*(\tau,\beta_i,\lambda_i|x)d\tau = 1-\frac{\alpha}{2}, \quad \frac{1}{N}\sum_{i=1}^{N}\int_{U_\tau}^{\infty}\pi^*(\tau,\beta_i,\lambda_i|x)d\tau = \frac{\alpha}{2},\\ &\frac{1}{N}\sum_{i=1}^{N}\int_{L_\lambda}^{\infty}\pi^*(\lambda,\beta_i,\tau_i|x)d\lambda = 1-\frac{\alpha}{2}, \quad \frac{1}{N}\sum_{i=1}^{N}\int_{U_\lambda}^{\infty}\pi^*(\lambda,\beta_i,\tau_i|x)d\lambda = \frac{\alpha}{2}.\end{aligned} \tag{4.5}$$

4.4 Highest Posterior Density Interval

A $(1-\alpha)\times 100\%$ HPD interval for β has been obtained by solving the following two nonlinear equations

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N}\int_{L_\beta}^{U_\beta}\pi^*(\beta,\tau_i,\lambda_i|x)d\beta = 1-\alpha, &\sum_{i=1}^{N}\pi^*(L_\beta,\tau_i,\lambda_i|x)\\ &= \sum_{i=1}^{N}\pi^*\left(U_\beta,\tau_i,\lambda_i\middle|x\right).\end{aligned} \tag{4.6}$$

Similarly, the $(1-\alpha)\times 100\%$ HPD interval for τ has been obtained by solving the following two nonlinear equations

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N}\int_{L_\tau}^{U_\tau}\pi^*(\tau,\beta_i,\lambda_i|x)d\tau = 1-\alpha, &\sum_{i=1}^{N}\pi^*(L_\tau,\beta_i,\lambda_i|x)\\ &= \sum_{i=1}^{N}\pi^*(U_\tau,\beta_i,\lambda_i|x).\end{aligned} \tag{4.7}$$

Finally, the $(1-\alpha)\times 100\%$ HPD interval for λ has been obtained by solving the following two nonlinear equations

$$\frac{1}{N}\sum_{i=1}^{N}\int_{L_\lambda}^{U_\lambda}\pi^*(\lambda,\beta_i,\tau_i|x)d\lambda = 1-\alpha, \sum_{i=1}^{N}\pi^*(L_\lambda,\beta_i,\tau_i|x) = \sum_{i=1}^{N}\pi^*(U_\lambda,\beta_i,\tau_i|x). \tag{4.8}$$

5. NUMERICAL COMPUTATIONS

In the following, the maximum likelihood and Bayesian estimates are compared based on a Monte Carlo simulation study.

1. For a given vector of prior parameters (c_1, c_2, c_3) the vector of population parameters (β, τ, λ) have been generated from the joint prior (3.2).
2. Making use of the generated population parameters, a progressive first-failure censored samples from the MW distribution with pdf (1.1) have been generated. To generate progressive first failure samples, the algorithm proposed by Balakrishnan and Aggarwala[4] has been used, with the fact that, the progressive first-failure censored sample $X_{1;m,n,k}^R, X_{2;m,n,k}^R, \dots, X_{m;m,n,k}^R$ with $cdf\ F(x)$, can be viewed as progressive type-II censored sample from a population with distribution function $1-(1-F(x))^k$. The number of items put on a life test has been assumed equal to $n \times k$, where n denotes the number of groups and k the number of items in each group. Using a progressive first-failure censoring scheme, only m observations are obtained from the test.
3. The $MLE's$ of β, τ and λ are computed as shown in section 3 using the software $Mathematica\ 8$ for solving the resulting nonlinear equations.
4. The $BE's$ for the parameter $\eta \equiv (\beta, \tau, \lambda)$ under SE and $Linex$ loss functions using $MCMC$ method are given, respectively, by using the formulas (3.4) and (3.5).
5. The above steps (2-4) are repeated 1000 times.
6. If $\hat{\theta}_j$ is an estimate of θ , based on sample $j, j = 1, 2, \dots, 1000$, then the average estimate over the 1000 samples is given by

   $\bar{\hat{\theta}} = \frac{1}{1000}\sum_{j=1}^{1000}\hat{\theta}_j.$
7. The $MSE's$ of $\hat{\theta}$ over the 1000 samples is given by

   $MSE(\hat{\theta}) = \frac{1}{1000}\sum_{j=1}^{1000}(\hat{\theta}_j - \theta)^2.$
8. From 6 and 7 the average estimates and the $MSE's$ for all parameters have been computed.
9. The ACI, BIC, CCI, HPD, lengths and finally the coverage probabilities $(CP's)$ for all parameters are computed.

The computations are shown in Tables 1 and 2.

Table 1
MSE′s of the Estimates under SEL and LINEX Loss Function
$(a = -3, 0.0001, 3)$, $(\beta = 2.5, \tau = 2.0, \lambda = 2.7)$, $(c_1 = 1.4, c_2 = 0.8, c_3 = 2.5)$
based on Progressive First-Failure Censored Scheme

(n, m, k)	$Method$			$MSE(\hat{\beta})$	$MSE(\hat{\tau})$	$MSE(\hat{\lambda})$
$R = (1, 2, 1, 1, 3, 1, 1, 2, 1, 2)$						
$(25, 10, 1)$	ML			0.3017	1.2013	1.0152
	$Bayes$	SEL		0.2438	1.0031	0.8161
		$LINEX$	$a = -3.0$	0.2819	1.1163	0.9032
			$a = 0.0001$	0.2438	1.0031	0.8161
			$a = 3.0$	0.1671	0.5721	0.4301
$R = (1, 1, 0, 0, 0, 1, 0, 1, 0, 0, 3, 0, 1, 0, 2)$						
$(25, 15, 3)$	ML			0.2541	0.8603	0.9068
	$Bayes$	SEL		0.1901	0.6691	0.8230
		$LINEX$	$a = -3.0$	0.2068	0.7105	0.8614
			$a = 0.0001$	0.1901	0.6691	0.8230
			$a = 3.0$	0.1005	0.4708	0.3552
$R = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$						
$(25, 25, 5)$	ML			0.1006	0.5104	0.7105
	$Bayes$	SEL		0.0916	0.2217	0.4319
		$LINEX$	$a = -3.0$	0.0951	0.4416	0.5115
			$a = 0.0001$	0.0916	0.2217	0.4319
			$a = 3.0$	0.0103	0.1017	0.1506

Table 2

$ACI's, BCI's, CCI's$ and $HPD's$ for The parameters β, τ and λ.

(n, m, k)	$Method$	(L_β, U_β) / $Length$ / CP	(L_τ, U_τ) / $Length$ / CP	(L_λ, U_λ) / $Length$ / CP
$R = (1, 2, 1, 1, 3, 1, 1, 2, 1, 2)$				
$(25, 10, 1)$	ACI	(0.2178,5.0164)	(0.6822,2.7594)	(1.5993,4.3504)
		4.7986	2.0771	2.7511
		95.35	96.21	95.71.4
	BCI	(0.9107,5.0233)	(0.8005,2.6524)	(1.5109,4.1728)
		4.1126	1.8519	2.6619
		95.35	96.21	95.71.4
	CCI	(1.3464,5.1343)	(1.3409,2.7701)	(1.8609,4.0924)
		3.7879	1.4292	2.2315
		98.0	95.4	96.9
	HPD	(1.7901,3.9010)	(1.0923,2.2021)	(1.5541,3.4347)
		2.1109	1.1098	1.8806
		95.76	95.45	95.98.01
$R = (1, 1, 0, 0, 0, 1, 0, 1, 0, 0, 3, 0, 1, 0, 2)$				
$(25, 10, 3)$	ACI	(0.7337,3.9824)	(0.5387,1.894)	(1.1332,3.2824)
		3.2387	1.3553	2.1482
		95.2	96.3	96.2
	BCI	(0.8213,3.3402)	(0.6617,1.9933)	(1.4101,3.4147)
		2.8015	1.3316	2.0046
		95.2	96.3	96.2
	CCI	(1.6038,3.5673)	(0.8091,2.1103)	(1.6583,3.4965)
		1.9635	1.3012	1.8382
		99.8	97.8	95.4
	HPD	(1.7709,3.5822)	(1.0733,2.1747)	(1.6198,3.4019)
		1.8113	1.1014	1.7821
		96.01	96.44	97.85
$R = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$				
$(25, 10, 5)$	ACI	(1.3309,4.2275)	(0.6914,1.9411)	(1.9415,3.9737)
		2.8966	1.2497	2.0322
		97.1	96.1	98.6
	BCI	(1.4016,2.9119)	(0.5809,1.6825)	(2.0131,3.9456)
		1.5103	1.1016	1.9325
		97.1	96.1	98.6
	CCI	(2.9566,3.9545)	(1.0393,2.0473)	(2.1147,3.1856)
		0.9979	1.0080	1.0709
		96.76	96.6	95.4
	HPD	(2.6520,3.5264)	(0.9187,1.8955)	(2.8319,3.4446)
		0.8744	0.9768	0.8127
		95.87	95.63	96.32

6. DATA ANALYSIS AND APPLICATION

In this section, a real life data set has been considered and the methods proposed in the previous sections have been illustrated. The real data set is from Nicholas and Padgett[14]. The data concerning tensile strength of 100 observations of carbon fibers, they are:

3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

These real data are analyzed using $\boldsymbol{Weibull(\alpha, \beta)}$ distribution and using $\boldsymbol{MW(\beta, \tau, \lambda)}$ by Ateya[1] and he found that the $\boldsymbol{MW}$ model fits these data better than the Weibull model. To illustrate the use of the estimation methods proposed in this paper, firstly the data have been ordered as follows

0.39, 0.81, 0.85, 0.98, 1.08, 1.12, 1.17, 1.18, 1.22, 1.25, 1.36, 1.41, 1.47, 1.57, 1.57, 1.59, 1.59, 1.61, 1.61, 1.69, 1.69, 1.71, 1.73, 1.80, 1.84, 1.84, 1.87, 1.89, 1.92, 2.00, 2.03, 2.03, 2.05, 2.12, 2.17, 2.17, 2.17, 2.35, 2.38, 2.41, 2.43, 2.48, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.68, 2.73, 2.74, 2.76, 2.77, 2.79, 2.81, 2.81, 2.82, 2.83, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.51, 3.56, 3.60, 3.65, 3.68, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90, 4.91, 5.08, 5.56.

Secondly, under the assumption that the carbon fibers are randomly grouped into 25 groups with $\boldsymbol{k = 4}$ carbon fibers within each group. The tensile strength of carbon fibers of the groups are:

{0.39, 0.81, 0.85, 0.98}, {1.08, 1.12, 1.17, 1.18}, {1.22, 1.25, 1.36, 1.41}, {1.47, 1.57, 1.57, 1.59}, {1.59, 1.61, 1.61, 1.69}, {1.69, 1.71, 1.73, 1.80}, {1.84, 1.84, 1.87, 1.89}, {1.92, 2.00, 2.03, 2.03}, {2.05, 2.12, 2.17, 2.17}, {2.17, 2.35, 2.38, 2.41}, {2.43, 2.48, 2.48, 2.50}, {2.53, 2.55, 2.55, 2.56}, {2.59, 2.67, 2.68, 2.73}, {2.74, 2.76, 2.77, 2.79}, {2.81, 2.81, 2.82, 2.83}, {2.85, 2.87, 2.88, 2.93}, {2.95, 2.96, 2.97, 2.97}, {3.09, 3.11, 3.11, 3.15}, {3.15, 3.19, 3.19, 3.22}, {3.22, 3.27, 3.28, 3.31}, {3.31, 3.33, 3.39, 3.39}, {3.51, 3.56, 3.60, 3.65}, {3.68, 3.68, 3.70, 3.75}, {4.20, 4.38, 4.42, 4.70}, {4.90, 4.91, 5.08, 5.56}.

Suppose that the pre-determined progressively first-failure censoring plan is applied using progressive censoring plan is applied using progressive censoring scheme

$$\boldsymbol{R} = (\mathbf{2}, \mathbf{2}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}).$$

The following progressively first-failure censored data of size ($\boldsymbol{m = 20}$) out of 25 groups of carbon fibers were observed: 0.39, 1.47, 1.84, 1.92, 2.05, 2.43, 2.53, 2.59, 2.74, 2.81, 2.85, 2.95, 3.09, 3.15, 3.22, 3.31, 3.51, 3.68, 4.20, 4.90. For this example 5 groups are censored and 20 first failure are observed. The estimates of the parameters $\boldsymbol{\beta}, \boldsymbol{\tau}$ and $\boldsymbol{\lambda}$ are obtained in Table 3. Moreover, the result of 95% $\boldsymbol{ACI}$, $\boldsymbol{BIC}$, $\boldsymbol{CCI}$ and $\boldsymbol{HPD}$ for $\boldsymbol{\beta}, \boldsymbol{\tau}$ and $\boldsymbol{\lambda}$ are given in Table 4

Table 3
Estimates of the Parameters β, τ and λ using ML and Bayes Methods (under SEL and $LINEX$ Loss Functions) ($a = 0, 1, 2$) based on Progressive First-Failure Censored Scheme from Real Data

(n, m, k)	$Method$			$\hat{\beta}$	$\hat{\tau}$	$\hat{\lambda}$
$(25, 20, 4)$	ML			1.9320	0.2155	2.3183
	$Bayes$	SEL		1.8162	0.2112	2.1516
		$LINEX$	$a = 0.00001$	1.8162	0.2112	2.1516
			$a = 1.0$	1.8218	0.2819	1.9915
			$a = 2.0$	1.9813	0.2812	1.9814

Table 4
Confidence Intervals for the Parameters β, τ and λ based on Progressive First-Failure Censored Scheme from Real Data

(n, m, k)	$Method$	(L_β, U_β)	(L_τ, U_τ)	(L_λ, U_λ)
		$Length$	$Length$	$Length$
$(25, 20, 4)$	ACI	(1.2182,2.7690)	(0.1205,0.3212)	(1.4128,3.7233)
		1.5518	0.2017	2.3105
	BCI	(1.4103,2.8211)	(0.1315,0.3121)	(1.1106,3.2623)
		1.4108	0.1806	2.1517
	CCI	(1.5011,2.8115)	(0.1481,0.3099)	(1.2306,3.1623)
		1.3104	0.1618	1.9317
	HPD	(1.4804,2.6363)	(0.1336,0.2749)	(1.1716,2.8869)
		1.1559	0.1413	1.7153

7. CONCLUDING REMARKS

In this paper, the estimation problem (point and interval) is studied based on progressive first failure censoring scheme of MW distribution. Also, a real data set is introduced as illustrative example. A simulation study is carried out to examine and compare the performance of the proposed methods for different sample sizes and different censoring schemes. From the results which are summarized in tables 1 and 2, the following can be observed.

1. The $MSE's$ of the $BE's$ based on SEL function and $LINEX$ loss function are less than that obtained for the $MLE's$ which means that the $BE's$ are better than the $MLE's$.
2. The $MSE's$ of the $BE's$ based on $LINEX$ loss function decrease by increasing a.
3. The $MSE's$ of the $BE's$ based on $LINEX$ loss function are the same as that obtained based on SEL function when $a \to 0$.
4. In all cases, the $CP's$ of all intervals of all methods close to the desired level of 0.95.
5. The length of the $ACI >$ that computed for $BCI >$ that computed for the $CCI >$ that computed for HPD interval.

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