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AN EMPIRICAL STUDY ON MODELING SKEWED DATA USING A MIXTURE OF PROBABILITY DISTRIBUTIONS

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ABSTRACT

In data analysis, we cannot expect the data under study to always follow a symmetric probability distribution; they can have a skewed probability distribution. Modeling skewed data is a challenging task for data scientists. A mixture of probability distributions is a better option for modelling skewed data. In this study, we generated a few mixtures of probability distributions using the exponential as the first fixed component and Gompertz, Lindley, and log-normal distributions as the second component for modelling skewed data. We also examined the characteristics of mixed probability distributions. The maximum likelihood method was used to estimate the unknown parameters. As a result, all three mixture models have a right-skewed pattern, which provides a better fit than existing distributions. Finally, we used real-time datasets to model skewed data.

KEYWORDS

Mixture model, Lifetime distributions, Goodness of fit, Parameter Estimation, Information measure.

I. INTRODUCTION

In this modern era, numerous probability distributions are available in the literature, and a new family of probability distributions is generated every day using various methods such as the transformation method, composite method, beta-generated method, method of adding parameters to an existing probability distribution, transformed-transformer method (*T-X* family), method of generating skewed probability distributions, and finite mixture models. In Statistics, data are expressed as a frequency distribution function that displays the range of potential values for a variable together with its frequency. Practically speaking, not all real datasets can be modeled well using traditional probability distributions. The development of a new class of flexible probability distributions is required for these types of datasets. We are developing a new class of distributions using a variety of techniques because these probability distributions are more flexible than traditional distributions. The finite mixture model is used by several authors to model skewed datasets.

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Let $X = \{x_i, i = 1,2, \dots, n\}$ be a random sample of size n obtained from an m-component finite mixture.

$$f(x_i;\theta) = \sum_{i=1}^{m} w_i g_i(x_i,\theta)$$

where,

$g_i(x_i,\theta) =$ probability density or mass function
w_i are nonnegative quantities
such that $w_1 + w_2 + \dots + w_m = 1$
(i.e.) $0 \le w_i \le 1$ for i =1,2,...,m

Furthermore, the two-component finite mixing model is.

$$f(x) = w_1 g_1(x) + w_2 g_2(x) \tag{1.1}$$

The first major analyses using mixture models were carried out by Karl Pearson [30], a well-known biometrician; who fitted a proportional mixture of two normal probability density functions with different means μ_1 and μ_2 and different variances σ_1^2 and σ_2^2 in proportions π_1 and π_2. The finite mixture model is also used by Lindley [22] to create the Lindley distribution. For the Lindley distribution, a two-component mixture model was used to generate the distribution. An exponential distribution with a scale parameter of θ and a gamma distribution with a shape parameter of 2 and a scale parameter of θ are the two components with proportions - $\frac{\theta}{\theta+1}$, $\frac{1}{\theta+1}$. The Table below presents a list of mixed models developed using exponential and gamma distributions with various weights.

Table 1
Two-Component Mixture of Probability Distributions

Author(s)	Model	Component 1	Component 2	Weightage $(w_2 = 1 - w_1)$	Ref.
Rama Shanker (2013)	Sushila	Exponential (θ/α)	Gamma (2, θ/α)	$w_1 = \frac{\theta}{\theta + 1}$	[38]
Rama Shanker et al., (2013)	Janardan distribution	Exponential (θ/α)	Gamma (2, θ/α)	$w_1 = \frac{\theta}{\theta + \alpha^2}$	[39]
Shanker (2015)	Akash	Exponential (θ)	Gamma (3, θ)	$w_1 = \frac{\theta^2}{\theta^2 + 2}$	[40]
Rama Shanker (2015)	Shanker distribution	Exponential (θ)	Gamma (2, θ)	$w_1 = \frac{\theta^2}{\theta^2 + 1}$	[32]
Shanker (2017)	Rama distribution	Exponential (θ)	Gamma (4, θ)	$w_1 = \frac{\theta^3}{\theta^3 + 6}$	[35]
Shanker and Shukla (2017)	Ishita distribution	Exponential (θ)	Gamma (3, θ)	$w_1 = \frac{\theta^3}{\theta^3 + 2}$	[42]
Shukla (2018)	Ram Awadh distribution	Exponential (λ)	Gamma (6, λ)	$w_1 = \frac{\lambda^6}{\lambda^6 + 120}$	[20]
Shukla (2018)	Pranav distribution	Exponential (θ)	Gamma (4, θ)	$w_1 = \frac{\theta^4}{\theta^4 + 6}$	[43]
Kamlesh Kumar Shukla and Rama Shanker (2019)	Shukla distribution	Exponential (θ)	Gamma (α+1, θ)	$w_1 = \frac{\theta^{\alpha+1}}{\theta^{\alpha+1} + \Gamma(\alpha + 1)}$	[19]
Shraa and Al-Omari (2019)	Darna distribution	Exponential (θ/α)	Gamma (3, θ/α)	$w_1 = \frac{2\alpha^2}{2\alpha^2 + \theta^2}$	[7]
Benrabia and Alzoubi (2022)	Alzoubi distribution	Exponential (β)	Gamma (α-1, β)	$w_1 = \frac{\alpha\beta}{\alpha\beta + 1}$	[24]
Mohammed Benrabia and Loai M.A. Alzoubi (2022)	Benrabia distribution	Exponential (β)	Gamma (α-1, β)	$w_1 = \frac{\alpha}{\alpha + \beta}$	[25]

The three-component mixture models are presented below:

Table 2
Three-Component Mixture Distribution

Author(s)	Model	Component 1 with the corresponding weightage	Component 2 with the corresponding weight	Component 3 with the corresponding weight	Ref.
Shanker (2016)	Aradhana distribution	Exponential (θ) $w_1 = \frac{\theta^2}{\theta^2 + 2\theta + 2}$	Gamma (2, θ) $w_2 = \frac{2\theta}{\theta^2 + 2\theta + 2}$	Gamma (2, θ) $w_3 = \frac{2}{\theta^2 + 2\theta + 2}$	[34]
Shanker (2015)	Sujatha distribution	Exponential (θ) $w_1 = \frac{\theta^2}{\theta^2 + \theta + 2}$	Gamma (2, θ) $w_2 = \frac{\theta}{\theta^2 + \theta + 2}$	Gamma (3, θ) $w_3 = \frac{2}{\theta^2 + \theta + 2}$	[41]

Likewise, four-component mixture models are available in the literature. Gharaibeh (2021) introduced the Gharaibeh distribution using four components i.e., exponential (θ), gamma $(2,\theta)$, gamma $(4,\theta)$, and gamma $(6,\theta)$ with proportions $w_1=\frac{\beta^6}{\beta^6+\beta^4+\beta^2+1}$, $w_2=\frac{\beta^4}{\beta^6+\beta^4+\beta^2+1}$, $w_3=\frac{\beta^2}{\beta^6+\beta^4+\beta^2+1}$, and, $w_4=\frac{1}{\beta^6+\beta^4+\beta^2+1}$. Rama Shanker [33] also introduced an Amarendra distribution and the four components are exponential (θ), gamma $(2,\theta)$, gamma $(3,\theta)$, and gamma $(4,\theta)$ with the proportions of $w_1=\frac{\theta^3}{\theta^3+\theta^2+2\theta+6}$, $w_2=\frac{\theta^2}{\theta^3+\theta^2+2\theta+6}$, $w_3=\frac{2\theta}{\theta^3+\theta^2+2\theta+6}$ and, $w_4=\frac{6}{\theta^3+\theta^2+2\theta+6}$ respectively.

When we change the proportion of each component, it gives different shapes to the models and changes the characteristics of the mixture distribution. Therefore, the key goal of this study is to propose a new mixture of probability distributions by changing the components and studying their essential characteristics. So, we fixed the first component as an exponential distribution with the scale parameter λ and the second component as the different choices of different distributions with proportions of α and $1-\alpha$.

This paper is structured as follows: A mixture of Exponential and Lindley (Exp-Lindley) distributions with some characteristics are introduced in Section 2. Section 3 deals with the properties of a mixture of Exponential and Gompertz (Exp-Gompertz). Section 4 presents a combination of exponential and log-normal (Exp-lognormal) distributions with their properties. Section 5 carries out simulation studies. Finally, real-time data are used for the proposed distributions in Section 6.

II. EXP-LINDLEY

The probability density function (pdf) of the Exp-Lindley distribution is

$$f(x;\theta,\lambda,\alpha)=\frac{\theta^2(1-\alpha)(x+1)e^{-\theta x}}{\theta+1}+\alpha\lambda e^{-\lambda x}, x\geq 0,\ \theta\geq 0, \lambda\geq 0, \alpha\geq 0 \tag{2.1}$$

Equation (2.1) is a mixture of exponential distribution with scale parameter λ and Lindley distribution with scale parameter θ, and their mixing proportions are α and $1-\alpha$.

The corresponding cumulative distribution function (cdf) of the Exp-Lindley is

$$F(x)=\frac{\left((\theta+1)\left(e^{\lambda x}-\alpha\right)e^{\theta x}+(\alpha-1)\theta(x+1)e^{\lambda x}+(\alpha-1)e^{\lambda x}\right)e^{-(\theta+\lambda)x}}{\theta+1} \tag{2.2}$$

Figure 1 shows the possible shapes of the pdf and cdf of the Exp-Lindley distribution for various parameter values. The Lindley and Exponential distributions are special cases of the Exp-Lindley distribution.

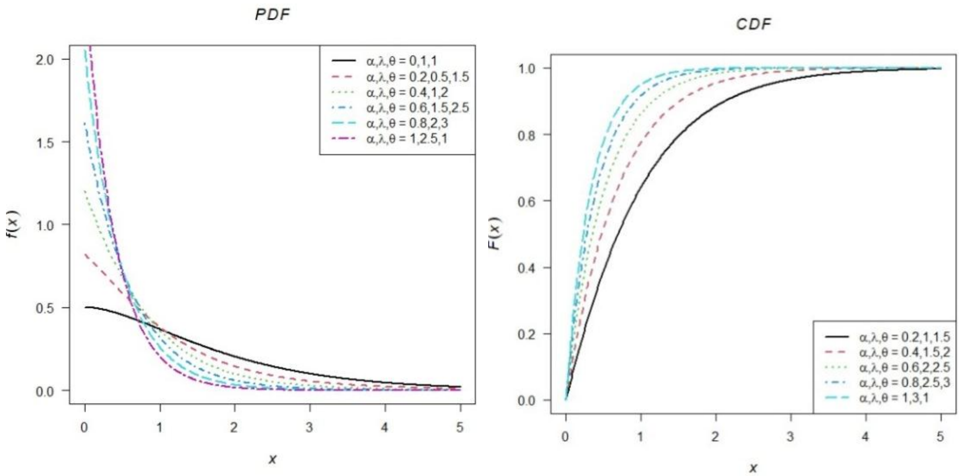

Figure 1: Various Shapes of pdf and cdf of the Exp-Lindley Distribution with Different Parameter Values

Let X be a continuous random variable with pdf $f(x)$ and $F(x)$. The survival function (sf) and Hazard Function (hf) of X are.

$$S(x) = \frac{\theta + 1 - \left((\theta+1)\left(e^{\lambda x} - \alpha\right)e^{\theta x} + (\alpha - 1)(\theta x + \theta + 1)e^{\lambda x}\right)e^{-(\theta+\lambda)x}}{\theta + 1} \tag{2.3}$$

$$h(x) = \frac{\theta^2(1-\alpha)(x+1)e^{-\theta x} + \alpha\lambda e^{-\lambda x}(\theta+1)}{(\theta+1) - \left((\theta+1)(e^{\lambda x} - \alpha)e^{\theta x} + (\alpha-1)(\theta x + \theta + 1)e^{\lambda x}\right)e^{-(\theta+\lambda)x}} \tag{2.4}$$

Figure 2 shows the various shapes of the survival and hazard functions of the Exp-Lindley distribution for various parameter values.

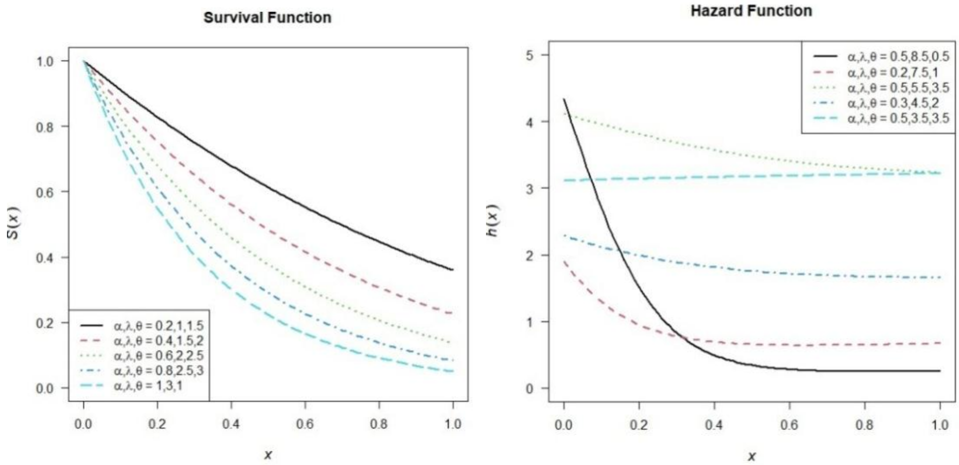

Figure 2: Various Shapes of the Survival and Hazard Functions of the Exp-Lindley Distribution for Different Parameter Values

The r^{th} moment about the origin (raw moments) has been obtained as

$$E(X^r) = \frac{\theta^2(1-\alpha)}{\theta+1}\left[\frac{\Gamma(r+1)}{\theta^{r+1}} + \frac{\Gamma(r+2)}{\theta^{r+2}}\right] + \alpha\frac{\Gamma(r+1)}{\lambda^r} \tag{2.5}$$

When $r = 1,2,3,4$ then we obtain the first four moments as follows:

The first four moments of Exp-Lindley distribution

$$Mean(\mu) = E(X) = \frac{(1-\alpha)(\theta+2)\lambda + \alpha(\theta+1)\theta}{(\theta+1)\theta\lambda}$$

$$\mu_2' = E(X^2) = \frac{2(\alpha\theta^3 + \alpha\theta^2 + (1-\alpha)\lambda^2\theta + (3-3\alpha)\lambda^2)}{(\theta+1)\theta^2\lambda^2}$$

$$\mu_3' = E(X^3) = \frac{6(\alpha\theta^4 + \alpha\theta^3 + (1-\alpha)\lambda^3\theta + (4-4\alpha)\lambda^3)}{(\theta+1)\theta^3\lambda^3}$$

$$\mu_4' = E(X^4) = \frac{24(\alpha\theta^5 + \alpha\theta^4 + (1-\alpha)\lambda^4\theta + (5-5\alpha)\lambda^4)}{(\theta+1)\theta^4\lambda^4}$$

The variance of the Exp-Lindley distribution is obtained as

$$Variance = \frac{\begin{array}{c}(\theta+1)(\lambda^2(1-\alpha)(2\theta+6) + 2\alpha(\theta+1)\theta^2) \\ -\big((1-\alpha)(\theta+2)\lambda + \alpha(\theta+1)\theta\big)^2\end{array}}{(\theta+1)^2\theta^2\lambda^2}$$

Using the above moments, the coefficient of variation and index of dispersion of the Exp-Lindley distribution were obtained using closed-form expressions. The index of dispersion (*DI*) is defined as the variance-to-mean ratio. If the *DI* value is less than one, the model is suitable for under-dispersed datasets. If the *DI* value is greater than 1, the model is suitable for over-dispersed datasets.

The coefficient of variation and index of dispersion for the Exp-Lindley distribution is obtained as

$$CV = \frac{\sigma}{\mu} = \frac{(\theta+1)\theta\lambda\left(\frac{(\theta+1)(\lambda^2(1-\alpha)(2\theta+6)+2\alpha(\theta+1)\theta^2) - \left((1-\alpha)(\theta+2)\lambda+\alpha(\theta+1)\theta\right)^2}{(\theta+1)^2\theta^2\lambda^2}\right)^{\frac{1}{2}}}{(1-\alpha)(\theta+2)\lambda+\alpha(\theta+1)\theta}$$

$$\mathrm{DI}(\gamma) = \frac{\sigma^2}{\mu} = \frac{(\theta+1)\theta\lambda\left(\frac{(\theta+1)(\lambda^2(1-\alpha)(2\theta+6)+2\alpha(\theta+1)\theta^2) - \left((1-\alpha)(\theta+2)\lambda+\alpha(\theta+1)\theta\right)^2}{(\theta+1)^2\theta^2\lambda^2}\right)}{(1-\alpha)(\theta+2)\lambda+\alpha(\theta+1)\theta}$$

The r^{th} incomplete moment for Exp-Lindley distribution is given as

$$\phi_r(x) = \frac{(\alpha-1)\Gamma(r+2,\theta y)-(\alpha-1)\Gamma(r+2,0)}{\theta^r(\theta+1)} + \frac{(\alpha-1)\left(\Gamma(r+1,\theta y)-\Gamma(r+1,0)\right)\theta^{1-r}}{\theta+1} - \frac{\alpha\left(\Gamma(r+1,\lambda y)-\Gamma(r+1,0)\right)}{\lambda^r} \tag{2.6}$$

The first incomplete moment of the Exp-Lindley distribution is

$$\phi_1(x) = \frac{\left[\begin{matrix}\left((\alpha\lambda\theta^2+\alpha\lambda\theta)y+\alpha\theta^2+\alpha\theta\right)e^{\theta y}+ \\ \left(\begin{matrix}(1-\alpha)\lambda\theta^2y^2+\left((1-\alpha)\lambda\theta^2+(2-2\alpha)\lambda\theta\right)y \\ +(1-\alpha)\lambda\theta+(2-2\alpha)\lambda\end{matrix}\right)e^{\lambda y}\end{matrix}\right]e^{-\theta y-\lambda y} + \alpha\theta^2+\left((1-\alpha)\lambda+\alpha\right)\theta+(2-2\alpha)\lambda}{\lambda\theta(1+\theta)}$$

The moment-generating function of the Exp-Lindley distribution is

$$M_X(t) = \sum_{i=0}^{\infty}\frac{t^i}{i!}\left(\frac{\theta^2(1-\alpha)}{\theta+1}\left[\frac{\Gamma(i+1)}{\theta^{i+1}}+\frac{\Gamma(i+2)}{\theta^{i+2}}\right]+\alpha\frac{\Gamma(i+1)}{\lambda^i}\right) \tag{2.7}$$

The characteristic function of the Exp-Lindley distribution is

$$\phi_X(t) = \sum_{i=0}^{\infty}\frac{it^k}{k!}\left(\frac{\theta^2(1-\alpha)}{\theta+1}\left[\frac{\Gamma(k+1)}{\theta^{k+1}}+\frac{\Gamma(k+2)}{\theta^{k+2}}\right]+\alpha\frac{\Gamma(k+1)}{\lambda^k}\right) \tag{2.8}$$

The cumulant generating function of the Exp-Lindley distribution is

$$K_X(t) = \prod_{i=0}^{\infty} log_e\left(\frac{t^i}{i!}\left(\frac{\theta^2(1-\alpha)}{\theta+1}\left[\frac{\Gamma(i+1)}{\theta^{i+1}}+\frac{\Gamma(i+2)}{\theta^{i+2}}\right]+\alpha\frac{\Gamma(i+1)}{\lambda^i}\right)\right) \quad (2.9)$$

The first and n^{th}-order statistics of the Exp-Lindley distribution are given by

$$f_{X_{(1)}}(x) = nf_X(x)[1-F_X(x)]^{(n-1)} = \frac{n\left[\theta^2(1-\alpha)(x+1)e^{-\theta x}+\alpha\lambda e^{-\lambda x}(\theta+1)\right]}{(\theta+1)}$$
$$\left[\frac{(\theta+1)-\left(\left(\begin{array}{c}(\theta+1)\left(e^{\lambda x}-\alpha\right)e^{\theta x}+\\ (\alpha-1)\theta(x+1)e^{\lambda x}+(\alpha-1)e^{\lambda x}\end{array}\right)e^{-(\theta+\lambda)x}\right)}{\theta+1}\right]^{(n-1)}$$

$$f_{X_{(n)}}(x) = nf_X(x)[F_X(x)]^{(n-1)} = \frac{n\left[\theta^2(1-\alpha)(x+1)e^{-\theta x}+\alpha\lambda e^{-\lambda x}(\theta+1)\right]}{(\theta+1)}$$
$$\left[\frac{\left(\begin{array}{c}(\theta+1)\left(e^{\lambda x}-\alpha\right)e^{\theta x}+\\ (\alpha-1)\theta(x+1)e^{\lambda x}+(\alpha-1)e^{\lambda x}\end{array}\right)e^{-(\theta+\lambda)x}}{\theta+1}\right]^{(n-1)}$$

Parameters θ, λ, and α are estimated using the Maximum Likelihood Estimation (*MLE*) method. Let $x_1, x_2, \ldots, x_n$ be a random sample from the Exp-Lindley distribution. Then the log-likelihood function is given by

$$\boldsymbol{g}(x) = \frac{\theta^2(1-\alpha)(x+1)e^{-\theta x}}{\theta+1}+\alpha\lambda e^{-\lambda x}$$

$$\begin{aligned}\boldsymbol{L}(x_i,\theta,\lambda,\alpha) &= \prod_{i=1}^{n} g(x_i,\theta,\lambda,\alpha)\\ &= \prod_{i=1}^{n}\left(\frac{\theta^2(1-\alpha)(x_i+1)e^{-\theta x_i}+(\theta+1)\alpha\lambda e^{-\lambda x_i}}{\theta+1}\right)\\ &= \left(\frac{n}{\theta+1}\prod_{i=1}^{n}\left[\theta^2(1-\alpha)(x_i+1)e^{-\theta x_i}+(\theta+1)\alpha\lambda e^{-\lambda x_i}\right]\right)\end{aligned}$$

The respective sample log-likelihood function is

$$\begin{aligned}log\,L\,(x_i,\theta,\lambda,\alpha) &= log\,n - log(\,\theta+1)\\ &\quad+\sum_{i=1}^{n} log\left[\theta^2(1-\alpha)(x_i+1)e^{-\theta x_i}+(\theta+1)\alpha\lambda e^{-\lambda x_i}\right]\end{aligned}$$

Now, by differentiating w.r.t. θ, λ, and α, we can write

$$\frac{\partial\, log\, L}{\partial\alpha} = \sum_{i=1}^{n}\frac{\lambda(\theta+1)e^{-\lambda x_i}-\theta^2(x_i+1)e^{-\theta x_i}}{[\theta^2(1-\alpha)(x_i+1)e^{-\theta x_i}+(\theta+1)\alpha\lambda e^{-\lambda x_i}]} = 0$$

$$\frac{\partial \log L}{\partial \theta} = \frac{-1}{(\theta+1)} \sum_{i=1}^{n} \frac{-x_i\theta^2(1-\alpha)(x_i+1)e^{-\theta x_i} + 2(1-\alpha)(x_i+1)\theta e^{-\theta x_i} + \alpha\lambda e^{-\lambda x_i}}{\theta^2(1-\alpha)(x_i+1)e^{-\theta x_i} + (\theta+1)\alpha\lambda e^{-\lambda x_i}} = 0$$

$$\frac{\partial \log L}{\partial \lambda} = \sum_{i=1}^{n} \frac{\alpha(\theta+1)e^{\theta x_i}(x_i\lambda - 1)}{[\theta^2(1-\alpha)(x_i+1)e^{-\theta x_i} + (\theta+1)\alpha\lambda e^{-\lambda x_i}]} = \mathbf{0}$$

The above nonlinear system of equations needs to be solved to obtain the ML estimates of the unknown parameters. Nonlinear optimization procedures are frequently more convenient for the numerical optimization of the sample likelihood function. R programming can be used to numerically solve these equations.

III.EXP-GOMPERTZ

The pdf for the Exp-Gompertz distribution is

$$f(x) = \theta\lambda e^{-\lambda x} + (1-\theta)b\eta\, exp(\eta + bx - \eta e^{bx}) \tag{3.1}$$

The mixture of exponential distribution with scale parameter λ and Gompertz distribution with scale parameter b and shape parameter η with their mixing proportions of θ and $1-\theta$ is given in Equation (3.1). The corresponding cdf of the Exp-Gompertz distribution is

$$F(x) = (e^{\eta}\theta - e^{\eta})e^{-\eta e^{bx}} + e^{-\lambda x}\left(e^{\lambda x} - \theta\right) \tag{3.2}$$

For $x > 0, \theta > 0, \lambda > 0, b > 0, \eta > 0$

Figure 3 displays the possible shapes of the pdf and cdf of the Exp-Gompertz distribution for various parameter values. The Gompertz distribution is a special case of the Exp-Gompertz distribution when $\theta = 0$; and when $\theta = 1$ it becomes an exponential distribution.

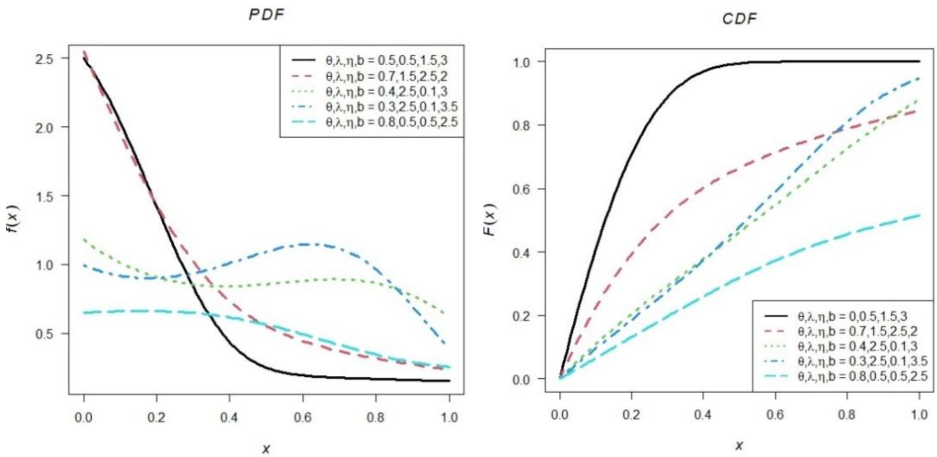

Figure 3: Various Shapes of pdf and cdf of Exp-Gompertz Distribution for Different Parameter Values

Let X be a continuous random variable with pdf $f(x)$ and $F(x)$. The survival function and hazard Function of X is

$$S(x) = 1 - \left((e^{\eta}\theta - e^{\eta})e^{-\eta e^{bx}} + e^{-\lambda x}(e^{\lambda x} - \theta) \right) \tag{3.3}$$

$$h(x) = \frac{\theta\lambda e^{-\lambda x} + (1-\theta)b\eta\, exp(\eta + bx - \eta e^{bx})}{1 - \left((e^{\eta}\theta - e^{\eta})e^{-\eta e^{bx}} + e^{-\lambda x}(e^{\lambda x} - \theta) \right)} \tag{3.4}$$

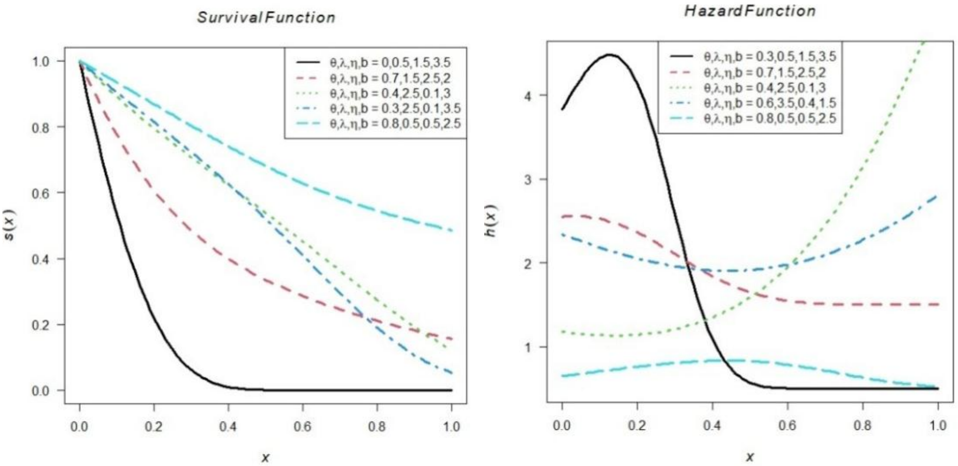

Figure 4: Various Shapes of sf and hf of Exp-Gompertz Distribution for Different Parameter Values

Figure 4 displays the various shapes of the sf and hf of the Exp-Gompertz distribution for various parameter values. The hazard function of the Exp-Gompertz distribution can have different shapes: decreasing hf, increasing hf, decreasing hf, and increasing hf.

The cumulative hazard function is given by

$$H(x) = -\,log\left(1 - \left((e^{\eta}\theta - e^{\eta})e^{-\eta e^{bx}} + e^{-\lambda x}(e^{\lambda x} - \theta) \right)\right) \tag{3.5}$$

The reversed hazard rate is given by

$$\tau(x) = \frac{\theta\lambda e^{-\lambda x} + (1-\theta)b\eta\, exp(\eta + bx - \eta e^{bx})}{(e^{\eta}\theta - e^{\eta})e^{-\eta e^{bx}} + e^{-\lambda x}(e^{\lambda x} - \theta)} \tag{3.6}$$

Wang et al. (2003) offered a log-odds rate-based model for time to failure, as well as some characterization of failure time distributions. The model can be used to study the distribution of time to failure by modeling the failure process in terms of the log odds rate. The odds function is expressed as

$$\pi_O(x) = \frac{(e^{\eta}\theta - e^{\eta})e^{-\eta e^{bx}} + e^{-\lambda x}\left(e^{\lambda x} - \theta\right)}{1 - \left((e^{\eta}\theta - e^{\eta})e^{-\eta e^{bx}} + e^{-\lambda x}(e^{\lambda x} - \theta)\right)} \tag{3.7}$$

The log-odds function is given by

$$\begin{aligned} LO(x) &= log\,\frac{(e^{\eta}\theta - e^{\eta})e^{-\eta e^{bx}} + e^{-\lambda x}\left(e^{\lambda x} - \theta\right)}{1 - \left((e^{\eta}\theta - e^{\eta})e^{-\eta e^{bx}} + e^{-\lambda x}(e^{\lambda x} - \theta)\right)} \\ &= log\left((e^{\eta}\theta - e^{\eta})e^{-\eta e^{bx}} + e^{-\lambda x}\left(e^{\lambda x} - \theta\right)\right) \\ &\quad - log\left(1 - \left((e^{\eta}\theta - e^{\eta})e^{-\eta e^{bx}} + e^{-\lambda x}\left(e^{\lambda x} - \theta\right)\right)\right) \end{aligned} \tag{3.8}$$

The log-odds rate is defined as

$$LOR(x) = \frac{\theta\lambda e^{-\lambda x} + (1-\theta)b\eta\, exp(\eta + bx - \eta e^{bx})}{\left(1 - \left((e^{\eta}\theta - e^{\eta})e^{-\eta e^{bx}} + e^{-\lambda x}(e^{\lambda x} - \theta)\right)\right)^2} \tag{3.9}$$

The MLE method is utilized to estimate unknown parameters $\theta, \lambda, b,$ and η as we used in Section 3.

IV. EXP-LOGNORMAL

The Exp-lognormal distribution is the mixture of Exponential distribution with scale parameter λ and lognormal distribution having location parameter μ and scale parameter σ with their mixing proportions of θ and $1-\theta$ is given in Equation (4.1). The pdf and cdf of the Exp-lognormal distributions are

$$f(x) = \theta\lambda e^{-\lambda x} + \frac{(1-\theta)e^{\frac{-(ln(x)-\mu)^2}{2\sigma^2}}}{x\sigma\sqrt{2\pi}} \tag{4.1}$$

$$F(x) = \frac{(\theta - 1)erf\left(\frac{\sqrt{2}\mu - \sqrt{2}\,ln(x)}{2\sigma}\right) + \theta + 1}{2} - \theta e^{-\lambda x} \tag{4.2}$$

For $x > 0, \theta > 0, \lambda > 0, \mu > 0, \sigma > 0$

Figure 5 displays the possible shapes of the pdf and cdf of the Exp-lognormal distribution for the various parameter values.

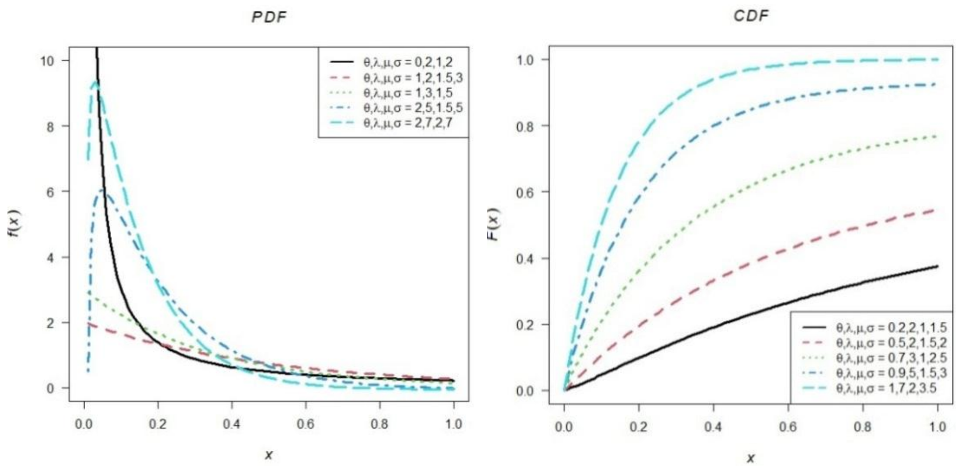

Figure 5: Various Shapes of pdf and cdf of Exp-Lognormal Distribution for Different Parameter Values

Let X be a continuous random variable with pdf $f(x)$ and $F(x)$. The survival function and hazard Function of X is given as

$$S(x) = \frac{2 - \left((\theta - 1)erf\left(\frac{\sqrt{2}\mu - \sqrt{2}\,ln(x)}{2\sigma} \right) + \theta + 1 \right) + 2\theta e^{-\lambda x}}{2} \tag{4.3}$$

$$h(x) = \frac{2\left(\theta\lambda e^{-\lambda x} + \frac{(1-\theta)e^{\frac{-(ln(x)-\mu)^2}{2\sigma^2}}}{x\sigma\sqrt{2\pi}} \right)}{2 - \left((\theta - 1)erf\left(\frac{\sqrt{2}\mu - \sqrt{2}\,ln(x)}{2\sigma} \right) + \theta + 1 \right) + 2\theta e^{-\lambda x}} \tag{4.4}$$

Figure 6 shows the different forms of the survival and hazard functions of the Exp-lognormal distribution for the different parameter values. The hazard function of the exp-lognormal distribution can have a decreasing and constant shape.

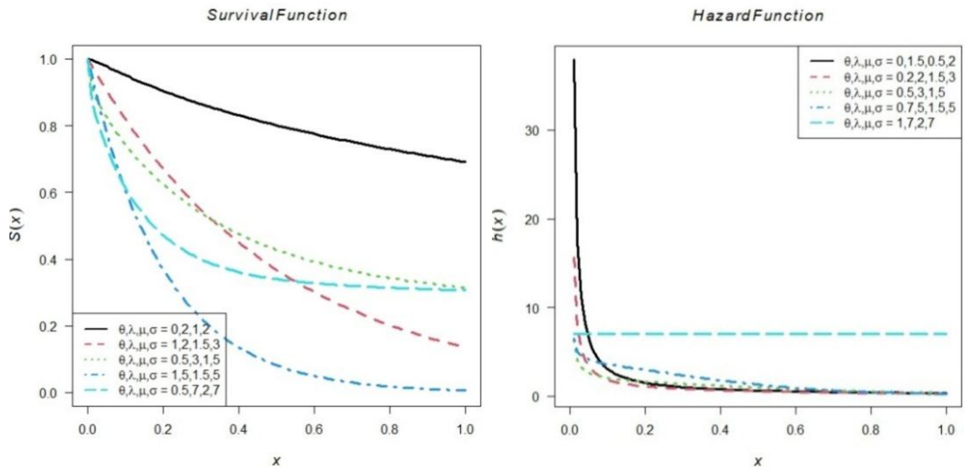

Figure 6: Various Shapes of the Survival Function and Hazard Function of Exp-Lognormal Distribution for Different Parameter Values

Let a random variable X~*Exp-lognormal* $(\theta, \lambda, \mu, \sigma)$ then Renyi entropy of X is defined as

$$I_R(\eta) = \frac{1}{1-\eta} log \int_0^{\infty} f^{\eta}(x)dx; \quad \eta > 0, \eta \neq 1$$

$$= \frac{1}{1-\eta} log \int_0^{\infty} \left(\frac{\theta\lambda x\sigma\sqrt{2\pi}e^{-\lambda x} + (1-\theta)e^{\frac{-(ln(x)-\mu)^2}{2\sigma^2}}}{x\sigma\sqrt{2\pi}} \right)^{\eta} dx$$

$$= \frac{1}{1-\eta} log \left(\frac{1}{\left(\sigma\sqrt{2\pi}\right)^{\eta}} \int_0^{\infty} \left(x^{-1} \left(\theta\lambda x\sigma\sqrt{2\pi}e^{-\lambda x} + (1-\theta)e^{\frac{-(ln(x)-\mu)^2}{2\sigma^2}} \right) \right)^{\eta} dx \right) \tag{4.5}$$

The application of stochastic ordering to compare the behaviors of positive continuous random variables is quite beneficial. If a random variable X is less than a random variable Y then

(i) Stochastic order $(X \leq_{st} Y)$ if $F_X(x) \geq F_Y(y)$ for all x
(ii) Hazard rate order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(y)$ for all x
(iii) Mean residual life order $(X \leq_{mrl} Y)$ if $m_X(x) \geq m_Y(y)$ for all x
(iv)Likelihood ratio order$(X \leq_{lr} Y)$ if $\frac{f_X(x)}{f_Y(y)}$decreases in x.

The stochastic ordering of distributions was discovered by Shaked and Shanthi Kumar (1994), who reached the following findings.

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y$$
$$\Downarrow$$
$$X \leq_{st} Y$$

The Exp-lognormal distribution was sorted according to the strongest 'likelihood ratio'. Let X~*Exp-lognormal* $(\theta_1, \lambda_1, \mu_1, \sigma_1)$ and Y~*Exp-lognormal* $(\theta_2, \lambda_2, \mu_2, \sigma_2)$. If, $\theta_1 \geq \theta_2,$ then $X \leq_{lr} Y$ hence $X \leq_{hr} Y$, $X \leq_{mlr} Y$ and $X \leq_{st} Y$. We have

$$\frac{f_X(x)}{f_Y(y)} = \frac{\sigma_2\left(\theta_1\lambda_1 x\sigma_1\sqrt{2\pi}e^{-\lambda_1 x} + (1-\theta_1)e^{\frac{-(ln(x)-\mu_1)^2}{2\sigma_1{}^2}}\right)}{\sigma_1\left(\theta_2\lambda_2 y\sigma_2\sqrt{2\pi}e^{-\lambda_2 y} + (1-\theta_2)e^{\frac{-(ln(y)-\mu_2)^2}{2\sigma_2{}^2}}\right)} \quad (4.6)$$

$$log\frac{f_X(x)}{f_Y(y)} = log\left[\frac{\sigma_2\left(\theta_1\lambda_1 x\sigma_1\sqrt{2\pi}e^{-\lambda_1 x} + (1-\theta_1)e^{\frac{-(ln(x)-\mu_1)^2}{2\sigma_1{}^2}}\right)}{\sigma_1\left(\theta_2\lambda_2 y\sigma_2\sqrt{2\pi}e^{-\lambda_2 y} + (1-\theta_2)e^{\frac{-(ln(y)-\mu_2)^2}{2\sigma_2{}^2}}\right)}\right] \quad (4.7)$$

$$log\frac{f_X(x)}{f_Y(y)} = log\left(\sigma_2\left(\theta_1\lambda_1 x\sigma_1\sqrt{2\pi}e^{-\lambda_1 x} + (1-\theta_1)e^{\frac{-(ln(x)-\mu_1)^2}{2\sigma_1{}^2}}\right)\right) - log\left(\sigma_1\left(\theta_2\lambda_2 y\sigma_2\sqrt{2\pi}e^{-\lambda_2 y} + (1-\theta_2)e^{\frac{-(ln(y)-\mu_2)^2}{2\sigma_2{}^2}}\right)\right)$$

$$\frac{d}{dx}log\frac{f_X(x)}{f_Y(y)} = \frac{\sigma_2\left(\begin{array}{c} -\frac{(1-\theta_1)e^{\frac{-(ln(x)-\mu_1)^2}{2\sigma_1{}^2}}(-ln(x)-\mu_1)}{\sigma^2 x} \\ -\theta_1\lambda^2{}_1 x\sigma_1\sqrt{2\pi}e^{-\lambda_1 x} + \theta_1\lambda_1\sigma_1\sqrt{2\pi}e^{-\lambda_1 x}\end{array}\right)}{\left[\sigma_2\left(\theta_1\lambda_1 x\sigma_1\sqrt{2\pi}e^{-\lambda_1 x} + (1-\theta_1)e^{\frac{-(ln(x)-\mu_1)^2}{2\sigma_1{}^2}}\right)\right]} - \frac{\sigma_1\left(\begin{array}{c} -\frac{(1-\theta_2)e^{\frac{-(ln(y)-\mu_2)^2}{2\sigma_2{}^2}}(-ln(y)-\mu_2)}{\sigma_2{}^2 y} \\ -\theta_2\lambda^2{}_2 y\sigma_2\sqrt{2\pi}e^{-\lambda_2 y} + \theta_2\lambda_2\sigma_2\sqrt{2\pi}e^{-\lambda_2 y}\end{array}\right)}{\left[\sigma_1\left(\theta_2\lambda_2 y\sigma_2\sqrt{2\pi}e^{-\lambda_2 y} + (1-\theta_2)e^{\frac{-(ln(y)-\mu_2)^2}{2\sigma_2{}^2}}\right)\right]} \quad (4.8)$$

Now if $\lambda_1 = \lambda_2 = \lambda$, $\mu_1 = \mu_2 = \mu$, $\sigma_1 = \sigma_2 = \sigma$, $\theta_1 \geq \theta_2$, then it implies $\frac{d}{dx}log\frac{f_X(x)}{f_Y(y)} \leq 0$. This means that $X \leq_{lr} Y$ and hence $X \leq_{hr} Y$, $X \leq_{mlr} Y$ and $X \leq_{st} Y$.

The maximum likelihood estimation approach is used to estimate the parameters θ, λ, μ, and σ as we mentioned in section 3.

V. SIMULATION STUDY

The performance of certain estimates over predetermined replications at different sample sizes is assessed in this section using simulation analysis. To do this, a simulation procedure is performed 1000 times with various sample sizes (n = 25, 50, 75, 100, 200, and 500). The purpose of this study is to assess how well the MLEs perform for the Exp-Lindley, Exp-Gompertz, and Exp-Lognormal distribution parameters. The inversion approach for creating random data from the suggested distributions fails because the equation $F(x) = u$, where u is an observation from the uniform distribution on (0,1), cannot be solved explicitly in x. We therefore used Monte Carlo simulation to generate the samples. R programming language is used to create samples from the proposed distributions. For the generated samples, we calculated the mean value, average biases (BIAS), and root-mean-square errors (RMSEs).

The results of the simulations are shown in Tables 3–5, along with the mean, bias, and root mean square error (RMSE) for the parameters of the Exp-Lindley, Exp-Gompertz, and Exp-Lognormal distributions, respectively. According to Tables 3–5, when sample size n grows, both bias and RMSE often decrease.

Table 3
Simulation Analysis: Mean, Bias, and RMSE Values for Exp-Lindley Distributions with various Sample Sizes

		Case (i): α=0.5, λ=0.9, θ=1.2			Case (ii): α=0.2, λ=0.5, θ=1		
n	Parameters	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
25	α	0.500194	0.181623	0.223342	0.499814	0.687172	0.257297
	λ	0.643842	0.316382	1.031072	0.871461	0.274705	2.678434
	θ	0.048155	0.023231	0.164498	0.053211	0.040089	0.078601
50	α	0.498073	0.133598	0.160337	0.482685	0.237683	0.159649
	λ	0.583805	0.170427	1.000245	0.556491	0.063526	1.132306
	θ	0.037649	0.004239	0.122779	0.052229	0.029515	0.053315
75	α	0.490104	0.075959	0.100396	0.405336	0.039511	0.103186
	λ	0.530404	0.098556	0.675153	0.539146	0.019419	0.345243
	θ	0.018821	0.004147	0.054381	0.050584	0.024524	0.046207
100	α	0.482748	0.055871	0.015065	0.399839	0.015992	0.089874
	λ	0.520125	0.037979	0.358396	0.524768	0.013950	0.272702
	θ	0.014824	0.001554	0.030165	0.046579	0.006415	0.035376
200	α	0.458752	0.049522	0.004258	0.301578	0.015179	0.041455
	λ	0.503649	0.015480	0.246597	0.506076	0.005527	0.084405
	θ	0.009854	0.000564	0.001269	0.003805	0.003714	0.005197

Table 4
Simulation Analysis: Mean, Bias, and RMSE Values for Exp-Gompertz Distributions with various Sample Sizes

		Case (i): θ=0.5, λ=0.8, η =1, β =1.5			Case (ii): θ=0.1, λ=0.5, η =1, β =0.5		
n	Parameters	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
25	θ	1.040784	0.697213	1.135451	0.257801	0.255498	0.299511
	λ	14.40554	6.701684	7.738910	7.432411	7.264812	6.079522
	η	0.674863	0.257596	1.365579	0.158241	0.054871	0.475635
	β	6.541210	2.149390	7.367740	5.114866	0.851141	2.902813
50	θ	0.773491	0.455551	0.608552	0.222767	0.150769	0.239547
	λ	8.833386	5.204027	5.296060	5.049495	6.444535	5.435474
	η	0.436487	0.222420	1.018249	0.092305	0.026607	0.207198
	β	4.800180	1.811460	6.946020	4.100889	0.690195	2.486313
75	θ	0.503769	0.165611	0.375133	0.273765	0.140879	0.200128
	λ	8.548397	2.965199	2.357460	4.790583	5.613676	4.921408
	η	0.279290	0.175145	0.980084	0.076895	0.014509	0.120236
	β	2.141877	0.884752	1.130480	3.93403	0.599944	1.218631
100	θ	0.334439	0.096434	0.348335	0.211156	0.027315	0.164801
	λ	4.404348	1.133457	1.653921	2.686431	4.192153	1.910273
	η	0.196789	0.153904	0.551145	0.069269	0.004441	0.119125
	β	1.296303	0.516518	0.736998	3.510842	0.484854	1.092859
200	θ	0.177295	0.057267	0.170949	0.148723	0.015485	0.157342
	λ	2.849778	0.159136	0.852861	1.347856	3.094763	0.130019
	η	0.037758	0.035116	0.062074	0.057672	0.003054	0.109089
	β	0.809634	0.027483	0.322992	3.423885	0.228846	0.986079

Table 5
Simulation Analysis: Mean, Bias, and RMSE Values for Exp-Lognormal Distributions with Various Sample Sizes

		Case (i): θ=0.1, λ=0.1, μ =1, σ =1.5			Case (ii): θ=0.1, λ=0.5, μ =2.5, σ =0.5		
n	Parameters	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
50	θ	0.515175	0.073546	0.078325	0.513665	0.006336	0.070306
	λ	3.758278	3.249234	0.899263	0.351814	0.090877	0.291851
	μ	2.131613	0.055316	0.321229	3.121311	0.001874	0.025949
	σ	0.062528	0.062421	0.349985	0.008551	0.008477	0.046100
75	θ	0.513458	0.035129	0.062996	0.505919	0.005252	0.057799
	λ	0.633478	0.528866	0.708708	0.350501	0.080426	0.245596
	μ	2.131593	0.000313	0.000481	3.022693	0.000131	0.000373
	σ	0.000154	0.000104	0.000141	9.80E-05	0.000194	0.000199
100	θ	0.510654	0.023315	0.054371	0.500819	0.001919	0.052781
	λ	0.234908	0.163217	0.124418	0.300584	0.057832	0.237483
	μ	2.131542	0.000148	0.000199	3.022311	0.000121	0.000277
	σ	9.58E-05	3.17E-05	7.47E-05	9.30E-05	3.09E-05	5.16E-05
200	θ	0.510301	0.022738	0.049393	0.500494	0.001451	0.037098
	λ	0.113242	0.076996	0.010288	0.270768	0.015535	0.200898
	μ	2.131521	6.39E-05	0.000157	3.022119	1.13E-04	0.000238
	σ	6.73E-05	1.14E-05	1.46E-05	8.46E-05	2.63E-05	4.02E-05
500	θ	0.506501	0.010282	0.036466	0.405002	0.001051	0.015745
	λ	0.030371	0.004695	0.000199	0.252151	0.007832	0.187451
	μ	2.076166	4.63E-05	6.62E-05	3.001258	8.05E-05	1.58E-04
	σ	5.80E-05	5.31E-06	1.34E-05	7.46E-05	1.63E-05	3.17E-05

VI. APPLICATION

In this section, two real-time datasets are used to illustrate the flexibility of the proposed distribution. These conclusions can be strengthened by a graphical study. You may analyze how well our datasets fit our distribution using the empirical cdf plots and pdf plots.

Dataset 1: It represents the uncensored data set corresponding to remission times (in months) of a random sample of 128 patients with bladder cancer patients. This data was previously used by Lee and Wang [21].

For this dataset, we compared the modeling fit of the proposed distribution with the existing mixture probability distributions. As we said in Section I, the Akash, Lindley, and Exp-Gamma distributions are a combination of the exponential and gamma distributions. Our proposed models are a combination of the exponential and Lindley distributions and a combination of the exponential and Gompertz distributions, and Table 7 shows that the proposed models have lower criteria values than the existing models and Figure 7 shows that the proposed model fits the data better than existing models.

The list of the distribution we have taken for comparison is (i) Exp-Lindley, (ii) Exp-Gompertz, (iii) Lognormal, (iv) Akash, (v) Exp-Gamma (Generalized Akash), and (vi) Lindley distribution. To compare the goodness of fit, we used Akaike Information Criteria, corrected Akaike information criteria, Bayesian information criteria, Kolmogorov-Smirnov, CVM, and Anderson Darling. The measures are computed and presented below in Table 6.

Table 6
Parameter Estimates of the Distributions

Model	Parameter Estimate	-2LL
Exp-Gompertz	$\hat{\alpha} = 0.9600, \hat{\lambda} = 0.1123$ $\hat{\eta} = 0.03496, \hat{b} = 7.8525$	803.0369
Exp-Lindley	$\hat{\alpha} = 0.9674, \hat{\lambda} = 0.1131$ $\hat{\theta} = 2.5257$	805.1309
Lognormal	$\hat{\mu} = 1.5109, \hat{\sigma} = 1.2819$	813.605
Akash	$\hat{\theta} = 0.3375$	903.990
Exp-Gamma (Generalized Akash)	$\hat{\theta} = 3.9329, \hat{\lambda} = 0.1114$ $\hat{\beta} = 0.5557$	805.1486
Lindley	$\hat{\theta} = 0.2129$	835.8477

Table 7
Information Criteria for Model Selection

Model	AIC	AICc	BIC	CVM	AD	KS
Exp-Gompertz	811.04	811.12	820.22	0.0886 (0.6446)	0.5364 (0.7097)	0.05832 (0.7767)
Exp-Lindley	811.13	811.99	819.69	0.1282 (0.464)	0.7363 (0.529)	0.0695 (0.5664)
Lognormal	817.60	818.56	823.309	0.3132 (0.1241)	1.8173 (0.1161)	0.0999 (0.155)
Akash	905.99	906.52	908.84	2.1502 (0.000)	17.867 (0.000)	0.2097 (0.000)
Exp-Gamma (Generalized Akash)	811.15	811.342	819.70	0.0901 (0.6363)	0.7236 (0.5391)	0.0567 (0.8055)
Lindley	837.85	838.24	840.70	0.8098 (0.0068)	5.8689 (0.0011)	0.1336 (0.0207)

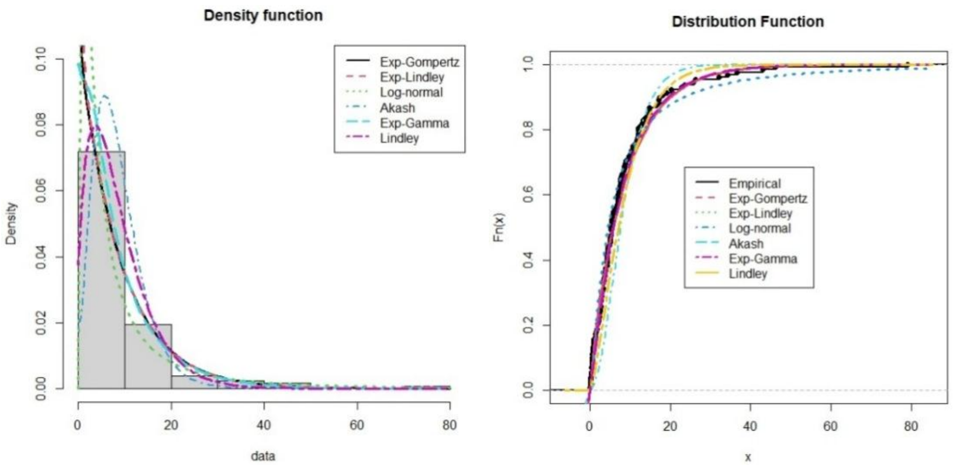

Figure 7: Model Fitting of Probability Distributions under Study

Dataset 2: It represents the waiting times (in minutes) before service for 100 bank customers. (Refer Ademola et al. [2]). For this dataset, we have compared the proposed model with existing mixture distributions like Akash, Exp-Gamma, and Janarthan distributions and with some other common distributions like Weibull, lognormal, and gamma distributions. The list of distributions we have taken for comparison is (i) Exp-Lognormal, (ii) Exp-Gamma, (iii) Lognormal, (iv) Akash, (v) Gamma, (vi) Weibull, and (vii) Janardan distributions. The estimated parameter values are tabulated and presented in Table 8 and the measures are computed and presented below in Table 9.

Table 8
Parameter Estimates of the Distributions

Model	Parameter Estimate	LL
Exp-lognormal	$\hat{\theta} = 0.1829, \hat{\lambda} = 0.1180$ $\hat{\mu} = 2.1005, \hat{\sigma} = 0.6823$	623.655
Exp-Gamma (Generalized Akash)	$\hat{\theta} = 0.85595, \hat{\lambda} = 0.0864,$ $\hat{\beta} = 0.3241$	637.332
Lognormal	$\hat{\mu} = 2.0211, \hat{\sigma} = 0.7811$	638.348
Akash	$\hat{\theta} = 0.2953$	641.9292
Gamma	$\hat{\lambda} = 2.0089, \hat{\beta} = 0.2034$	634.6002
Weibull	$\hat{k} = 1.4585, \hat{\lambda} = 10.9553$	637.4614
Janardan	$\hat{\theta} = 3.6588, \hat{\alpha} = 18.1497$	634.7764

Table 9
Information Criteria for model selection

Model	AIC	BIC	AICc	CVM	AD	KS
Exp-lognormal	628.655	628.655	628.234	0.0226 (0.9941)	0.1605 (0.9977)	0.0453 (0.9865)
Exp-Gamma (Generalized Akash)	643.3321	651.1476	643.5821	0.0380 (0.9439)	0.2557 (0.9673)	0.0535 (0.9372)
Lognormal	642.348	647.558	642.4717	0.05422 (0.8514)	0.4088 (0.8396)	0.0564 (0.9078)
Akash	643.9292	646.5344	643.97	0.2168 (0.2373)	1.3789 (0.2082)	0.10023 (0.2676)
Gamma	638.6002	643.8103	638.724	0.0288 (0.9805)	0.1856 (0.9939)	0.0425 (0.9936)
Weibull	641.4614	646.6717	641.5851	0.0610 (0.8086)	0.4056 (0.8428)	0.0578 (0.8921)
Janardan	638.7764	643.9868	638.9001	0.0272 (0.9847)	0.1835 (0.9943)	0.0413 (0.9956)

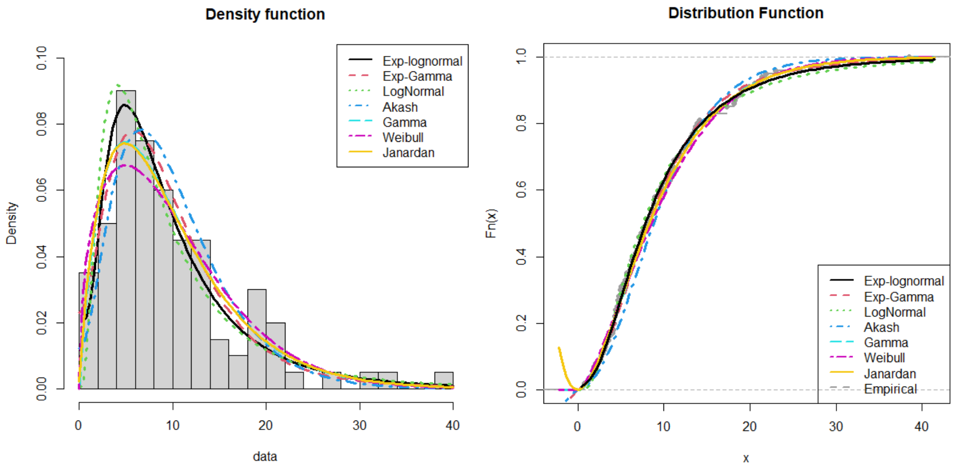

Figure 8: Model Fitting of Probability Distributions under Study

CONCLUSION

In this study, a few mixed probability distributions are developed to model lifetime data. The developed distributions can cover right-skewed and left-skewed unimodal data at specific parameter values. Proposed mixture models have better flexibility than existing mixture distributions like Lindley, Akash, and Janardhan distributions. In light of this, we can conclude that changing the component has positive effects rather than altering the proportions of the mixed distribution. And we develop the formulations for essential statistical quantities, including mean, variance, moments, moment-generating functions, etc. Further, a simulation study was also conducted for all the proposed models, and the parameter estimation of the proposed probability distributions was estimated using the method of maximum likelihood estimation. A real-time dataset was utilized to show the usefulness of the proposed mixture of distributions for modeling skewed data.

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