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ESTIMATION OF STRESS- STRENGTH RELIABILITY FROM EXPONENTIATED INVERSE RAYLEIGH DISTRIBUTION BASED ON NEOTERIC RANKED SET SAMPLING APPROACH

Yahya, M.[1§] and **Shaaban, M.**[2]

[1] Modern Academy for Computer Science and Management Technology, 304 Saqr Qureish St., New Maadi, Egypt.

[2] The High Institute for Tourism, Hotels & Computer, El-Seyouf, Alexandria, Egypt.

[§] Corresponding author Email: dr.marwamyr@gmail.com

ABSTRACT

Ranked set sampling (RSS) is considered an alternative sampling approach to simple random sampling (SRS). There are many modifications of RSS design such as percentile ranked set sampling (PRSS) and neoteric ranked set sampling (NRSS). In this article, estimation parameters of stress strength model when both the random- stress and strength follow exponentiated inverse Rayleigh distribution (EIRD) based on selective ranked set sampling approaches is studied. Maximum likelihood method of estimation is derived to compare the performance of the estimators using the different sampling designs, namely, RSS, PRSS, and NRSS approaches to the estimators-based SRS technique. Massively simulation study is conducted to compare the different estimators; in addition, two real data sets are used to verify the results.

KEYWORD

Neoteric ranked set sampling, percentile ranked set sampling, stress strength model, Exponentiated Inverse Rayleigh Distribution.

1. INTRODUCTION

Inverse Rayleigh distribution is initiated from Rayleigh distribution by using the transformation $X = \frac{1}{V}$, where the random variable V has Rayleigh distribution and X has inverse Rayleigh distribution. Rao at al. (2019) introduced the new distribution by using the proposed method in Nadarajah and Kotz (2006) using the reliability function.

$$F(x) = 1 - (R(x))^{\propto},$$

where $R(x)$ is the reliability function of the Rayleigh distribution. The probability density and cumulative distribution functions denoted by (PDF) and (CDF) of the EIRD are given as follows:

$$f(x) = \frac{2\alpha\sigma^2}{x^3} e^{-(\sigma/x)^2} (1 - e^{-(\sigma/x)^2})^{\alpha-1};\ x \geq 0, \alpha > 0, \sigma > 0, \tag{1}$$

and

$$F(x) = 1 - (1 - e^{-(\sigma/x)^2})^{\alpha};\ x \geq 0, \alpha > 0, \sigma > 0, \tag{2}$$

respectively, where σ is the shape parameter and α is the scale parameter, it is obvious that, when $\alpha = 1$, inverse Rayleigh distribution can be obtained as a special case of EIRD.

In 1952, McIntyre introduced RSS as an alteration of the traditional SRS approach to estimate the pasture yield. The RSS process involves randomly drawing n sets of size n from the target population. The units included in each set assumed to be ranked visually or by free-cost method.

The first set of n units ranked lowest is appraised. From the second set, the second lowest unit is appraised. The process is continued until from the n^{th} set of n units the n^{th} ranked unit is appraised. Repeat the steps r times until a sample size $\tau = nr$ is obtained, where n, r and τ are the set size, number of cycles and total of sample size respectively.

Let $\{X_{i(i)s}, i = 1,2,..,n; s = 1,2,..,r\}$ be a ranked set sample where n is the set size and r is the number of cycles, then the PDF can be written as follows

$$f_i(x_{i(i)s}) = \frac{n!}{(i-1)!\,(n-i)!}\left[F(x_{i(i)s})\right]^{i-1}\left[1 - F(x_{i(i)s})\right]^{n-i} f(x_{i(i)s}). \quad -\infty < x_{i(i)s} < \infty \tag{3}$$

PRSS is suggested by Muttlak (2003) as a modification of RSS approach. It differs from RSS method in that after a sample of n sets of size n is drawn from the population under study and ranking the units in relation to a variable of interest, the selection of the units is through determining two values O and t where $0 < O \leq 0.5$ and $t = 1 - O$ so that if the sample size is odd, we choose from the first $(n-1)/2$ samples the $O(n+1)th$ smallest ranked unit and from the other $(n-1)/2$ samples the $t(n+1)th$, and choose the median from the last sample. If the sample size is even, choose from the first $n/2$ samples the $O(n+1)th$ smallest ranked unit and from the other $n/2$ samples the $t(n+1)th$ smallest ranked unit.

For odd set sizes, let $\{X_{i(n_1)s}, i = 1,2,..,V-1; s = 1,2,..,r\} \cup \{X_{i(n_2)s}, i = V, \ldots, n-1; s = 1,2,..,r\} \cup \{X_{n(V)s}, s = 1,2,..,r\}$ be a PRSS where n is the set size, r is the number of cycles, $V = \frac{n+1}{2}$, n_1 and n_2 are the nearest integer values of $O(n+1)th$ and $t(n+1)th$, then the PDFs of n_1 and n_2 order statistics are given as follows

$$f_{n_1}(x_{i(n_1)s}) = \frac{n!}{(n-n_1)!\,(n_1-1)!}\left[F(x_{i(n_1)s})\right]^{n_1-1} \left[1 - F(x_{i(n_1)s})\right]^{n-n_1} f(x_{i(n_1)s}). \quad -\infty < x_{i(n_1)s} < \infty \tag{4}$$

and

$$f_{n_2}(x_{i(n_2)s}) = \frac{n!}{(n-n_2)!\,(n_2-1)!}\left[F(x_{i(n_2)s})\right]^{n_2-1} \left[1 - F(x_{i(n_2)s})\right]^{n-n_2} f(x_{i(n_2)s}). \quad -\infty < x_{i(n_2)s} < \infty \tag{5}$$

The PDF of the $X_{n(V)s}$ is given by

$$f_{n(V)s}(x_{n(V)s}) = \frac{n!}{[(V-1)!]^2}\left[F(x_{n(V)s})\right]^{V-1}\left[1-F(x_{n(V)s})\right]^{V-1} f(x_{n(V)s}). \qquad (6)$$
$$-\infty < x_{n(V)s} < \infty$$

For even set sizes, let $\{X_{i(n_1)s}, i = 1,2,..,F; s = 1,2,..,r\} \cup \{X_{i(n_2)s}, i = F+1,\dots,n;$ $s = 1,2,..,r\}$ be a PRSS where n is the set size, r is the number of cycles, $F = \frac{n}{2}$, the PDFs of n_1 and n_2 order statistics are given in Equations (4) and (5).

Zamanzade and AL-Omari (2016) have suggested the NRSS as a new modification of the usual RSS. Similar to the RSS procedure firstly, select n^2 units from the target population. Secondly, instead of dividing them into n samples each of size n, rank the n^2 selected units in an increasing magnitude depending on personal judgement or any low-priced method. If n is an odd number, select $\left[\frac{n+1}{2} + (i-1)n\right] th$ ranked unit for $i = 1,2,\dots,n$. In case of even set size, the way of selection will differ if i is an even or odd number as follows

a) If i is an odd number select $\left[\frac{n}{2} + (i-1)n\right] th$ ranked unit.
b) If i is an even number select $\left[\frac{n+2}{2} + (i-1)n\right] th$ ranked unit.

Lastly, repeat this procedure r times to obtain a NRSS of size $\tau = nr$.

Let $\{X_{(k(i))s}, i = 1,2,..,n \; and \; s = 1,2,..,r\}$ be a NRSS data, Sabry et al. (2020) derived the pdf of $X_{(k(i))s}$ as follows:

$$f_{k(i)}\left(X_{(k(i))s}\right) = \frac{w!}{(k(i)-k(i-1)-1)!} f\left(x_{(k(i))s};\theta\right)$$
$$[F\left(x_{(k(i))s};\theta\right) - F\left(x_{(k(i-1))s};\theta\right)]^{k(i)-k(i-1)-1}, \qquad (7)$$

Sabry et al. (2020) obtained the maximum likelihood function as follows:

$$L_{NRSS}(\theta) = \frac{w!}{\prod_{i=1}^{n+1}(k(i)-k(i-1)-1)!}\prod_{i=1}^{n} f\left(x_{(k(i))s};\theta\right)$$
$$\times \prod_{i=1}^{n+1}[F\left(x_{(k(i))s};\theta\right) - F\left(x_{(k(i-1))s};\theta\right)]^{k(i)-k(i-1)-1} \qquad (8)$$

where

$$k(i) = \begin{cases} \frac{n+1}{2} + (i-1)n & n \text{ odd} \\ \frac{n}{2} + (i-1)n & n \text{ even}, i \text{ even} \\ \frac{n+2}{2} + (i-1)n & n \text{ even}, i \text{ odd}, \end{cases} \qquad (9)$$

$k(0) = 0, k(i+1) = w+1, x_{(k(0))} = -\infty, x_{(k(i+1))} = \infty$ and $w = n^2$.

Sabry and Shabban (2020) used maximum likelihood method of estimation to estimate the parameters of the inverse Weibull distribution using NRSS and DNRSS techniques, they proved that DNRSS and NRSS are more efficient than the MLEs based on SRS and RSS approaches. Taconeli and Cabral (2018) proposed some different two-stage sampling approaches, they proved that all proposed sampling techniques are superior over RSS and NRSS. Shabban and Yahya (2020) made a comparison between the performance of NRSS in estimating the parameters of exponentiated Gumbel distribution, and the performance of RSS and SRS techniques. They concluded that NRSS is better than the usual RSS and SRS techniques.

The term stress strength reliability refers to the failure of the system if the random stress Y overcomes the random strength X. Stress strength model has been widely studied based on the traditional sampling technique SRS. Alotaibi et al. (2020), estimated the parameters of stress-strength reliability model based on different statistical procedures using SRS technique. Rao et al. (2019) constructed MLEs, asymptotic distribution and asymptotic confidence interval in order to estimate stress-strength reliability from exponentiated inverse Rayleigh distribution. Rao et al. (2021) obtained Bayesian and non-Bayesian estimation for multicomponent stress-strength model from exponentiated inverse Rayleigh distribution. Kotz et al. (2003) provide some important and practical results on the theory and application of stress-strength reliability model. Examples of estimation parameters for stress strength model from some probability distributions can be reviewed through (Downtown (1973), Bhattacharyya and Johnson (1974), Awad and Gharraf. (1986), Surles and Padgett (2001), Mokhlis (2005), Baklizi. (2008), Sarhan et al. (2015), Kizilasaln and Nadar (2015), Hassan et al. (2020), and Jaber and Karam (2021).

Stress strength model based on RSS and its modifications gain the attention of many authors. Salman and Hamad (2021), suggested different estimation methods such as maximum likelihood, moment, least square and shrinkage methods, to estimate the reliability stress-strength model from power Lomax distribution based on RSS, Hassan at al. (2021), introduced the reliability estimation for stress-strength model when the stress and the strength variables are distributed as generalized inverted exponential distribution using MRSS approach. Al-Omari et al. (2020) obtained MLEs for $P(Y < X)$ when X and Y are independent identical exponentiated Pareto distribution based on RSS and MRSS. Akgul et al. (2021) introduced MLEs of stress-strength reliability in case of generalized inverse Lindely distribution based on SRS, RSS, and PRSS. See for more examples Kayal, et al. (2020), Hassan, et al. (2021), Akgül, Acıtaş, and Şenoğlu (2018), and Akgül and Şenoğlu (2020).

This study is organized as follows, In Section 2, MLEs of stress strength model, $R = P(Y < X)$ based on SRS data is reviewed. In Section 3, 4, and 5 MLEs based RSS, PRSS, and NRSS are obtained. In Section 6 and 7, simulation study and two real data sets are considered. In Section 8, conclusion results are discussed.

2. MAXIMUM LIKELIHOOD ESTIMATION BASED ON SRS

Rao et al. (2019) discussed the reliability estimation of stress strength model $R = P(Y < X)$ when the random strength X and the random stress Y are distributed as $EIRD(\sigma,\alpha)\ and\ EIRD(\sigma,\beta)$ respectively based on SRS data. Let X and Y are two independent identical random variables where $X \sim EIRD(\sigma, \alpha)$ and $Y \sim EIRD(\sigma, \beta)$, then $R = P(Y < X)$ is given as follows

$$R = \frac{1}{1+\delta}, \text{where } \delta = \frac{\alpha}{\beta}. \tag{10}$$

Rae et al. (2019) introduced the MLE of R denoted by R_{SRS}. Let $X_1, X_2, \dots, X_{\tau_1}$ is a SRS data from $EIRD(\sigma, \alpha)$, and $Y_1, Y_2, \dots, Y_{\tau_2}$ is a SRS data from $EIRD(\sigma, \beta)$. Then, the log-likelihood denoted by l_{SRS} of the observed sample will be as follows

$$l_{SRS} \propto \tau_1 \ln \propto + \tau_2 \ln \beta \\ + \alpha \ln \sum_{i=1}^{\tau_1} \ln\left(1 - e^{-(\sigma/x_i)^2}\right) + \beta \sum_{j=1}^{\tau_2} \ln\left(1 - e^{-(\sigma/y_j)^2}\right) \tag{11}$$

The MLEs $\hat{\propto}_{SRS}, and\ \hat{\beta}_{SRS}$ of α and β are the values which maximize Equation (11). The first derivative of the natural logarithm of the function with respect to α and β respectively based on SRS technique is given by

$$\frac{\partial l_{SRS}}{\partial \propto} = \frac{\tau_1}{\propto} + \sum_{i=1}^{\tau_1} \ln\left(1 - e^{-(\sigma/x_i)^2}\right) = 0, \tag{12}$$

$$\frac{\partial l_{SRS}}{\partial \beta} = \frac{\tau_2}{\beta} + \sum_{j=1}^{\tau_2} \ln\left(1 - e^{-(\sigma/y_j)^2}\right) = 0. \tag{13}$$

Therefore $\hat{\propto}_{SRS}$, and $\hat{\beta}_{SRS}$ are obtained respectively as follows:

$$\hat{\propto}_{SRS} = \frac{-\tau_1}{\sum_{i=1}^{\tau_1} \ln\left(1 - e^{-(\sigma/x_i)^2}\right)}, \hat{\beta}_{SRS} = \frac{-\tau_2}{\sum_{j=1}^{\tau_2} \ln\left(1 - e^{-(\sigma/y_j)^2}\right)}. \tag{14}$$

Then by using the invariance property of the maximum likelihood method of estimation, then MLE of R denoted by $\hat{R}_{SRS}$ is given as follows

$$\hat{R}_{SRS} = \frac{1}{1 + \hat{\delta}_{SRS}}, where\ \hat{\delta} = \frac{\hat{\propto}_{SRS}}{\hat{\beta}_{SRS}} \tag{15}$$

3. MAXIMUM LIKELIHOOD ESTIMATION BASED ON RSS

In this section, reliability estimation of R based on RSS technique is derived. Let $\{X_{\mathrm{i(i)s}}, \mathrm{i} = 1,2, \dots \mathrm{n}$ and $s = 1, \dots r\}$ be a ranked set sample from $EIRD(\sigma, \alpha)$, and $\{Y_{j(j)s}, \mathrm{j} = 1,2, \dots \mathrm{m}$ and $s = 1, \dots r\}$ be a ranked set sample from $EIRD(\sigma, \beta)$. Then using Equation (3) the PDFs of $X_{\mathrm{i(i)s}}$ and $Y_{j(j)s}$ respectively can be written as follows

$$f\left(x_{i(i)s}\right) = \frac{2n! \propto \sigma^2 e^{-(\sigma/x_{i(i)s})^2}}{(i-1)!\,(n-i)!\, x_{i(i)s}^3} \\ \left[1 - \left(1 - e^{-(\sigma/x_{i(i)s})^2}\right)^{\propto}\right]^{i-1} \left[1 - e^{-(\sigma/x_{i(i)s})^2}\right]^{\propto(n-i+1)-1}, \tag{16}$$

and

$$f\left(y_{j(j)s}\right)=\frac{2m!\,\beta\sigma^{2}e^{-(\sigma/y_{j(j)s})^{2}}}{(j-1)!\,(m-j)!\,y_{j(j)s}^{3}}$$
$$\left[1-\left(1-e^{-(\sigma/y_{j(j)s})^{2}}\right)^{\beta}\right]^{j-1}\left[1-e^{-(\sigma/y_{j(j)s})^{2}}\right]^{\beta(m-j+1)-1}. \tag{17}$$

Let $\{X_{1(1)s}, X_{2(2)s}, \dots, X_{n(n)s}, s=1,\dots r\}$ is a ranked set sample from $EIRD(\sigma,\alpha)$, and $\{Y_{1(1)s}, Y_{2(2)s}, \dots, Y_{m(m)s}, s=1,\dots r\}$ is a ranked set sample from $EIRD(\sigma,\beta)$. Then, likelihood function of the observed sample denoted by L_{RSS} can be obtained using (16) and (17) as follows:

$$L_{RSS}=\prod_{s=1}^{r}\left[\prod_{i=1}^{n}f\left(x_{i(i)s}\right)\prod_{j=1}^{m}f\left(y_{j(j)s}\right)\right],$$

The log-likelihood function, denoted by l_{RSS} can be obtained as follows:

$$l_{RSS}\propto\tau_{1}\ln\propto+\tau_{2}\ln\beta$$
$$+\sum_{s=1}^{r}\sum_{i=1}^{n}(i-1)\ln\left[1-\left(1-e^{-(\sigma/x_{i(i)s})^{2}}\right)^{\propto}\right]$$
$$+\sum_{s=1}^{r}\sum_{j=1}^{m}(j-1)\ln\left[1-\left(1-e^{-(\sigma/y_{j(j)s})^{2}}\right)^{\beta}\right]$$
$$+\propto\sum_{s=1}^{r}\sum_{i=1}^{n}(n-i+1)\ln\left[1-e^{-(\sigma/x_{i(i)s})^{2}}\right]$$
$$+\beta\sum_{s=1}^{r}\sum_{j=1}^{m}(m-j+1)\ln\left[1-e^{-(\sigma/y_{j(j)s})^{2}}\right]. \tag{18}$$

In order to obtain MLEs of the parameters $\propto$ and β denoted by $\hat{\propto}_{RSS}$ and $\hat{\beta}_{RSS}$, we first should obtain the first derivative of Equation (18) with respect to $\propto$ and β as follows:

$$\frac{\partial l_{RSS}}{\partial\propto}=\frac{\tau_{1}}{\propto}-\sum_{s=1}^{r}\sum_{i=1}^{n}(i-1)\frac{\left(1-e^{-(\sigma/x_{i(i)s})^{2}}\right)^{\propto}\ln\left(1-e^{-(\sigma/x_{i(i)s})^{2}}\right)}{1-\left(1-e^{-(\sigma/x_{i(i)s})^{2}}\right)^{\propto}}$$
$$+\sum_{s=1}^{r}\sum_{i=1}^{n}(n-i+1)\ln\left[1-e^{-(\sigma/x_{i(i)s})^{2}}\right]=0, \tag{19}$$

$$\frac{\partial l_{RSS}}{\partial\beta}=\frac{\tau_{2}}{\beta}-\sum_{s=1}^{r}\sum_{j=1}^{m}(j-1)\frac{\left(1-e^{-(\sigma/y_{j(j)s})^{2}}\right)^{\beta}\ln\left(1-e^{-(\sigma/y_{j(j)s})^{2}}\right)}{1-\left(1-e^{-(\sigma/y_{j(j)s})^{2}}\right)^{\beta}}$$
$$+\sum_{s=1}^{r}\sum_{j=1}^{m}(m-j+1)\ln\left[1-e^{-(\sigma/y_{j(j)s})^{2}}\right]=0. \tag{20}$$

Once the MLEs $\hat{\propto}_{RSS}$ and $\hat{\beta}_{RSS}$ form Equations (19) and (20), then the MLE of R based on RSS will be obtained as follows

$$\hat{R}_{RSS} = \frac{1}{1+\hat{\delta}_{RSS}}, \text{where } \hat{\delta}_{RSS} = \frac{\hat{\propto}_{RSS}}{\hat{\beta}_{RSS}}. \quad (21)$$

4. MAXIMUM LIKELIHOOD ESTIMATION BASED ON PRSS

As we mentioned previously, when samples are drawn using PRSS technique, then the method of withdrawal differs if the size of the sample drawn is even or odd. In this section, MLEs of $R = P(Y < X)$ is obtained based on PRSS approach depending on two cases, the first when the sample size is odd and the second when the sample size is even.

Odd Set Size

Let $\{X_{i(n_1)s}, i = 1,2,..,V-1; s = 1,2,..,r\} \cup \{X_{i(n_2)s}, i = V,\dots,n-1; s = 1,2,..,r\} \cup \{X_{n(V)s} = 1,2,..,r\}$ be a percentile ranked set sample from $EIRD(\sigma,\alpha)$, then using Equations (4), and (5), the PDFs of $x_{i(n_1)s}$, $x_{i(n_2)s}$ are given as follows:

$$f_{n_1}\left(x_{i(n_1)s}\right) = \frac{2n! \propto \sigma^2 e^{-(\sigma/x_{i(n_1)s})^2}}{(n-n_1)!\,(n_1-1)!\,x^3_{i(n_1)s}}\left[1-e^{-(\sigma/x_{i(n_1)s})^2}\right]^{\propto(n-n_1+1)-1}$$
$$\left[1-\left(1-e^{-(\sigma/x_{i(n_1)s})^2}\right)^{\propto}\right]^{n_1-1}, \quad (22)$$

and

$$f_{n_2}\left(x_{i(n_2)s}\right) = \frac{2n! \propto \sigma^2 e^{-(\sigma/x_{i(n_2)s})^2}}{(n-n_2)!\,(n_2-1)!\,x^3_{i(n_2)s}}\left[1-e^{-(\sigma/x_{i(n_2)s})^2}\right]^{\propto(n-n_2+1)-1}$$
$$\left[1-\left(1-e^{-(\sigma/x_{i(n_2)s})^2}\right)^{\propto}\right]^{n_2-1}. \quad (23)$$

The PDF of the $X_{n(V)s}$ is given by using Equation (6) as follows:

$$f_{n(V)s}\left(x_{n(V)s}\right) = \frac{2n! \propto \sigma^2 e^{-(\sigma/x_{n(V)s})^2}}{[(V-1)!]^2 x^3_{n(V)s}}\left[1-e^{-(\sigma/x_{n(V)s})^2}\right]^{\propto V-1}$$
$$\left[1-\left(1-e^{-(\sigma/x_{n(V)s})^2}\right)^{\propto}\right]^{V-1}. \quad (24)$$

Let $\{Y_{j(m_1)s}, j = 1,2,..,L-1; s = 1,2,..,r\} \cup \{Y_{j(m_2)s}, j = L,\dots,m-1; s = 1,2,..,r\} \cup \{Y_{m(L)s} = 1,2,..,r\}$ be a percentile ranked set sample from $EIRD(\sigma,\beta)$, then using Equations (4), and (5), the PDFs of $y_{j(m_1)s}$, $y_{j(m_2)s}$ are given as follows:

$$f_{m_1}\left(y_{j(m_1)s}\right) = \frac{2m!\,\beta\sigma^2 e^{-(\sigma/y_{j(m_1)s})^2}}{(m_1-1)!\,(m_1-j)!\,y^3_{j(m_1)s}}$$
$$\left[1-e^{-(\sigma/y_{j(m_1)s})^2}\right]^{\beta(m-m_1+1)-1}\left[1-\left(1-e^{-(\sigma/y_{j(m_1)s})^2}\right)^{\beta}\right]^{m_1-1}. \quad (25)$$

and

$$f_{m_2}\left(y_{j(m_2)s}\right) = \frac{2m!\,\beta\sigma^2 e^{-(\sigma/y_{j(m_2)s})^2}}{(m_2-1)!\,(m_2-j)!\,y^3_{j(m_2)s}}$$

$$\left[1-e^{-(\sigma/y_{j(m_2)s})^2}\right]^{\beta(m-m_2+1)-1}\left[1-\left(1-e^{-(\sigma/y_{j(m_2)s})^2}\right)^{\beta}\right]^{m_2-1}. \quad (26)$$

The PDF of the $Y_{m(L)s}$ is given by using Equation (6) as follows:

$$f\left(y_{m(L)s}\right)=\frac{2m!\,\beta\sigma^2 e^{-(\sigma/y_{j(L)s})^2}}{[(L-1)!]^2!\,y_{m(L)s}^3}$$
$$\left[1-e^{-(\sigma/y_{m(L)s})^2}\right]^{\beta L-1}\left[1-\left(1-e^{-(y_{m(L)s})^2}\right)^{\beta}\right]^{L-1}. \quad (27)$$

The likelihood function of the observed sample based on PRSS data denoted by L_{PRSS} can be obtained using (22) to (27) as follows:

$$L_{PRSS}=\prod_{s=1}^{r}\left[f(x_{n(V)s})\prod_{i=1}^{V-1}f(x_{i(n_1)s})\prod_{i=V}^{n-1}f(x_{i(n_2)s})\right]$$
$$\times\prod_{s=1}^{r}\left[f(y_{m(L)s})\prod_{j=1}^{L-1}f(y_{j(m_1)s})\prod_{j=L}^{m}f(y_{j(m_2)s})\right],$$

The log-likelihood function, denoted by l_{PRSS} can be derived as follows:

$$l_{PRSS}\propto\tau_1\ln\propto+\tau_2\ln\beta+\propto\sum_{s=1}^{r}\sum_{i=1}^{V-1}(n-n_1+1)\ln\left(1-e^{-(\sigma/x_{i(n_1)s})^2}\right)$$
$$+\sum_{s=1}^{r}\sum_{i=1}^{V-1}(n_1-1)\times\ln\left[1-\left(1-e^{-(\sigma/x_{i(n_1)s})^2}\right)^{\propto}\right]$$
$$+\propto\sum_{s=1}^{r}\sum_{i=V}^{n-1}(n-n_2+1)\ln\left(1-e^{-(\sigma/x_{i(n_2)s})^2}\right)$$
$$+\sum_{s=1}^{r}\sum_{i=1}^{n-1}(n_2-1)\times\ln\left[1-\left(1-e^{-(\sigma/x_{i(n_2)s})^2}\right)^{\propto}\right]$$
$$+\sum_{s=1}^{r}(\propto V-1)\ln\left(1-e^{-(\sigma/x_{n(V)s})^2}\right)$$
$$+\sum_{s=1}^{r}(V-1)\ln\left[1-\left(1-e^{-(\sigma/x_{n(V)s})^2}\right)^{\propto}\right]$$
$$+\beta\sum_{s=1}^{r}\sum_{j=1}^{m}(m-m_1+1)\ln\left(1-e^{-(\sigma/y_{j(m_1)s})^2}\right)$$
$$+\sum_{s=1}^{r}\sum_{j=1}^{L-1}(m_1-1)\ln\left[1-\left(1-e^{-(\sigma/y_{j(m_1)s})^2}\right)^{\beta}\right]$$
$$+\beta\sum_{s=1}^{r}\sum_{J=L}^{m-1}(m-m_2+1)\ln\left(1-e^{-(\sigma/y_{j(m_2)s})^2}\right)$$

$$+\sum_{s=1}^{r}\sum_{j=L}^{m-1}(m_2-1)\ln\left[1-\left(1-e^{-(\sigma/y_{j(m_2)s})^2}\right)^{\beta}\right]$$
$$+\sum_{s=1}^{r}(\beta L-1)\ln\left(1-e^{-(\sigma/y_{m(L)s})^2}\right)$$
$$+\sum_{s=1}^{r}(L-1)\ln\left[1-\left(1-e^{-(\sigma/y_{m(L)s})^2}\right)^{\propto}\right]. \tag{28}$$

MLEs of R based on PRSS data denoted by $\hat{R}_{PRSS}$ can be obtained by deriving the first derivative of Equation (28) with respect to $\propto$ and β denoted by $\hat{\propto}_{PRSS}$ and $\hat{\beta}_{PRSS}$ as follows

$$\frac{\partial l_{PRSS}}{\partial \propto}=\frac{\tau_1}{\propto}+\sum_{s=1}^{r}\sum_{i=1}^{V-1}(n-n_1+1)\ln\left(1-e^{-(\sigma/x_{i(n_1)s})^2}\right)$$
$$-\sum_{s=1}^{r}\sum_{i=1}^{V-1}(n_1-1)\times\frac{\left(1-e^{-(\sigma/x_{i(n_1)s})^2}\right)^{\propto}\ln\left(1-e^{-(\sigma/x_{i(n_1)s})^2}\right)}{1-\left(1-e^{-(\sigma/x_{i(n_1)s})^2}\right)^{\propto}}$$
$$+\sum_{s=1}^{r}\sum_{i=V}^{n-1}(n-n_2+1)\ln\left(1-e^{-(\sigma/x_{i(n_2)s})^2}\right)$$
$$-\sum_{s=1}^{r}\sum_{i=V}^{n-1}(n_2-1)\frac{\left(1-e^{-(\sigma/x_{i(n_2)s})^2}\right)^{\propto}\ln\left(1-e^{-(\sigma/x_{i(n_2)s})^2}\right)}{1-\left(1-e^{-(\sigma/x_{i(n_2)s})^2}\right)^{\propto}}$$
$$+\sum_{s=1}^{r}V\ln\left[1-e^{-(\sigma/x_{n(V)s})^2}\right]$$
$$-\sum_{s=1}^{r}(V-1)\frac{\left(1-e^{-(\sigma/x_{n(V)s})^2}\right)^{\propto}\ln\left(1-e^{-(\sigma/x_{n(V)s})^2}\right)}{1-\left(1-e^{-(\sigma/x_{n(V)s})^2}\right)^{\propto}}, \tag{29}$$

$$\frac{\partial l_{PRSS}}{\partial \beta}=\frac{\tau_2}{\beta}+\sum_{s=1}^{r}\sum_{j=1}^{L-1}(m-m_1+1)\ln\left(1-e^{-(\sigma/y_{j(m_1)s})^2}\right)$$
$$-\sum_{s=1}^{r}\sum_{j=1}^{L-1}(m_1-1)\times\frac{\left(1-e^{-(\sigma/y_{j(m_1)s})^2}\right)^{\beta}\ln\left(1-e^{-(\sigma/y_{j(m_1)s})^2}\right)}{1-\left(1-e^{-(\sigma/y_{j(m_1)s})^2}\right)^{\beta}}$$
$$+\sum_{s=1}^{r}\sum_{j=L}^{m-1}(m-m_2+1)\ln\left(1-e^{-(\sigma/y_{j(m_2)s})^2}\right)$$
$$-\sum_{s=1}^{r}\sum_{j=L}^{m-1}(m_2-1)\frac{\left(1-e^{-(\sigma/y_{j(m_2)s})^2}\right)^{\beta}\ln\left(1-e^{-(\sigma/y_{j(m_2)s})^2}\right)}{1-\left(1-e^{-(\sigma/y_{j(m_2)s})^2}\right)^{\beta}}$$
$$+\sum_{s=1}^{r}L\ln\left[1-e^{-(\sigma/y_{m(L)s})^2}\right]$$

$$-\sum_{s=1}^{r}(L-1)\frac{\left(1-e^{-(\sigma/y_{m(L)s})^2}\right)^{\beta}\ln\left(1-e^{-(\sigma/y_{m(L)s})^2}\right)}{1-\left(1-e^{-(\sigma/y_{m(L)s})^2}\right)^{\beta}}, \quad (30)$$

Even Set Sizes

Let $\{X_{i(n_1)s}, i=1,2,..,F; s=1,2,..,r\} \cup \{X_{i(n_2)s}, i=F+1,\dots,n; s=1,2,..,r\}$ where $F=\frac{n}{2}$ be a percentile ranked set sample from $EIRD(\sigma,\alpha)$ in case of even set size, the PDFs of $x_{i(n_1)s}$ and $x_{i(n_2)s}$ are defined in Equations (22), and (23). Let $\{Y_{j(m_1)s}, j=1,2,..,L; s=1,2,..,r\} \cup \{Y_{j(m_2)s}, j=L+1,\dots,m; s=1,2,..,r\}$ where $L=\frac{m}{2}$ be a percentile ranked set sample from $EIRD(\sigma,\beta)$, the PDFs of $y_{j(m_1)s}$, $y_{j(m_2)s}$ are defined in Equations (25), and (26). Then the likelihood function of the observed sample based on PRSS in case of even set size can be obtained as follows

$$L_{PRSS}=\prod_{s=1}^{r}\left[\prod_{i=1}^{F}f(x_{i(n_1)s})\prod_{i=F+1}^{n}f(x_{i(n_2)s})\right] \times\prod_{s=1}^{r}\left[\prod_{j=1}^{L}f(y_{j(m_1)s})\prod_{j=L+1}^{L}f(y_{j(m_2)s})\right],$$

The log-likelihood function, denoted by l_{PRSS} can be derived as follows

$$\begin{aligned}l_{PRSS}\propto{}&\tau_1\ln\propto+\tau_2\ln\beta+\propto\sum_{s=1}^{r}\sum_{i=1}^{F}(n-n_1+1)\ln\left(1-e^{-(\sigma/x_{i(n_1)s})^2}\right)\\&+\propto\sum_{s=1}^{r}\sum_{i=F+1}^{n}(n-n_2+1)\ln\left(1-e^{-(\sigma/x_{i(n_2)s})^2}\right)\\&+\sum_{s=1}^{r}\sum_{i=1}^{F}(n_1-1)\ln\left[1-\left(1-e^{-(\sigma/x_{i(n_1)s})^2}\right)^{\propto}\right]\\&+\sum_{s=1}^{r}\sum_{i=F+1}^{n}(n_2-1)\ln\left[1-\left(1-e^{-(\sigma/x_{i(n_2)s})^2}\right)^{\propto}\right]\\&+\beta\sum_{s=1}^{r}\sum_{j=L+1}^{m}(m-m_2+1)\ln\left(1-e^{-(\sigma/y_{j(m_2)s})^2}\right)\\&+\sum_{s=1}^{r}\sum_{j=1}^{L}(m_1-1)\ln\left[1-\left(1-e^{-(\sigma/y_{j(m_1)s})^2}\right)^{\beta}\right]\\&+\sum_{s=1}^{r}\sum_{j=L+1}^{m}(m_2-1)\ln\left[1-\left(1-e^{-(\sigma/y_{j(m_2)s})^2}\right)^{\beta}\right].\\&+\beta\sum_{s=1}^{r}\sum_{j=1}^{L}(m-m_1+1)\ln\left(1-e^{-(\sigma/y_{j(m_1)s})^2}\right)\end{aligned} \quad (31)$$

MLEs of R based on PRSS data denoted by $\hat{R}_{PRSS}$ can be obtained by deriving the first derivative of Equation (31) with respect to $\propto$ and β denoted by $\hat{\propto}_{PRSS}$ and $\hat{\beta}_{PRSS}$ as follows:

$$\frac{\partial l_{PRSS}}{\partial \propto} = \frac{\tau_1}{\propto} + \sum_{s=1}^{r}\sum_{i=1}^{L}(n - n_1 + 1)\ln\left(1 - e^{-(\sigma/x_{i(n_1)s})^2}\right) + \sum_{s=1}^{r}\sum_{i=F+1}^{n}(n - n_2 + 1) \times \ln\left(1 - e^{-(\sigma/x_{i(n_2)s})^2}\right) - \sum_{s=1}^{r}\sum_{i=1}^{F}(n_1 - 1)\frac{\left(1 - e^{-(\sigma/x_{i(n_1)s})^2}\right)^{\propto}\ln\left(1 - e^{-(\sigma/x_{i(n_1)s})^2}\right)}{1 - \left(1 - e^{-(\sigma/x_{i(n_1)s})^2}\right)^{\propto}} - \sum_{s=1}^{r}\sum_{i=F+1}^{n}(n_2 - 1)\frac{\left(1 - e^{-(\sigma/x_{i(n_2)s})^2}\right)^{\propto}\ln\left(1 - e^{-(\sigma/x_{i(n_2)s})^2}\right)}{1 - \left(1 - e^{-(\sigma/x_{i(n_2)s})^2}\right)^{\propto}}, \quad (32)$$

$$\frac{\partial l_{PRSS}}{\partial \beta} = \frac{\tau_1}{\beta} + \sum_{s=1}^{r}\sum_{j=1}^{L}(m - m_1 + 1)\ln\left(1 - e^{-(\sigma/y_{j(m_1)s})^2}\right) + \sum_{s=1}^{r}\sum_{j=L+1}^{m}(m - m_2 + 1) \times \ln\left(1 - e^{-(\sigma/y_{j(m_2)s})^2}\right) - \sum_{s=1}^{r}\sum_{j=1}^{L}(m_1 - 1)\frac{\left(1 - e^{-(\sigma/y_{j(m_1)s})^2}\right)^{\beta}\ln\left(1 - e^{-(\sigma/y_{j(m_1)s})^2}\right)}{1 - \left(1 - e^{-(\sigma/y_{j(m_1)s})^2}\right)^{\beta}} - \sum_{s=1}^{r}\sum_{j=L+1}^{m}(m_2 - 1)\frac{\left(1 - e^{-(\sigma/y_{j(m_2)s})^2}\right)^{\beta}\ln\left(1 - e^{-(\sigma/y_{j(m_2)s})^2}\right)}{1 - \left(1 - e^{-(\sigma/y_{j(m_2)s})^2}\right)^{\beta}}, \quad (33)$$

Once the MLEs of $\propto$ and β denoted by $\hat{\propto}_{PRSS}$ and $\hat{\beta}_{PRSS}$, $\hat{R}_{PRSS}$ can be obtained using Equation (29) and (30) in case of odd set sizes, and Equations (32) and (33) in case of even set sizes as follows

$$\hat{R}_{PRSS} = \frac{1}{1 + \hat{\delta}_{PRSS}}, \text{where } \hat{\delta}_{PRSS} = \frac{\hat{\propto}_{PRSS}}{\hat{\beta}_{PRSS}}. \quad (34)$$

5. MAXIMUM LIKELIHOOD ESTIMATION BASED ON NRSS

This section is concerned with maximum likelihood estimation of R based on $NRSS$, Let $\{X_{(k(i))s}, i = 1,2,..,n \text{ and } s = 1,2,..,r\}$ be a NRSS data from $EIRD(\sigma,\alpha)$, and $\{Y_{(k(j))s}, j = 1,2,..,m \text{ and } s = 1,2,..,r\}$ be a NRSS data from $EIRD(\sigma,\beta)$ by using Equation (7), the PDFs of $X_{(k(i))s}$ and $Y_{(k(j))s}$ can be written respectively as follows:

$$f_{k(i)}\left(X_{(k(i))s}\right)=\frac{w_1!}{(k(i)-k(i-1)-1)!}\frac{2\propto\sigma^2}{x_{k(i)s}^3}e^{-(\sigma/x_{k(i)s})^2}\left(1-e^{-(\sigma/x_{k(i)s})^2}\right)^{\propto-1}$$
$$\times\left[\left(1-e^{-(\sigma/x_{k(i-1)s})^2}\right)^{\propto}-\left(1-e^{-(\sigma/x_{k(i)s})^2}\right)^{\propto}\right]^{k(i)-k(i-1)-1}, \quad (35)$$

and

$$\left(y_{(k(j))s}\right)=\frac{w_2!}{(k(j)-k(j-1)-1)!}\frac{2\beta\sigma^2}{y_{k(j)s}^3}e^{-(\sigma/y_{k(j)s})^2}\left(1-e^{-(\sigma/y_{k(j)s})^2}\right)^{\beta-1}$$
$$f_{k(j)}\times\left[\left(1-e^{-(\sigma/y_{k(j-1)s})^2}\right)^{\beta}-\left(1-e^{-(\sigma/y_{k(j)s})^2}\right)^{\beta}\right]^{(k(j)-k(j-1)-1)} \quad (36)$$

where $k(0)=0,\ k(i+1)=w_1+1,\ k(j+1)=w_2+1,\ x_{(k(0))}=y_{(k(0))}=-\infty$, $x_{(k(i+1))}=y_{(k(j+1))}=\infty$ and $w_1=n^2$, and $w_2=m^2$ and $k(.)$ is defined in Equation (9).

Let $\{X_{(k(1))s}, X_{(k(2))s}, \dots, X_{(k(n))s}\ ;\ s=1,2,..,r\}$ is $NRSS$ data from $EIRD(\sigma,\alpha)$ and $\{y_{(k(1))s}, y_{(k(2))s}, \dots, y_{(k(m))s};\ s=1,2,..,r\}$ is $NRSS$ data from $EIRD(\sigma,\beta)$. Then the likelihood function of R denoted by L_{NRSS} based on $NRSS$ approach can be derived using (35) and (36) as follows:

$$L_{NRSS}=\prod_{s=1}^{r}\left[\prod_{i=1}^{n}f_{k(i)}\left(x_{(k(i))s}\right)\prod_{j=1}^{m}f_{k(j)}\left(y_{(k(j))s}\right)\right]$$

$$L_{NRSS}=\prod_{s=1}^{r}\left[\frac{w_1!}{\prod_{i=1}^{n+1}(k(i)-k(i-1)-1)!}\prod_{i=1}^{n}\frac{2\propto\sigma^2}{x_{k(i)s}^3}e^{-(\sigma/x_{k(i)s})^2}\left(1-e^{-(\sigma/x_{k(i)s})^2}\right)^{\propto-1}\right.$$
$$\times\prod_{i=1}^{n+1}\left[\left(1-e^{-(\sigma/x_{k(i-1)s})^2}\right)^{\propto}-\left(1-e^{-(\sigma/x_{k(i)s})^2}\right)^{\propto}\right]^{k(i)-k(i-1)-1}\Big]$$
$$\times\prod_{s=1}^{r}\left[\frac{w_2!}{\prod_{j=1}^{m+1}(k(j)-k(j-1)-1)!}\prod_{j=1}^{m}\frac{2\beta\sigma^2}{y_{k(j)s}^3}e^{-(\sigma/y_{k(j)s})^2}\left(1-e^{-(\sigma/y_{k(j)s})^2}\right)^{\beta-1}\right.$$
$$\times\prod_{j=1}^{m+1}\left[\left(1-e^{-(\sigma/y_{k(j-1)s})^2}\right)^{\beta}-\left(1-e^{-(\sigma/y_{k(j)s})^2}\right)^{\beta}\right]^{(k(j)-k(j-1)-1)}\Big]$$

The log-likelihood function based on NRSS approach denoted by l_{NRSS} is given by

$$l_{NRSS}\propto\tau_1 ln\propto+\tau_2\, ln\beta+(\propto-1)\sum_{s=1}^{r}\sum_{i=1}^{n}\ln\left(1-e^{-(\sigma/x_{k(i)s})^2}\right)$$
$$+(\beta-1)\sum_{s=1}^{r}\sum_{j=1}^{m}\ln\left(1-e^{-(\sigma/y_{k(j)s})^2}\right)$$
$$+\sum_{s=1}^{r}\sum_{i=1}^{n+1}(k(i)-k(i-1)-1)\ln\left[\left(1-e^{-(\sigma/x_{k(i-1)s})^2}\right)^{\propto}-\left(1-e^{-(\sigma/x_{k(i)s})^2}\right)^{\propto}\right]$$

$$+\sum_{s=1}^{r}\sum_{j=1}^{m+1}(k(j)-k(j-1)-1)$$
$$\ln\left[\left(1-e^{-(\sigma/y_{k(j-1)s})^2}\right)^{\beta}-\left(1-e^{-(\sigma/y_{k(j)s})^2}\right)^{\beta}\right]. \tag{37}$$

MLEs of $\propto$ and β based on NRSS denoted by $\hat{\propto}_{NRSS}$ and $\hat{\beta}_{NRSS}$ can be obtained by using the first derivative of Equation (37)

$$\frac{\partial l_{NRSS}}{\partial \propto}=\frac{\tau_1}{\propto}+\sum_{s=1}^{r}\sum_{i=1}^{n}\ln\left(1-e^{-(\sigma/x_{k(i)s})^2}\right)+\sum_{s=1}^{r}\sum_{i=1}^{n+1}(k(i)-k(i-1)-1)$$
$$\times\frac{\left(1-e^{-(\sigma/x_{k(i-1)s})^2}\right)^{\propto}\ln\left(1-e^{-(\sigma/x_{k(i-1)s})^2}\right)-\left(1-e^{-(\sigma/x_{k(i)s})^2}\right)^{\propto}\ln\left(1-e^{-(\sigma/x_{k(i)s})^2}\right)}{\left(1-e^{-(\sigma/x_{k(i-1)s})^2}\right)^{\propto}-\left(1-e^{-(\sigma/x_{k(i)s})^2}\right)^{\propto}}, \tag{38}$$

and

$$\frac{\partial l_{NRSS}}{\partial \beta}=\frac{\tau_2}{\beta}+\sum_{s=1}^{r}\sum_{j=1}^{m}\ln\left(1-e^{-(\sigma/y_{k(j)s})^2}\right)+\sum_{s=1}^{r}\sum_{j=1}^{m+1}(k(j)-k(j-1)-1)$$
$$\times\frac{\left(1-e^{-(\sigma/y_{k(j-1)s})^2}\right)^{\beta}\ln\left(1-e^{-(\sigma/y_{k(j-1)s})^2}\right)-\left(1-e^{-(\sigma/y_{k(j)s})^2}\right)^{\beta}\ln\left(1-e^{-(\sigma/y_{k(j)s})^2}\right)}{\left(1-e^{-(\sigma/y_{k(j-1)s})^2}\right)^{\beta}-\left(1-e^{-(\sigma/y_{k(j)s})^2}\right)^{\beta}}. \tag{39}$$

It is noted that, it is not easy to obtain a closed form for the previous two equations (38) and (39) so to obtain the MLEs of $\propto$ and β based on NRSS approach, denoted by $\propto_{NRSS}$ and β_{NRSS} an iterative method will be used. Then the MLE of R based on NRSS is given by

$$\hat{R}_{NRSS}=\frac{1}{1+\hat{\delta}_{NRSS}}\text{, where } \hat{\delta}_{NRSS}=\frac{\hat{\propto}_{NRSS}}{\hat{\beta}_{NRSS}} \tag{40}$$

6. SIMULATION STUDY

In this section, we study a Monte Carlo simulation to compare the performance of the maximum likelihood method of estimation using different sample sizes where $\tau_1 = \tau_2 = 30, 40, 50\ 60$ and 70, the set sizes $(n = m) = (3,3), (4,4), (5,5), (6,6)$ and $(7,7)$, and $r = 10$. The simulation study is carried on 1000 simulated samples from EIRD stress-strength variables X and Y where $X \sim EIRD(\sigma, \alpha)$ and $Y \sim EIRD(\sigma, \beta)$, the data is generated from the EIRD for $\sigma = 1$ and different values of $\delta = \frac{\beta}{\alpha}$ where $\delta = 0.20, 0.80, 1, 3, 6, 9$. The performance of the MLEs is studied in terms of mean square errors (MSEs) and efficiencies based on different sampling approaches namely; SRS, RSS, PRSS and NRSS. The comparison between the different sampling techniques based on different sample sizes and different parameters values is made using two criteria, bias and MSE, which are calculated as follows:

$$bais = \sum_{i=1}^{10,000} \frac{\hat{\theta} - \theta}{\theta};\; MSE(\hat{\theta}) = \frac{1}{10,000} \sum_{k=1}^{10,000} (\hat{\theta} - \theta).$$

The efficiency to SRS estimators was calculated for each RSS-based design, by

$$eff_{RSS}(\hat{\theta}) = \frac{MSE_{SRS}(\hat{\theta})}{MSE_{RSS}(\hat{\theta})}, eff_{PRSS}(\hat{\theta}) = \frac{MSE_{SRS}(\hat{\theta})}{MSE_{PRSS}(\hat{\theta})}$$

and $eff_{NRSS}(\hat{\theta}) = \frac{MSE_{SRS}(\hat{\theta})}{MSE_{NRSS}(\hat{\theta})}$.

Simulation results are shown in Tables 1 – 3 and figures 1 – 3. Tables 1 and 2 give biases and MSEs for the estimators $\hat{R}_{SRS}, \hat{R}_{RSS}, \hat{R}_{PRSS}$ and $\hat{R}_{NRSS}$, while efficiencies are listed in Table 3. By examining Tables 1-2 thus it can be concluded that

- All biases of the different estimators are very small.
- In all cases, MSEs for all estimators $\hat{R}_{SRS}, \hat{R}_{RSS}, \hat{R}_{PRSS}$ and $\hat{R}_{NRSS}$ decrease as the set size increase.
- In all cases, MSEs of the estimator $\hat{R}_{NRSS}$ based on NRSS data are smaller than the rest of the estimators studied, namely, $\hat{R}_{SRS}, \hat{R}_{RSS}$, and $\hat{R}_{PRSS}$.
- In almost all cases, MSEs of the estimator $\hat{R}_{PRSS}$ when $O = 45\%$ are less than MSEs when $O = 15\%$ and 35% based on PRSS and the estimator $\hat{R}_{RSS}$ based on RSS.
- In all cases, MSEs based on NRSS, PRSS, and RSS are smaller than MSEs based on SRS.

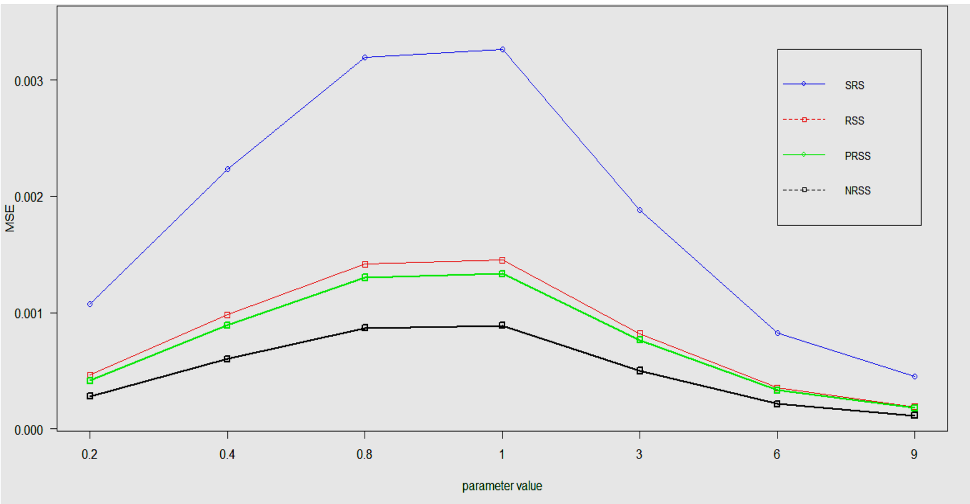

Figure 1: MSEs for $\hat{R}_{SRS}, \hat{R}_{RSS}, \hat{R}_{PRSS}$ and $\hat{R}_{NRSS}$ based on SRS, RSS, PRSS and NRSS when $(\tau_1, \tau_2) = (40, 40)$

Table 1
Biases for $\hat{R}_{SRS}, \hat{R}_{RSS}, \hat{R}_{PRSS}$ and $\hat{R}_{NRSS}$ based on SRS, RSS, PRSS and NRSS

δ	(τ_1, τ_2)	$\hat{R}_{SRS}$	$\hat{R}_{RSS}$	$\hat{R}_{PRSS}$			$\hat{R}_{NRSS}$
				15%	35%	45%	
0.20	(30,30)	-0.0043110	-0.0030670	-0.0012910	-0.0012910	-0.0013600	-0.0017090
	(40,40)	-0.0039980	-0.0016860	-0.0013130	-0.0002317	-0.0002317	-0.0013500
	(50,50)	-0.0035560	-0.0002855	0.0001380	-0.0000829	-0.0001312	-0.0000102
	(60,60)	-0.0025150	0.0003109	-0.0014100	-0.0009176	-0.0012360	-0.0000298
	(70,70)	-0.0013630	-0.0006557	0.0001617	0.0000924	-0.0002551	0.0002080
0.40	(30,30)	-0.004669	-0.003597	-0.0009939	-0.0009939	-0.001174	-0.001892
	(40,40)	-0.004551	-0.0019000	-0.001299	0.0001808	0.0001808	-0.0016300
	(50,50)	-0.004281	0.0000439	0.0006108	0.0002384	0.0001541	-0.0002447
	(60,60)	-0.0029260	0.0007178	-0.0004844	-0.0010880	-0.001568	0.0001098
	(70,70)	-0.0013270	-0.0007520	0.0004703	0.00006622	0.0001696	-0.0001906
0.80	(30,30)	-0.0030440	-0.0029240	0.0002306	0.0002306	0.0001072	-0.0013060
	(40,40)	-0.0034310	-0.0013810	-0.0005695	0.0010550	0.0010550	-0.0014110
	(50,50)	-0.0036980	0.0005594	0.0013940	0.0008678	0.0007461	0.0006664
	(60,60)	-0.0023210	0.0012880	-0.0015870	-0.0008982	0.0015010	0.0003808
	(70,70)	-0.0005299	-0.0005708	0.0009435	0.0004050	0.0001244	-0.0000447
1	(30,30)	-0.0021680	-0.0024610	0.0007385	0.0007385	0.0003550	-0.0009757
	(40,40)	-0.0027470	-0.0010760	-0.0002233	0.00136500	0.00136500	-0.0012310
	(50,50)	-0.0032260	0.0007836	0.0016440	0.0010840	0.0009544	0.0008062
	(60,60)	-0.0019220	0.0014530	-0.0014230	-0.0007619	-0.0013800	0.0004735
	(70,70)	-0.0001570	0.0004582	0.0010880	0.00052510	0.00024270	0.0000205
3	(30,30)	0.0014990	-0.0001687	0.0022720	0.0022720	0.00184500	0.0004371
	(40,40)	0.0004058	0.0002815	0.0010300	0.0020330	0.00203300	-0.0002570
	(50,50)	-0.0006767	0.0013270	0.0020230	0.0015130	0.00139500	0.0010510
	(60,60)	0.0000060	0.0015940	-0.0004484	-0.000730	-0.0005660	0.0006534
	(70,70)	0.0011710	0.00006023	0.0012670	0.0007829	0.00057650	0.0002379
6	(30,30)	0.0018950	0.0003620	0.0019760	0.0019760	0.0016590	0.0006161
	(40,40)	0.0009766	0.0004912	0.0010130	0.0016150	0.0016150	0.0000191
	(50,50)	0.0000516	0.0010770	0.0015460	0.0011870	0.0011040	0.0008128
	(60,60)	0.0004151	0.0011830	-0.0001195	0.0000924	-0.0002386	0.0005109
	(70,70)	0.0011320	0.0001527	0.0009549	0.0006209	0.0004875	0.0002179
9	(30,30)	0.0016710	0.0004087	0.0016000	0.0016000	0.0013550	0.0005509
	(40,40)	0.0009319	0.0004521	0.0008459	0.0012730	0.0012730	0.0006926
	(50,50)	0.0001848	0.0008536	0.0012030	0.0009316	0.0008692	0.0006348
	(60,60)	0.0004274	0.0009118	-0.0000366	0.0001094	-0.0001368	0.0004003
	(70,70)	0.0009418	0.0001456	0.0007396	0.0004887	0.0003911	0.0001785

Table 2
MSEs for $\hat{R}_{SRS}, \hat{R}_{RSS}, \hat{R}_{PRSS}$ *and* $\hat{R}_{NRSS}$ based on SRS, RSS, PRSS and NRSS

δ	(τ_1, τ_2)	$\hat{R}_{SRS}$	$\hat{R}_{RSS}$	$\hat{R}_{PRSS}$			$\hat{R}_{NRSS}$
				15%	35%	45%	
0.20	(30,30)	0.0013490	0.0007337	0.0007256	0.0007256	0.0006611	0.0004954
	(40,40)	0.0010710	0.0004624	0.0005042	0.0004156	0.0004156	0.0002818
	(50,50)	0.0007614	0.0003035	0.0003250	0.0002868	0.0002759	0.0001826
	(60,60)	0.0006178	0.0002076	0.0002624	0.0002078	0.0001983	0.0001221
	(70,70)	0.0005412	0.0001685	0.0001851	0.0001608	0.0001633	0.0000910
0.40	(30,30)	0.0028070	0.0015380	0.0015350	0.0015350	0.0014030	0.0010520
	(40,40)	0.0022350	0.0009819	0.0010720	0.0008905	0.0008905	0.0006007
	(50,50)	0.0015970	0.0006516	0.0006976	0.0006162	0.0005942	0.0003935
	(60,60)	0.0013070	0.0004464	0.0007455	0.0004453	0.0004237	0.0002633
	(70,70)	0.0011500	0.0003614	0.0003983	0.0003458	0.0003509	0.0001976
0.80	(30,30)	0.0040150	0.0021970	0.0022180	0.0022180	0.0020350	0.0015210
	(40,40)	0.0031950	0.0014180	0.0015520	0.0013010	0.0013010	0.0008696
	(50,50)	0.0022810	0.0009530	0.0010200	0.0009018	0.0008722	0.0005767
	(60,60)	0.0018830	0.0006526	0.0008091	0.0006482	0.0006137	0.0003860
	(70,70)	0.0016670	0.0005260	0.0005829	0.0005049	0.0005122	0.0002890
1	(30,30)	0.0041110	0.0022400	0.0022700	0.0022700	0.0020900	0.0015600
	(40,40)	0.0032660	0.0014510	0.0015900	0.0013350	0.0013350	0.0008894
	(50,50)	0.0023270	0.0009783	0.0010460	0.0009260	0.0008950	0.0005920
	(60,60)	0.0019260	0.0006694	0.0008273	0.0006639	0.0006275	0.0003961
	(70,70)	0.0017080	0.0005387	0.0009810	0.0005176	0.0005251	0.0002962
3	(30,30)	0.0024060	0.0012640	0.0013070	0.0013070	0.0012030	0.0008816
	(40,40)	0.0018810	0.0008198	0.0009062	0.0007656	0.0007656	0.0005000
	(50,50)	0.0013140	0.0005597	0.0005983	0.0005302	0.0005153	0.0003372
	(60,60)	0.0010940	0.0003803	0.0004639	0.0003748	0.0003512	0.0002248
	(70,70)	0.0009773	0.0003033	0.0003404	0.0002929	0.0002968	0.0001672
6	(30,30)	0.0010700	0.0005461	0.0005699	0.0005699	0.0005248	0.0003801
	(40,40)	0.0008261	0.0003531	0.0003922	0.0003315	0.0003315	0.0002142
	(50,50)	0.0005685	0.0002418	0.0002584	0.0002291	0.0002230	0.0001451
	(60,60)	0.0004736	0.0001634	0.0001985	0.0001607	0.0001499	0.0000964
	(70,70)	0.0004241	0.0001298	0.0001464	0.0001256	0.0001272	0.0000715
9	(30,30)	0.0005907	0.0002972	0.0003113	0.0003113	0.0002868	0.0002065
	(40,40)	0.0004531	0.0001917	0.0002135	0.0001804	0.0001804	0.0001161
	(50,50)	0.0003096	0.0001314	0.0001404	0.0001245	0.0001213	0.0000786
	(60,60)	0.0002579	0.0000886	0.0001075	0.0000871	0.0000811	0.0000522
	(70,70)	0.0002311	0.0000702	0.0000794	0.0000680	0.0000689	0.0000387

In almost all cases, MSEs of all estimators increase as the value of δ increases up to $\delta = 1$, then MSEs of all estimators decrease as the value of $\propto$ increases. Thus, we can conclude that, MSEs of all estimators increase when β is less than or equal $\propto$ then, and MSEs of all estimators decease when β is greater than $\propto$ (See Figure (1)).

Table 3
Efficiencies for $\hat{R}_{RSS}, \hat{R}_{PRSS}$ *and* $\hat{R}_{NRSS}$ based on SRS, RSS, PRSS and NRSS

δ	(τ_1, τ_2)	$\hat{R}_{RSS}$	$\hat{R}_{PRSS}$			$\hat{R}_{NRSS}$
			15%	35%	45%	
0.20	(30,30)	1.839	1.859	1.859	2.040	2.723
	(40,40)	2.315	2.124	2.576	2.576	3.800
	(50,50)	2.509	2.342	2.655	2.760	4.170
	(60,60)	2.976	2.355	2.974	3.115	5.061
	(70,70)	3.212	2.925	3.365	3.315	5.899
0.40	(30,30)	1.825	1.829	1.829	2.000	2.667
	(40,40)	2.276	2.086	2.510	2.510	3.721
	(50,50)	2.450	2.289	2.591	2.687	4.057
	(60,60)	2.927	2.573	2.934	3.084	4.962
	(70,70)	3.182	2.888	3.326	3.277	5.819
0.80	(30,30)	1.828	1.810	1.810	1.973	2.639
	(40,40)	2.253	2.058	2.456	2.456	3.674
	(50,50)	2.393	2.237	2.529	2.615	3.955
	(60,60)	2.885	2.327	2.905	3.068	4.878
	(70,70)	3.169	2.860	3.301	3.254	5.769
1	(30,30)	1.830	1.806	1.806	1.962	2.628
	(40,40)	2.252	2.054	2.446	2.446	3.672
	(50,50)	2.379	2.224	2.513	2.596	3.931
	(60,60)	2.877	2.328	2.900	3.069	4.861
	(70,70)	3.171	2.855	3.300	3.253	5.765
3	(30,30)	1.904	1.841	1.841	2.001	2.730
	(40,40)	2.295	2.076	2.457	2.457	3.763
	(50,50)	2.347	2.196	2.478	2.550	3.896
	(60,60)	2.877	2.358	2.919	3.115	4.866
	(70,70)	3.222	2.871	3.336	3.293	5.844
6	(30,30)	1.960	1.878	1.878	2.039	2.816
	(40,40)	2.340	2.106	2.492	2.492	3.856
	(50,50)	2.352	2.200	2.482	2.550	3.919
	(60,60)	2.898	2.386	2.947	3.159	4.910
	(70,70)	3.268	2.896	3.376	3.334	5.928
9	(30,30)	1.988	1.898	1.898	2.060	2.861
	(40,40)	2.363	2.123	2.512	2.512	3.904
	(50,50)	2.356	2.205	2.487	2.554	3.935
	(60,60)	2.910	2.399	2.962	3.181	4.935
	(70,70)	3.291	2.910	3.397	3.355	6.970

From Table 3, we can conclude that

- All efficiencies of all estimators $\hat{R}_{RSS}, \hat{R}_{PRSS}$ *and* $\hat{R}_{NRSS}$ are greater than one.
- The efficiencies of the estimator $\hat{R}_{NRSS}$ based on NRSS data are greater than the efficiencies of the other estimators, namely $\hat{R}_{RSS}, \hat{R}_{PRSS}$.
- In almost all cases, the efficiencies of $\hat{R}_{PRSS}$ when $O = 45\%$, are greater than the corresponding estimators when $O = 15\%$ and when $O = 35\%$ based on PRSS, and $\hat{R}_{RSS}$ based on RSS data.

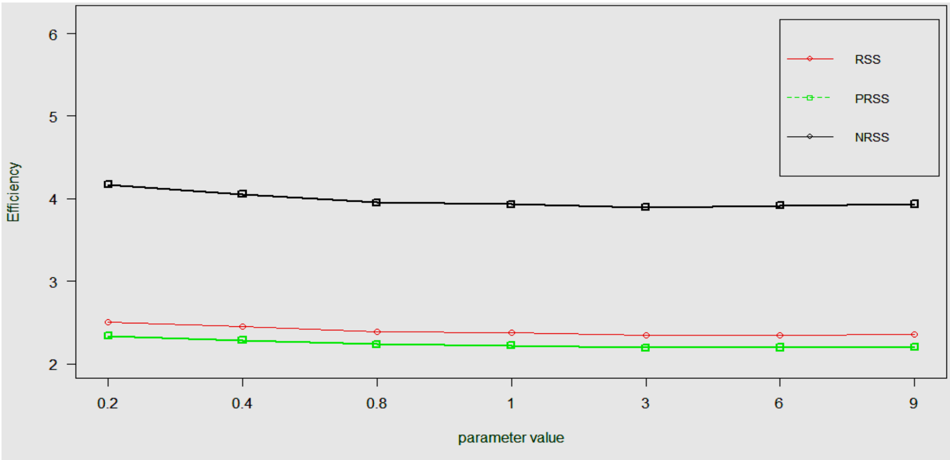

Figure 2: Efficiencies for $\hat{R}_{RSS}, \hat{R}_{PRSS}$ *and* $\hat{R}_{NRSS}$ based on SRS, RSS, PRSS and NRSS when $O = 15\%$ and $(\tau_1, \tau_2) = (50, 50)$

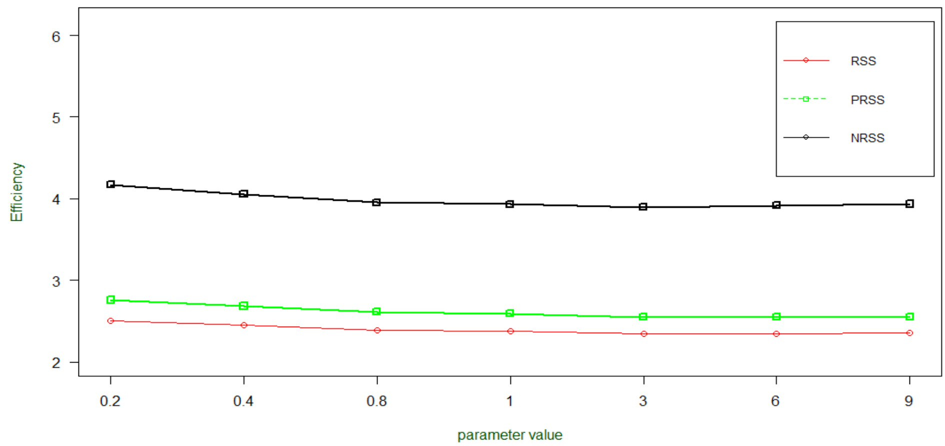

Figure 3: Efficiencies for $\hat{R}_{RSS}, \hat{R}_{PRSS}$ *and* $\hat{R}_{NRSS}$ based on SRS, RSS, PRSS and NRSS when $O = 45\%$ and $(\tau_1, \tau_2) = (50, 50)$

7. DATA ANALYSIS

In this section, real data application is used to verify the results of the study. For the purpose of research, two sets of real data that were used by Rao et al. (2019) to estimate $R = P(Y < X)$ in case of SRS technique are used to compare the performance of the different sampling techniques namely, SRS, RSS, PRSS and NRSS. The two sets of real data obtained from Lawless (2003) represent the failure times of two electrical insulation where insulation is adverted to a continuously increasing voltage stress. They are listed below as follows

- Data Set I (X): $21.8, 70.7, 24.4, 138.6, 151.9, 12.3, 95.5, 98.1, 43.2, 28.6, 46.9.$
- Data Set II (Y): $219.3, 79.4, 86.0, 150.2, 21.7, 18.5, 121.9, 40.5, 147.1, 4.87$

Rao et al. (2019) proved that EIRD provide a good fit for both data sets, they calculated MLEs for $\hat{\propto}_{SRS}$ *and* $\hat{\beta}_{SRS}$ based on SRS data where $\hat{\propto}_{SRS}$ *and* $\hat{\beta}_{SRS}$ are equal to 0.6226 and 0.5225 respectively. To illustrate the estimation of stress- strength model for EIRD, we need to draw samples based on RSS, PRSS and NRSS techniques. Random samples of $n = m = 3$ and $r = 1$ are drawn from data sets I and II. Table 4 represent biases and MSEs of the different estimators $\hat{R}_{SRS}$, $\hat{R}_{RSS}$, $\hat{R}_{PRSS}$and $\hat{R}_{NRSS}$included in this study as follows

Table 4
Biases and MSEs for $\hat{R}_{SRS}$, $\hat{R}_{RSS}$, $\hat{R}_{PRSS}$ and $\hat{R}_{NRSS}$ based on SRS, RSS, PRSS and NRSS

	$\hat{R}_{SRS}$	$\hat{R}_{RSS}$	$\hat{R}_{PRSS}$			$\hat{R}_{NRSS}$
			15%	35%	45%	
Biases	0.067000	0.016000	0.084000	0.084000	0.015000	0.004100
MSEs	0.004492	0.002486	0.000706	0.000706	0.0002361	0.000016

It is clear that the results included in Table 4 agree with the results contained in Tables 1-3. It can be easily concluded that $\hat{R}_{NRSS}$ demonstrate better performance than $\hat{R}_{SRS}$, $\hat{R}_{RSS}$ and $\hat{R}_{PRSS}$.

8. CONCLUSION

This manuscript is devoted to estimate the parameters of stress strength reliability model from EIRD based on NRSS technique. Maximum likelihood method of estimation is used to obtain MLEs of R based on different sampling techniques. A comparative study is conducted to compare the performance of NRSS with RSS, PRSS and SRS designs. Extensive simulation study and real data application are used to reach the results of the study, and therefore we can conclude that, in all cases, MSEs decrease as the set size increase. MSEs of all sampling techniques based on RSS and its modifications are less than MSEs based on SRS approach. MSEs for R based on NRSS data are the lowest compared to the other methods studied in this research. Efficiencies increase as the set size increase, the efficiencies of NRSS approach are higher than the other sampling techniques.

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