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STATISTICAL ANALYSIS OF HAQ DISTRIBUTION: ESTIMATION AND APPLICATIONS

Muhammad Ahsan-ul-Haq
College of Statistical Sciences, University of the Punjab
Lahore, Pakistan
Email: ahsanshani36@gmail.com

ABSTRACT

A new one-parameter distribution named "Haq distribution" is proposed using a mixture approach. Some statistical properties are derived including ordinary moments, moment generating function, incomplete moments, index of dispersion, and mode. Some reliability measures of the proposed distribution are derived, hazard function, cumulative hazard function, reversed hazard function, mills ratio, stress strength reliability, and mean residual life. The parameter of Haq distribution is estimated using five different estimation methods, maximum likelihood, moments, Anderson darling, Cramer von-Misses, and least-squares method. The performance of these methods is assessed using a comprehensive simulation study. In the end, three data sets from different areas are utilized for the application of the proposed distribution.

KEYWORDS

Mixture distribution, lifetime model, moments, statistical inference.

1. INTRODUCTION

Modeling and analysis of lifetime data are essential in many fields, including medical, engineering, insurance, and finance. The exponential and gamma probability distributions, as well as their extensions, are well-known probability distributions that are used to analysis of lifetime datasets. Both distributions have several intriguing structural features, such as exponential distribution taking memoryless and constant hazard rate properties. The exponential distribution may also be used to describe time-to-event data or wait periods.

Let X be a random variable that follows the xgamma distribution, which is a one-parameter distribution with a density function.

$$f(x) = \frac{\theta^2}{(1+\theta)^2}\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}, \quad x > 0, \theta > 0.$$

It was first introduced by (Sen, Maiti, & Chandra, 2016). Various researchers have generalized, extended, modified, and applied xgamma distribution in reliability and other fields of knowledge. Some extensions of xgamma distribution are; quasi xgamma distribution (Sen & Chandra, 2017), weighted xgamma distribution (Sen, Chandra, & Maiti, 2017), log-xgamma distribution (Altun and Hamedani, 2018), quasi xgamma-

Poisson distribution (Sen, Korkmaz, & Yousof, 2018), discrete xgamma distributions (Maiti, Dey, & Sarkar, 2018), quasi xgamma-geometric distribution (Sen, Afify, Al-Mofleh, & Ahsanullah, 2019), discrete quasi xgamma-geometric distribution (Mazucheli, Bertoli, Oliveira, & Menezes, 2020), Half-logistic xgamma distribution (Bantan, Hassan, Elsehetry, & Kibria, 2020), inverse xgamma distribution (Yadav, Maiti, & Saha, 2021) and unit-xgamma distribution (Hashmi, Ahsan-ul-Haq, Zafar, & Khaleel, 2022).

A new probability distribution can be proposed using the mixture of two known probability distributions. A random variable X is said to have the following mixture distribution if its probability density function $f(x)$ is of the form

$$f(x)=\sum_{j=1}^{h} p_j\, f_j(x)$$

where $f_j(x)$ is each probability density function, and $p_1, p_2, p_3, \dots, p_h$ are the non-negative mixing proportions and $\sum_{j=1}^{h} p_j = 1$. Various authors proposed new probability distributions using this approach taking different mixing approaches such as Lindley distribution (Lindley, 1958), Xgamma distribution (Sen et al., 2016), X Lindley distribution (Chouia & Zeghdoudi, 2021), and new extended gamma distribution (Altun, Korkmaz, El-Morshedy, & Eliwa, 2021).

The notion of this work is based on a unique mixture of exponential and xgamma distribution to the proposed Haq distribution. The motivation behind this work is the Haq distribution is simple and easy to use for modeling, and the mathematical expressions of moments, index of dispersion, and reliability characteristics are tractable.

The paper is arranged accordingly: Section 2 is devoted to the derivation of the new distribution and the shape of the distribution. Section 3 deals with the mathematical properties including ordinary moments, moment generating function, incomplete moments, skewness, kurtosis, and index of dispersion. Some reliability measures such as hazard function, the shape of the hazard function, mean residual life, stress strength reliability, mils ratio, and cumulative hazard function are also derived in this section. Section 4 deals with parameter estimation using methods of maximum likelihood, moments, Anderson darling, Cramer Von-Misses, and least-squares. Finally, we provide two illustrated examples of Haq distribution with other distributions to show that the proposed distribution is flexible.

2. DERIVATION OF NEW DISTRIBUTION

A new probability distribution is proposed using a mixture of two known probability distributions called the Haq distribution. We have to consider $f_1(x)$ to follow an exponential distribution with scale parameter θ and $f_2(x)$ to follow xgamma distribution with scale parameter θ with proportions $p_1 = \theta/(1+\theta)$ and $p_2 = 1/(1+\theta)$, respectively. Now the density function of the new distribution is given by

$$f(x)=\left(\frac{\theta}{1+\theta}\right)^2\left(2+\theta+\frac{\theta x^2}{2}\right)e^{-\theta x}, \quad x>0, \theta>0, \tag{1}$$

The corresponding cumulative distribution function is given by

$$F(x) = 1 - \frac{\left(1 + 2\theta + \theta^2 + \theta x + \frac{(\theta x)^2}{2}\right)}{(1+\theta)^2} e^{-\theta x}. \tag{2}$$

2.1. Shape of the Haq distribution

The limiting behavior of pdf at lower and upper limits is given by

$$\lim_{x\to 0} f(x) = \frac{\theta^2(2+\theta)}{(1+\theta)^2} \ \& \ \lim_{x\to\infty} f(x) = 0$$

The pdf assumes the limiting value of $\left(\frac{\theta^2(2+\theta)}{(1+\theta)^2}\right)$ at the origin and then approaching the value of zero at infinity.

The first derivative of pdf is

$$\frac{d}{dx} f(x) = \frac{\theta^3}{(1+\theta)^2}\left(-(2+\theta) + x - \frac{\theta x^2}{2}\right) e^{-\theta x}.$$

gives

$$\hat{x} = \frac{1 + \sqrt{1 - 4\theta - 2\theta^2}}{\theta}$$

It follows that,

- For $0 < \theta < \frac{1}{2}\left(\sqrt{6} - 2\right)$, $\frac{d}{dx} f(x) = 0$ implies that $\hat{x}$ is the unique critical point at which pdf is maximized.
- For $\theta \geq \frac{1}{2}\left(\sqrt{6} - 1\right)$, the pdf is decreasing function.

Therefore, the mode of Haq distribution is given by

$$Mod(X) = \begin{cases} \frac{1 + \sqrt{1 - 4\theta - 2\theta^2}}{\theta}, & 0 < \theta < \frac{1}{2}\left(\sqrt{6} - 2\right) \\ 0, & otherwise \end{cases} \tag{3}$$

The different shapes of the pdf are illustrated in Figure 1, for selected values of the parameters.

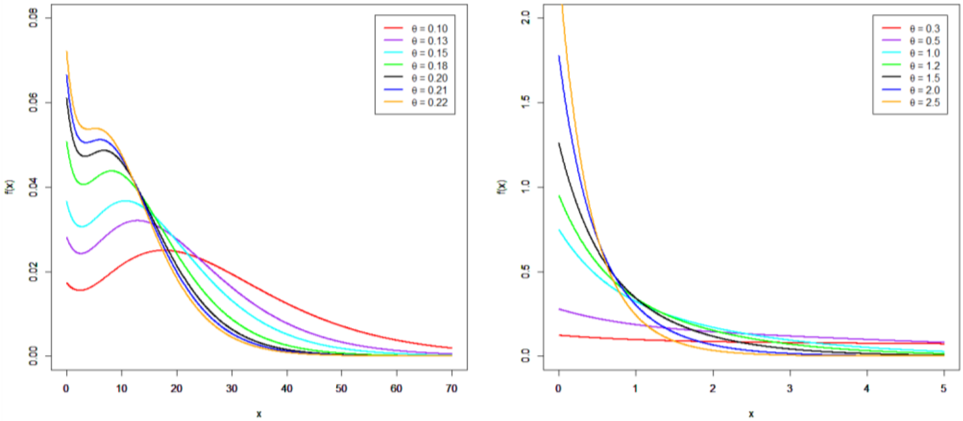

Figure 1: pdf Plots for Haq Distribution

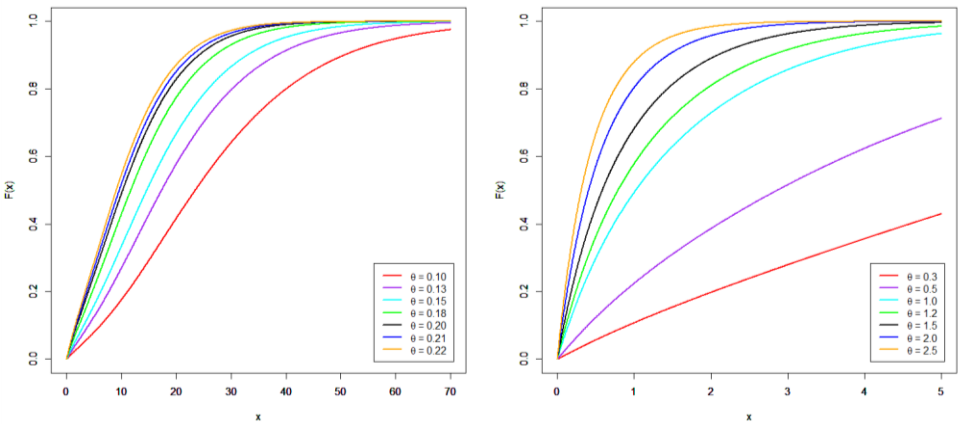

Figure 2: cdf Plots for Haq Distribution

3. MATHEMATICAL PROPERTEIS

This section is devoted to the derivation of some statistical properties including ordinary moments and associated measures, incomplete moments, moment generating function, hazard function, stress strength reliability, and mean residual life.

3.1. Moments and Associated Measures

In distribution theory, moments are useful for addressing important properties of a probability distribution such as mean, variance, skewness, and kurtosis.

Theorem 1:

If $X \sim Haq(X;\theta)$, *with* $\theta > 0$ *then the rth moments of* X *is given by*

$$\mu_r' = \frac{r!}{\theta^r (1+\theta)^2}\left(\frac{2\theta(2+\theta)+(r+1)(r+2)}{2}\right)$$

Proof:

Using the pdf of Haq distribution μ_r' can be written as

$$\mu_r' = \left(\frac{\theta}{1+\theta}\right)^2 \int_0^\infty x^r \left(2+\theta+\frac{\theta x^2}{2}\right) e^{-\theta x} dx$$

$$\mu_r' = \left(\frac{\theta}{1+\theta}\right)^2 \left[(2+\theta)\int_0^\infty x^r\, e^{-\theta x} dx + \frac{\theta}{2}\int_0^\infty x^{r+2}\, e^{-\theta x} dx\right]$$

Ordinary moments take the following final form after simple computations:

$$\mu_r' = \frac{r!}{\theta^r(1+\theta)^2}\left(\frac{2\theta(2+\theta)+(r+1)(r+2)}{2}\right). \tag{4}$$

The first four moments can be obtained by substituting $r = 1,2,3$ and 4 in (4) and then using the relationship between moments about the origin and then moments about the mean. The first four moments about the origin can be obtained as:

$$\mu_1' = \frac{\theta^2+2\theta+3}{\theta(1+\theta)^2} \tag{5}$$

$$\mu_2' = \frac{2(\theta^2+2\theta+6)}{\theta^2(1+\theta)^2} \tag{6}$$

$$\mu_3' = \frac{6(\theta^2+2\theta+10)}{\theta^3(1+\theta)^2} \tag{7}$$

$$\mu_4' = \frac{24(\theta^2+2\theta+15)}{\theta^4(1+\theta)^2} \tag{8}$$

The moments of about mean, variance, skewness, and kurtosis are obtained as

$$Var(x) = \mu_2' - (\mu_1')^2 = \frac{\theta^4+4\theta^3+12\theta^2+16\theta+3}{\theta^2(1+\theta)^4}$$

$$\mu_3 = \frac{2\Big(3+\theta(2+\theta)\big(9+\theta(2+\theta)(15+\theta(2+\theta))\big)\Big)}{\theta^3(1+\theta)^6}$$

$$\mu_4 = \frac{3\big(1+3\theta(2+\theta)\big)\Big(15+\theta(2+\theta)\big(27+\theta(2+\theta)(21+\theta(2+\theta))\big)\Big)}{\theta^4(1+\theta)^8}$$

$$CS = \frac{3+\theta(2+\theta)\big(9+\theta(2+\theta)(15+\theta(2+\theta))\big)}{\sqrt{2}\theta^3(1+\theta)^6\left(\frac{6+\theta(2+\theta)}{\theta^2(1+\theta)^2}\right)^{3/2}}$$

$$CK = \frac{3\big(1+3\theta(2+\theta)\big)\Big(15+\theta(2+\theta)\big(27+\theta(2+\theta)(21+\theta(2+\theta))\big)\Big)}{4(1+\theta)^4\big(6+\theta(2+\theta)\big)^2}$$

The index of dispersion (DI) is given as

$$DI = \frac{Var(x)}{Mean} = \frac{\theta^4 + 4\theta^3 + 12\theta^2 + 16\theta + 3}{\theta(1+\theta)^2(\theta^2 + 2\theta + 3)} \tag{9}$$

Table 1 represents summary measures, mean, variance, skewness, and kurtosis of Haq distribution. The distribution is skewed and leptokurtic. Figure 3 displays the plot of the index of dispersion. The DI is monotonically decreasing with parameter θ. We also noted that the proposed distribution is under-dispersed, equi-dispersed, and over-dispersed.

Table 1
Some Numerical Results of the Mean, Variance Skewness, and Kurtosis

θ	Mean	Variance	Skewness	Kurtosis
0.1	26.529	322.66	1.0831	4.7340
0.3	7.2781	34.998	1.2364	5.0024
0.5	3.7778	11.506	1.4267	5.6119
0.8	2.0216	3.8606	1.6603	6.6127
1.0	1.5000	2.2500	1.7778	7.2222
1.5	0.8800	0.8256	1.9685	8.4009
2.0	0.6111	0.4043	2.0635	9.1210
2.5	0.4653	0.2341	2.1064	9.5159
3.0	0.3750	0.1510	2.1220	9.7087
4.0	0.2700	0.0771	2.1191	9.7935
5.0	0.2111	0.0465	2.1027	9.7271
10.0	0.1017	0.0105	2.0429	9.3316

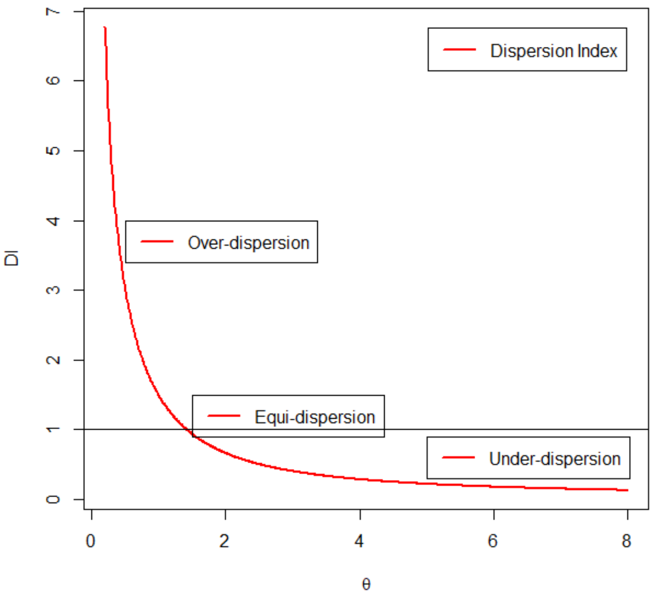

Figure 3: Index of Dispersion for Haq Distribution

3.2. Incomplete Moments

The incomplete moments can be obtained as

$$\varphi_r(t) = \int_0^t x^r f(x)dx$$

$$\begin{aligned}\varphi_r(t) &= \frac{\theta^2}{(1+\theta)^2}\int_0^t x^r\left(2+\theta+\frac{\theta x^2}{2}\right)e^{-\theta x}\,dx\\ &= \frac{\theta^2}{(1+\theta)^2}\left[(2+\theta)\int_0^t x^r e^{-\theta x}\,dx+\frac{\theta^2}{2}\int_0^t x^{r+2}e^{-\theta x}\,dx\right]\\ &= \frac{1}{(1+\theta)^2}\left[\frac{(1+\theta)^2}{\theta}\left\{t^r(t\theta)^{-r}\big(\Gamma(1+r)-\Gamma(1+r,t\theta)\big)\right\}\right.\\ &\qquad\left.+\frac{1}{2\theta}\left\{t^r(t\theta)^{-r}\big(\Gamma(3+r)-\Gamma(3+r,t\theta)\big)\right\}\right]\end{aligned}$$

$$\varphi_r(t) = \frac{t^r\theta(t\theta)^{-r}\begin{pmatrix}\Gamma(3+r)+2(1+\theta)^2\big(\Gamma(1+r)-\Gamma(1+r,t\theta)\big)\\ -\Gamma(3+r,t\theta)\end{pmatrix}}{2(1+\theta)^2}.$$

3.3. Moment Generating Function

The moment generating function of X can be obtained as

$$M_X(t) = \int_0^\infty e^{tx}f(x)dx$$

$$M_X(t) = \frac{\theta^2}{(1+\theta)^2}\left[(2+\theta)\int_0^\infty e^{-x(\theta-t)}\,dx+\frac{\theta}{2}\int_0^\infty x^2e^{-x(\theta-t)}\,dx\right]$$

$$M_X(t) = \frac{\theta^2}{(1+\theta)^2}\left[\frac{(2+\theta)}{(\theta-t)}+\frac{\theta}{(\theta-t)^3}\right].$$

4. RELIABILITY CHARACTERISTICS

In this section, we derived reliability properties including survival function, hazard rate, cumulative hazard rate, reversed hazard rate, mills ratio, mean residual life, and stress-strength reliability.

4.1. Survival, hazard, cumulative hazard, reversed hazard functions

The survival function (sf) of X is as follows

$$S(x) = P(X \geq x) = \frac{\left(1+2\theta+\theta^2+\theta x+\frac{(\theta x)^2}{2}\right)}{(1+\theta)^2}e^{-\theta x}. \qquad (10)$$

The hazard function (hrf) of X can be defined as

$$h(x) = P(X = x | X \geq x) = \frac{\theta^2 \left(2 + \theta + \frac{\theta x^2}{2}\right)}{\left(1 + 2\theta + \theta^2 + \theta x + \frac{(\theta x)^2}{2}\right)}. \quad (11)$$

Remarks on the Hazard Function

i) The limits of h(x) are;

- $\lim_{x \to 0} h(x) = \frac{\theta^2(2+\theta)}{(1+\theta)^2}$
- $\lim_{x \to \infty} h(x) = \theta$

ii) The hazard function is partially upward bathtub behavior

The proof follows from the first derivative of the hazard function

$$h'(x) = \frac{2\theta^3\big(x(2 + x\theta) - 2(2 + \theta)\big)}{(2 + 2(2 + x)\theta + (2 + x^2)\theta^2)^2}$$

As $h'(x) < 0$ at lower limit and $\lim_{x \to \infty} h(x) = \theta^2$ if follows the hazard rate of Haq distribution partially bathtub with the critical point

$$x_0 = \frac{-1 + \sqrt{1 + 4\theta + 2\theta^2}}{\theta}.$$

Figure 4 displays some possible shapes of hrf for selected parameter values.

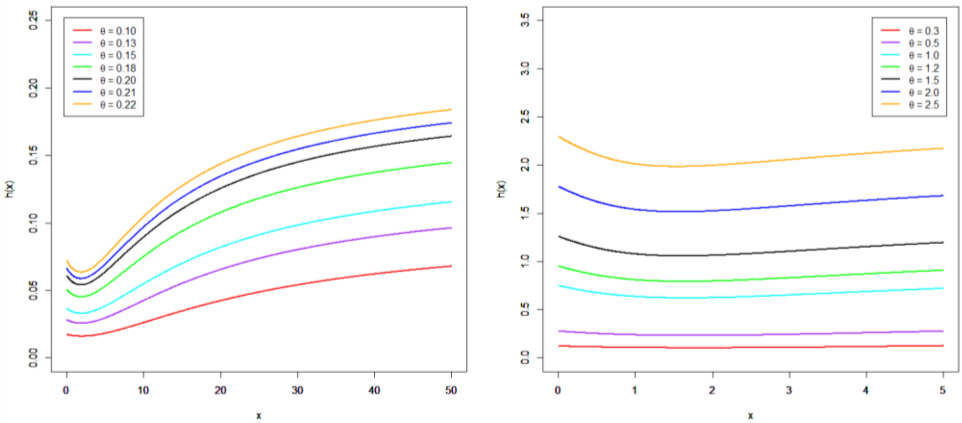

Figure 4: hrf Plots for Haq Distribution

The cumulative hazard function is

$$H(x) = -\log\big(S(x)\big) = -\log\left\{\frac{\left(1 + 2\theta + \theta^2 + \theta x + \frac{(\theta x)^2}{2}\right)}{(1 + \theta)^2} e^{-\theta x}\right\}. \quad (12)$$

The reversed hazard function of Haq distribution is

$$r(x)=\frac{f(x)}{F(x)}=\frac{\left(\frac{\theta}{1+\theta}\right)^2\left(2+\theta+\frac{\theta x^2}{2}\right)e^{-\theta x}}{1-\frac{\left(1+2\theta+\theta^2+\theta x+\frac{(\theta x)^2}{2}\right)}{(1+\theta)^2}e^{-\theta x}}. \tag{13}$$

The Mills ratio of X is

$$M=\frac{S(x)}{f(x)}=\frac{\left(1+2\theta+\theta^2+\theta x+\frac{(\theta x)^2}{2}\right)}{\theta^2\left(2+\theta+\frac{\theta x^2}{2}\right)}. \tag{14}$$

4.2. Stress Strength Reliability

Theorem 2:

If $X_1 \sim Haq(X;\theta_1)$ *and* $X_2 \sim Haq(X;\theta_2)$ *be independent random variables, the stress-strength reliability is*

$$R=1-\frac{\theta^2}{(1+\theta)^2(1+\alpha)^2}\left\{\frac{(2+\theta)(1+\alpha)^2}{(\alpha+\theta)}+\frac{\alpha(2+\theta)}{(\alpha+\theta)^2}+\frac{\alpha^2(2+\theta)}{(\alpha+\theta)^3}+\frac{\theta(1+\alpha)^2}{(\alpha+\theta)^3}+\frac{3\alpha\theta}{(\alpha+\theta)^4}+\frac{6\theta\alpha^2}{(\alpha+\theta)^5}\right\}.$$

Proof:

The stress strength reliability of X is given by

$$R=P(X_1<X_2)=\int_0^\infty f_1(x)\,F_2(x)dx$$

Using Eq. (1) and Eq. (2), R can be written as

$$R=\int_0^\infty\left(\frac{\theta_1}{1+\theta_1}\right)^2\left(2+\theta_1+\frac{\theta_1x^2}{2}\right)e^{-\theta_1x}\left\{1-\frac{\left((1+\theta_2)^2+\theta_2x+\frac{(\theta_2x)^2}{2}\right)}{(1+\theta_2)^2}e^{-\theta_2x}\right\}dx$$

Hence, it reduces to

$$R = 1 - \frac{{\theta_1}^2}{(1+\theta_1)^2(1+\theta_2)^2}\Bigg[(2+\theta_1)(1+\theta_2)^2\int_0^\infty e^{-x(\theta_1+\theta_2)}\,dx + (2+\theta_1)\theta_2\int_0^\infty xe^{-x(\theta_1+\theta_2)}\,dx + \frac{(2+\theta_1){\theta_2}^2}{2}\int_0^\infty xe^{-x(\theta_1+\theta_2)}\,dx + \frac{(1+\theta_2)^2\theta_1}{2}\int_0^\infty x^2e^{-x(\theta_1+\theta_2)}\,dx + \frac{\theta_2\theta_1}{2}\int_0^\infty x^3e^{-x(\theta_1+\theta_2)}\,dx + \frac{{\theta_2}^2\theta_1}{4}\int_0^\infty x^4e^{-x(\theta_1+\theta_2)}\,dx\Bigg]$$

After simple computations, we get the proof of the theorem.

4.3. Mean Residual Life Function

Theorem 3:

If $X \sim Haq(X;\theta)$, with $\theta > 0$ then the mean residual life function of X is given by

$$m(t) = \frac{t(2+t^2)\theta^3 + (2+4t+5t^2)\theta^2 + 4(1+3t)\theta + 12}{2\theta^2\left(1+2\theta+t\theta+\theta^2+\frac{t^2\theta^2}{2}\right)} - t$$

Proof:

$$m(t) = E(X-t|X>t) = \frac{\int_t^\infty xf(x)dx}{S(t)} - t \quad for \quad t>0$$

We have

$$\int_t^\infty xf(x)dx = \frac{1}{(1+\theta)^2}\int_t^\infty x\left((1+\theta)^2 + \theta x + \frac{(\theta x)^2}{2}\right)e^{-\theta x}\,dx$$

On simplification, the above equation takes the form

$$\int_t^\infty xf(x)dx = \frac{1}{(1+\theta)^2}\left[(1+\theta)^2\frac{e^{-t\theta}(1+t\theta)}{\theta^2} + \theta\frac{e^{-t\theta}\big(2+t\theta(2+t\theta)\big)}{\theta^3} + \frac{\theta^2}{2}\frac{e^{-t\theta}\Big(6+t\theta\big(6+t\theta(3+t\theta)\big)\Big)}{\theta^4}\right]$$

$$\int_t^\infty xf(x)dx = \frac{e^{-t\theta}(12+4(1+3t)\theta + (2+4t+5t^2)\theta^2 + t(2+t^2)\theta^3)}{2\theta^2(1+\theta)^2}$$

Hence, the simplified form of $m(t)$ is as follows

$$m(t) = \frac{t(2+t^2)\theta^3 + (2+4t+5t^2)\theta^2 + 4(1+3t)\theta + 12}{2\theta^2\left(1+2\theta+t\theta+\theta^2+\frac{t^2\theta^2}{2}\right)} - t. \qquad (15)$$

5. ESTIMATION

In this section, we estimate the parameter using five frequently used estimation methods via complete samples.

5.1. Maximum Likelihood Estimation

The log-likelihood function of a random sample from Haq distribution is

$$l(\theta) = 2n\log(\theta) - 2n\log(1+\theta) + \sum_{i=1}^{n}\log\left(2+\theta+\frac{\theta {x_i}^2}{2}\right) - \theta\sum_{i=1}^{n}x_i \quad (16)$$

By taking the partial derivative of the above Equation (16), we get

$$\frac{\partial l(\theta)}{\partial\theta} = \frac{2n}{\theta} - \frac{2n}{1+\theta} + \sum_{i=1}^{n}\frac{\left(1+\frac{{x_i}^2}{2}\right)}{\left(2+\theta+\frac{\theta {x_i}^2}{2}\right)} - \sum_{i=1}^{n}x_i \quad (17)$$

The ML estimate of parameter θ of the Haq distribution can be obtained by maximizing the above Eq. (17). The solution of the above equation is not possible in a closed form, so by using numerical computation, the solution of the log-likelihood will provide the MLE of θ.

5.2. Method of Moments

While using the method of moments (MOM) for the parameter of Haq distribution, we have to first equate the population moment to the corresponding sample moment and then solve the non-linear equation

$$\frac{\theta^2+2\theta+3}{\theta(1+\theta)^2} = \frac{1}{n}\sum_{i=1}^{n}x_i \quad (18)$$

with respect to θ. The **uniroot** function using R (R Core Team, 2020) can be used.

5.3. Least Squares Estimator

Let $x_{(1)}, x_{(2)}, x_{(3)}, \ldots, x_{(n)}$ be the ordered statistics sample of size n. The Least Square estimator can be obtained by minimizing

$$LS(\theta) = \sum_{i=1}^{n}\left[1 - \frac{\left(1+2\theta+\theta^2+\theta x+\frac{(\theta x)^2}{2}\right)}{(1+\theta)^2}e^{-\theta x} - \frac{i}{n+1}\right]^2 \quad (19)$$

with respect to θ.

5.4. Anderson Darling Estimator

The Anderson Darling (AD) estimator is a minimum distance-based estimator. The AD estimator of the Haq distribution parameter $\hat{\theta}$ can be obtained by minimizing the below expression with respect to θ.

$$AD(\theta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log\left(F\left(x_{(i)}|\theta\right)\right) + \log\left(1-F\left(x_{(i)}|\theta\right)\right)\right] \quad (20)$$

5.5. Crammer Von-Misses Estimator

The Cramer-von Misses (CVM) estimator is a minimum distance-based estimator frequently used for the estimation of parameters. The CVM estimator of the Haq distribution parameter $\hat{\theta}$ can be obtained by minimizing the below expression with respect to θ.

$$CVM(\theta) = \frac{1}{12n} + \sum_{i=1}^{n}\left[\log\left(F\left(x_{(i)}|\theta\right)\right) - \frac{2i-1}{2n}\right]^2 \tag{21}$$

5.6. Simulation Study

In this section, we conducted a simulation study to evaluate the accuracy of the five estimators discussed in Section 4. We generated samples of size n=25, 50, 100, 200, and 500 for the Haq distribution and then calculated the average values of the MLE, MM, AD, CVM, and LSE of θ, average absolute biases (ABBs), mean square errors (MSEs) and mean relative errors (MREs) when θ=0.5, 1.0 and 2.5. Each sample is repeated 10,000 times. The ABBs, MREs and MSEs are given by

$$ABBs(\theta) = \frac{1}{N}\sum_{i=1}^{N}\left|\hat{\theta} - \theta\right|, MREs(\theta) = \frac{1}{N}\sum_{i=1}^{N}\frac{\left|\hat{\theta} - \theta\right|}{\theta} \;\&\; MSEs(\theta) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta} - \theta\right)^2$$

The results reported in Tables 2–4 are computed using the **mle** library routine of R software.

Table 2
Simulation Results for Haq Distribution when $\boldsymbol{\theta = 0.5}$

n	Est.	MLE	MME	ADE	CVME	LSE
25	AVEs	0.5656	0.5709	0.5042	0.5075	0.5678
	ABSs	0.0656	0.0709	0.0042	0.0075	0.0678
	MREs	0.1675	0.0868	0.1169	0.1258	0.1858
	MSEs	0.0121	0.0127	0.0056	0.0067	0.0163
50	AVEs	0.5585	0.5627	0.5037	0.5050	0.5633
	ABSs	0.0585	0.0627	0.0037	0.0050	0.0633
	MREs	0.1348	0.0709	0.0819	0.0915	0.1475
	MSEs	0.0073	0.0079	0.0028	0.0034	0.0094
100	AVEs	0.5568	0.5579	0.5029	0.5031	0.5619
	ABSs	0.0568	0.0579	0.0029	0.0031	0.0619
	MREs	0.1196	0.0608	0.0616	0.0599	0.1308
	MSEs	0.0051	0.0054	0.0015	0.0015	0.0062
200	AVEs	0.5538	0.5539	0.4995	0.5006	0.5610
	ABSs	0.0538	0.0539	0.0005	0.0006	0.0610
	MREs	0.1091	0.0544	0.0416	0.0419	0.1235
	MSEs	0.0039	0.0038	0.0007	0.0007	0.0049
500	AVEs	0.5518	0.5543	0.5001	0.5001	0.5603
	ABSs	0.0518	0.0543	0.0001	0.0001	0.0603
	MREs	0.1035	0.0543	0.0257	0.0277	0.1207
	MSEs	0.0030	0.0033	0.0003	0.0003	0.0041

Table 3
Simulation Results for Haq Distribution when $\theta = 1.0$

n	Est.	MLE	MME	ADE	CVME	LSE
25	AVEs	1.0247	1.3807	1.0121	1.0331	1.2268
	ABSs	0.0247	0.3807	0.0121	0.0331	0.2268
	MREs	0.1229	0.3915	0.1304	0.1505	0.2531
	MSEs	0.0254	0.2557	0.0295	0.0413	0.1011
50	AVEs	1.0093	1.3416	1.0104	1.0083	1.2329
	ABSs	0.0093	0.3416	0.0104	0.0083	0.2329
	MREs	0.0844	0.3441	0.0936	0.1022	0.2392
	MSEs	0.0116	0.1622	0.0142	0.0176	0.0810
100	AVEs	1.0028	1.3247	1.0022	1.0009	1.2314
	ABSs	0.0028	0.3247	0.0022	0.0009	0.2314
	MREs	0.0604	0.3251	0.0646	0.0695	0.2322
	MSEs	0.0057	0.1262	0.0066	0.0075	0.0672
200	AVEs	1.0054	1.3048	1.0010	1.0016	1.2361
	ABSs	0.0054	0.3048	0.0010	0.0016	0.2361
	MREs	0.0407	0.3048	0.0455	0.0476	0.2361
	MSEs	0.0027	0.1024	0.0033	0.0035	0.0626
500	AVEs	1.0021	1.3025	1.0011	1.0016	1.2348
	ABSs	0.0021	0.3025	0.0011	0.0016	0.2348
	MREs	0.0266	0.3025	0.0291	0.0313	0.2348
	MSEs	0.0011	0.0956	0.0013	0.0016	0.0578

Table 4
Simulation Results for Haq Distribution when $\theta = 1.5$

n	Est.	MLE	MME	ADE	CVME	LSE
25	AVEs	1.5430	2.7320	1.5490	1.5231	1.8649
	ABSs	0.0430	1.2320	0.0490	0.0231	0.3649
	MREs	0.1396	1.2372	0.1512	0.1498	0.2608
	MSEs	0.0836	2.6172	0.0888	0.0921	0.2488
50	AVEs	1.5198	2.5619	1.5083	1.5237	1.8822
	ABSs	0.0198	1.0619	0.0083	0.0237	0.3822
	MREs	0.0908	1.0620	0.0948	0.1086	0.2579
	MSEs	0.0300	1.4805	0.0328	0.0443	0.1994
100	AVEs	1.5113	2.4734	1.5050	1.5143	1.8894
	ABSs	0.0113	0.9734	0.0050	0.0143	0.3894
	MREs	0.0653	0.9734	0.0724	0.0758	0.2599
	MSEs	0.0148	1.0945	0.0187	0.0211	0.1813
200	AVEs	1.5028	2.4758	1.5010	1.5075	1.8889
	ABSs	0.0028	0.9758	0.0010	0.0075	0.3889
	MREs	0.0435	0.9758	0.0490	0.0518	0.2593
	MSEs	0.0067	1.0182	0.0085	0.0099	0.1651
500	AVEs	1.5045	2.4478	1.4978	1.4994	1.8924
	ABSs	0.0045	0.9478	0.0022	0.0006	0.3924
	MREs	0.0290	0.9478	0.0311	0.0323	0.2616
	MSEs	0.0031	0.9242	0.0035	0.0038	0.1602

Tables 3–5 show that the AVEs are closer to the true value of parameters when the sample size n increases. Further, for all estimators, ABBs, MREs, and MSEs decrease as the sample size increases.

6. APPLICATION

In this part, we compare the Haq distribution to Xgamma, Lindley, and new extended gamma (NEG) distributions using three data sets.

Data Set I:

The first data set is about the total annual rainfall (in inches) from January 1880 to 1916 recorded at Los Angeles Civic Center (Selim, 2018). The observations are:

1.33, 1.43, 1.01, 1.62, 3.15, 1.05, 7.72, 0.2, 6.03, 0.25, 7.83, 0.25, 0.88, 6.29, 0.94, 5.84, 3.23, 3.7, 1.26, 2.64, 1.17, 2.49, 1.62, 2.1, 0.14, 2.57, 3.85, 7.02, 5.04, 7.27, 1.53, 6.7, 0.07, 2.01, 10.35, 5.42, and 13.3.

Data Set III:

The third data set is the accelerated life testing of 40 items with a change in stress from 100 to 150 at time = 15 (Murthy, Xie, & Jiang, 2004). The observations are:

0.13, 0.62, 0.75, 0.87, 1.56, 2.28, 3.15, 3.25, 3.55, 4.49, 4.50, 4.61, 4.79, 7.17, 7.31, 7.43, 7.84, 8.49, 8.94, 9.40, 9.61, 9.84, 10.58, 11.18, 11.84, 13.28, 14.47, 14.79, 15.54, 16.90, 17.25, 17.37, 18.69, 18.78, 19.88, 20.06, 20.10, 20.95, 21.72, and 23.87.

Data Set IV:

The fourth data set is about censored tri-modal data containing 30 items that are tested when the test is stopped after the 20th failure (Murthy et al., 2004). The observations are:

0.032, 0.035, 0.104, 0.169, 0.196, 0.260, 0.326, 0.445, 0.449, 0.496, 0.543, 0.544, 0.577, 0.648, 0.666, 0.742, 0.757, 0.808, 0.857, 0.858, 0.882, 1.005, 1.025, 1.472, 1.916, 2.313, 2.457, 2.530, 2.543, 2.617, 2.835, 2.940, 3.002, 3.158, 3.430, 3.459, 3.502, 3.691, 3.861, 3.952, 4.396, 4.744, 5.346, 5.479, 5.716, 5.825, 5.847, 6.084, 6.127, 7.241, 7.560, 8.901, 9.000, 10.482, and 11.133.

The descriptive measures for all data sets are presented in Table 5. The boxplot is also presented in Figure 5 to describe the nature of the datasets.

Table 5
Some Summary Statistics of the Data Sets

Data	Min.	Q1	Q2	Q3	Mean	Max.	Var.	Skewness	Kurtosis
Set I	0.0700	1.1700	2.4900	5.8400	3.4950	13.300	10.009	1.1525	3.8905
Set II	0.1300	4.4970	9.5050	16.988	10.446	23.870	48.862	0.2208	1.8103
Set III	0.0320	0.6570	2.5300	4.5700	3.0180	11.133	8.1484	1.0657	3.4249

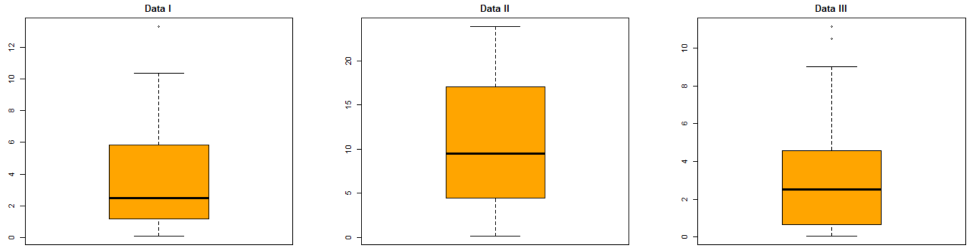

Figure 5: Boxplots for Data Sets

The TTT (Total Test Time) plots (Aarset, 1987) for these data sets are presented in Figure 5. The purpose of the TTT plot is to identify the failure rate behavior of data sets, increasing, decreasing, or constant hazard rate.

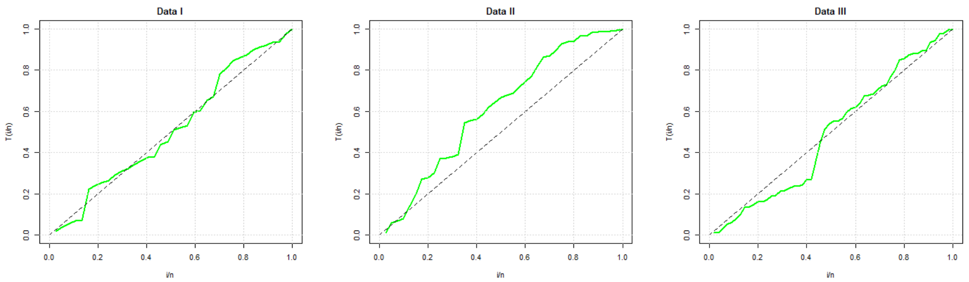

Figure 6: TTT Plots for Data Sets

Table 6 provides the maximum likelihood estimates for each considered distribution. For the selection of the best-fitted distribution, certain discrimination measures are considered. The discrimination measures:

- The Akaike information criterion (AIC):
  $AIC = 2k - 2l$
- The Bayesian information criterion (BIC)
  $BIC = k\log(n) - 2l$

where l represents the log-likelihood function, and k is the number of distribution parameters.

The maximum value of log-likelihood and minimum values of AIC and BIC represents the best-fitted distribution. Figures 6 and 7 show a graphic representation of the competing models. For this purpose, we plotted the density function, cumulative distribution function, and probability-probability (P-P) functions.

Table 6
The Estimated Values and Model Selection Values for each Data Set

Data Set	Model	ML Estimate	LogLik.	AIC	BIC
I	Haq	$\hat{\theta} = 0.52967$	**-82.970**	**167.94**	**169.55**
	Xgamma	$\hat{\theta} = 0.62612$	-83.740	169.48	171.09
	Lindley	$\hat{\theta} = 0.47956$	-83.942	169.88	171.49
	NEG	$\hat{\theta} = 0.46941$ $\hat{\alpha} = 1.08880$	-83.011	168.02	169.63
III	Haq	$\hat{\theta} = 0.22459$	**-130.35**	**262.69**	**264.38**
	Xgamma	$\hat{\theta} = 0.24653$	-130.61	263.22	264.91
	Lindley	$\hat{\theta} = 0.17706$	-131.96	265.92	267.61
	NEG	$\hat{\theta} = 0.18852$ $\hat{\alpha} = 1.49748$	-132.05	266.11	267.79
IV	Haq	$\hat{\theta} = 0.59177$	**-114.78**	**231.56**	**233.24**
	Xgamma	$\hat{\theta} = 0.70026$	-116.41	234.83	236.51
	Lindley	$\hat{\theta} = 0.54573$	-118.11	238.21	239.90
	NEG	$\hat{\theta} = 0.51367$ $\hat{\alpha} = 0.90689$	-114.81	231.61	233.30

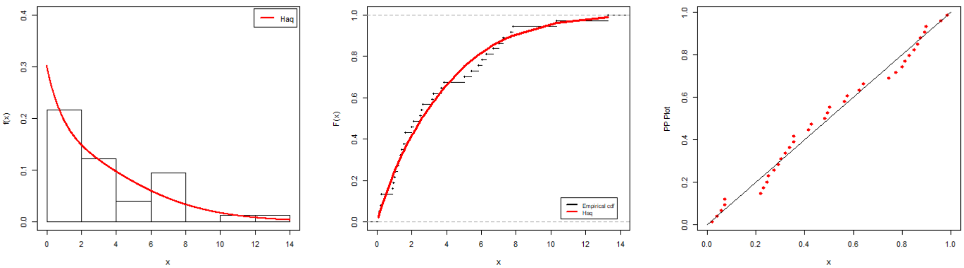

Figure 7: Empirical and Fitted pdf, cdf, and P-P Plots for the First Data Set

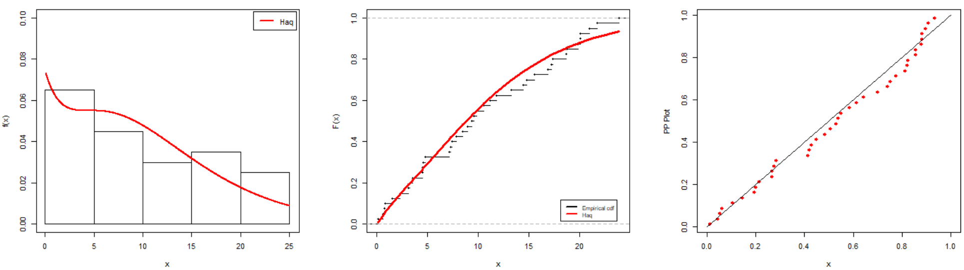

Figure 8: Empirical and Fitted pdf, cdf, and P-P Plots for the Second Data Set

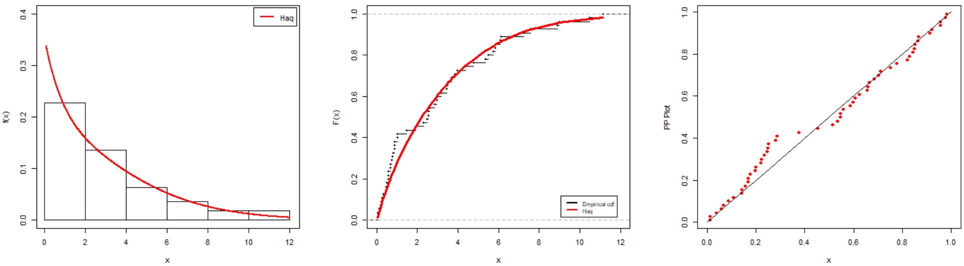

Figure 9: Empirical and Fitted pdf, cdf, and P-P Plots for the Third Data Set

According to Table 6, the Haq distribution has the highest log-likelihood and the lowest AIC and BIC discriminatory values for both datasets. Figures 6-7 show that the Haq distribution fits the failure times and reanalysis of oral irrigator datasets well. As evidenced by applications to real-world data sets, the Haq distribution outperforms the Xgamma, Lindley, and new extended gamma distributions. The Haq distribution provides a new avenue for modeling real-world data sets.

7. CONCLUSION

A new one-parameter lifetime distribution called “Haq” is introduced for modeling failure time datasets. The proposed distribution is a mixture of xgamma and exponential distributions. Some of the mathematical properties including the moments, shape of the density function, mode, dispersion index, incomplete moments, and moment generating function are obtained. Further, reliability properties including hazard function, the shape of hazard rate, cumulative hazard rate, reversed hazard rate, mills ratio, mean residual life, and stress strength reliability are derived. Five frequentist methods; maximum likelihood, least squares, method of moments, Anderson darling and Cramer-von Misses are used for estimation of the model parameter. The selection of efficient methods is varied via a simulation study. Two real datasets are analyzed to show the flexibility of the Haq distribution against xgamma and Lindley distributions. Based on the model adequacy measures proposed distribution provides better fits.

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