Pak. J. Statist.
2022 Vol. 38(3), 359-376

ON MIXTURE OF MAXWELL AND WEIGHTED MAXWELL DISTRIBUTIONS: PROPERTIES, ESTIMATION AND APPLICATIONS

A. A. Bhat[§] and **S.P. Ahmad**
Department of Statistics, University of Kashmir, Srinagar, India.
Email: bhatajaz.msst@gmail.com
sprvz@yahoo.com
[§] Corresponding author

ABSTRACT

It is well established that classical distributions lack the flexibility to model the characteristics of a complex random phenomenon. This fact motivates clever generalizations of these distributions by applying various mathematical schemes. In this article, we propose a continuous mixture distribution which is obtained by mixing Maxwell distribution and Weighted Maxwell distribution. The proposed distribution is named as Mixture of Maxwell and Weighted Maxwell distributions. Various statistical properties of the proposed model such as reliability analysis, moments, moment generating function, Renyi entropy and order statistics are presented. The MLE technique has been implemented to estimate the parameters. Finally, two real life data sets have been considered to check the flexibility and usefulness of the distribution.

KEYWORDS

Maxwell Distribution, Weighted Maxwell Distribution, Mixture Distribution, Reliability Analysis, Renyi Entropy and real life data.

1. INTRODUCTION

The Maxwell distribution was first introduced by James Maxwell (1880) and gave the distribution of velocities among the molecules of a gas. In reliability theory, the one parameter Maxwell distribution was first studied by Tyagi and Bhattacharya (1989). Thereafter, numerous researchers have worked on this distribution in various fields of science and technology. Chaturvedi and Rani (1998) generalized the one parameter Maxwell distribution by introducing one more parameter and estimated the reliability function using classical and Bayesian estimation procedure. Krishna and Malik (2012) obtained the reliability estimate of the Maxwell distribution under type II and progressive type II censoring schemes. Kazmi et al. (2012) estimated the parameters of Maxwell distribution using the Bayes estimates under different types of loss functions.

A random variable X is said to have Maxwell distribution with scale parameter θ, denoted by $X \sim MD(\theta)$, if the probability density function (PDF) and cumulative distribution function (CDF) are of the form

$$m_1(x;\theta) = \sqrt{\frac{2}{\pi}}\theta^{\frac{3}{2}}x^2 exp\left(-\frac{\theta x^2}{2}\right); x > 0, \theta > 0, \tag{1}$$

$$M_1(x;\theta) = \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{\theta x^2}{2}\right).$$

where $\gamma(a,x) = \int_0^x x^{a-1}\, e^{-x} dx$ is the lower incomplete gamma function.

However, the use of Maxwell distribution is restricted to the situation of increasing failure rate which decreases its flexibility and compatibility to model certain lifetime phenomena. This distribution can be used for modeling positively skewed data sets, but inability to model lifetime data that have both positively and negatively skewed. For this reason, many researchers have developed some extensions of Maxwell distribution by applying different generalization techniques. For instance, Amusan (2010) extended Maxwell distribution by adding two shape parameters using Beta-G family of distributions. Sharma et al. (2016a) generalized classical Maxwell distribution using T-X family of distributions introduced by Alzaatreh et al. (2013) by using Weibull as baseline distribution. Singh et al. (2018) introduced power Maxwell distribution and discussed its statistical properties and estimation techniques. Chaturvedi and Vyas (2019) developed Gamma-Maxwell distribution by using Gamma-G generator. Mathew and Chesneau (2020) generalized length-biased Maxwell distribution by using Marshall-Olkin scheme of generalization. Dar et al. (2018) studied the characterization and estimation of weighted Maxwell distribution (WMD) by taking weight function $w(x) = x^{\omega}$, where $\omega > 0$ is the weight parameter. The PDF and CDF of weighted Maxwell distribution, denoted by $X \sim WMD(\theta,\omega)$, are respectively given as

$$m_2(x;\theta,\omega) = \frac{\theta^{\left(\frac{\omega+3}{2}\right)}x^{(\omega+2)}exp\left(-\frac{\theta x^2}{2}\right)}{2^{\left(\frac{\omega+1}{2}\right)}\Gamma\left(\frac{\omega+3}{2}\right)}; x > 0, \theta, \omega > 0, \tag{2}$$

$$M_2(x;\theta,\omega) = \frac{\gamma\left(\left(\frac{\omega+3}{2}\right),\frac{\theta x^2}{2}\right)}{\Gamma\left(\frac{\omega+3}{2}\right)}.$$

Sometimes, a single probability model may not provide us the desired outcome when the observed data may be assumed to have obtained from a mixture of two or more populations. In such situations, the finite mixture of some known and suitable probability models are very much flexible and effective to analyze the different failure pattern of sub-populations. The finite mixture distributions have wide range of applications in various fields of research such as medicine, finance, psychology, reliability and life testing among others. The mixture distribution was first considered by Pearson (1894) while estimated the parameters of the two component mixture normal distribution. The numbers of researchers such as McLachlan and Peel (2004), Cordeiro et al. (2012), Marco et al. (2013), Bhat et al. (2018), Iqbal and Iqbal (2020), Bhat et al. (2021) and Satsayamon Suksaengrakcharoen and Winai Bodhisuwan (2014) have developed mixture type of distributions and estimate the parameters of the distributions using different techniques.

In the present work, a finite mixture of Maxwell distribution and weighted Maxwell distribution is developed. The main objective of this research article is to study the mathematical properties of the proposed model and to estimate the unknown parameters of the model. Other motivation of this study regarding the advantages comes from its flexibility to model the variety of data sets.

2. PROPOSED DISTRIBUTION

Let us consider a two component mixture of Maxwell distribution and weighted Maxwell distribution with their mixing proportions α and $(1-\alpha)$ respectively. Then, the PDF of a two component mixture distribution is expressed as

$$m(x) = \alpha m_1(x) + (1-\alpha) m_2(x), \tag{3}$$

where $m_k(x), k = 1,2$ are the two PDF's involved in the mixture distribution. Inserting (1) and (2) in (3), the PDF of the new distribution is obtained as

$$m(x;\alpha,\omega,\theta) = \theta^{\frac{3}{2}} x^2 exp\left(-\frac{\theta x^2}{2}\right)\left\{\alpha\sqrt{\frac{2}{\pi}} + \frac{(1-\alpha)\theta^{\frac{\omega}{2}} x^{\omega}}{2^{\left(\frac{\omega+1}{2}\right)}\Gamma\left(\frac{\omega+3}{2}\right)}\right\}. \tag{4}$$

The probability density function (4) is called the mixture of Maxwell and weighted Maxwell distribution, denoted by $MMWM(\alpha,\omega,\theta)$ distribution. The CDF corresponds to (4) can be obtained as

$$M(x;\alpha,\omega,\theta) = \frac{2\alpha}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{\theta x^2}{2}\right) + (1-\alpha)\frac{\gamma\left(\left(\frac{\omega+3}{2}\right),\frac{\theta x^2}{2}\right)}{\Gamma\left(\frac{\omega+3}{2}\right)}. \tag{5}$$

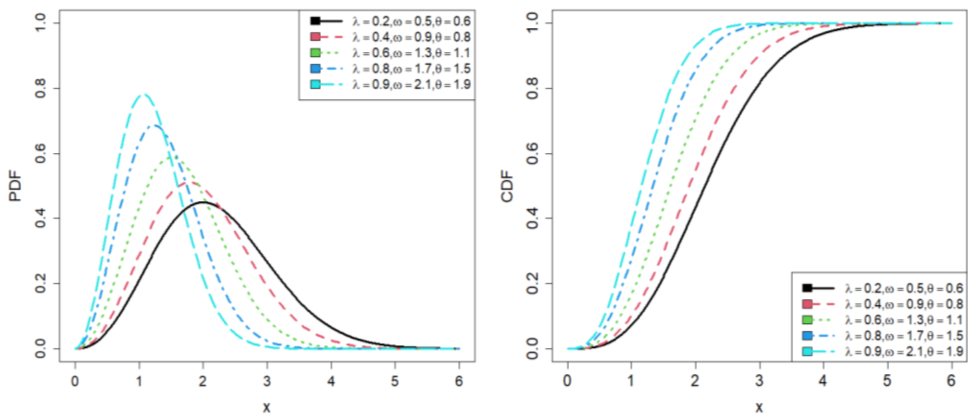

Figure 1: PDF and CDF Plots of the $MMWM(\alpha,\omega,\theta)$ Distribution for Different Values of Parameters

Table 1
Sub-Models of the $MMWM(\alpha, \omega, \theta)$ Distribution

Model	Parameter Restriction	PDF	Abbr.
Maxwell Distribution	$\alpha = 1$ $\omega = 0$	$\sqrt{\frac{2}{\pi}}\theta^{\frac{3}{2}}x^2 exp\left(-\frac{\theta x^2}{2}\right)$	$MD(\theta)$
Weighted Maxwell Distribution	$\alpha = 1$	$\frac{\theta^{\left(\frac{\omega+3}{2}\right)}x^{(\omega+2)}exp\left(-\frac{\theta x^2}{2}\right)}{2^{\left(\frac{\omega+1}{2}\right)}\Gamma\left(\frac{\omega+3}{2}\right)}$	$WMD(\omega, \theta)$
Length Biased Maxwell Distribution	$\alpha = 0$ $\omega = 1$	$\frac{\theta^2 x^3 exp\left(-\frac{\theta x^2}{2}\right)}{2}$	$LBMD(\theta)$
Area Biased Maxwell Distribution	$\alpha = 0$ $\omega = 2$	$\frac{1}{3}\sqrt{\frac{2}{\pi}}\theta^{\frac{5}{2}}x^4 exp\left(-\frac{\theta x^2}{2}\right)$	$ABMD(\theta)$

3. STATISTICAL PROPERTIES

In this section, the various statistical properties of $MMWM(\alpha, \omega, \theta)$ distribution are unfolded.

3.1 Reliability and Hazard Functions

The characteristics based on reliability function and failure rate function is very helpful to study the behavior of any lifetime distribution. The expression for reliability $R(x; \alpha, \omega, \theta)$ and failure rate function $H(x; \alpha, \omega, \theta)$ of the proposed distribution are respectively given as

$$R(x; \alpha, \omega, \theta) = 1 - \left\{\frac{2\alpha}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{\theta x^2}{2}\right) + (1-\alpha)\frac{\gamma\left(\left(\frac{\omega+3}{2}\right), \frac{\theta x^2}{2}\right)}{\Gamma\left(\frac{\omega+3}{2}\right)}\right\}.$$

$$H(x; \alpha, \omega, \theta) = \frac{\theta^{\frac{3}{2}}x^2 exp\left(-\frac{\theta x^2}{2}\right)\left\{\alpha\sqrt{\frac{2}{\pi}} + \frac{(1-\alpha)\theta^{\frac{\omega}{2}}x^{\omega}}{2^{\left(\frac{\omega+1}{2}\right)}\Gamma\left(\frac{\omega+3}{2}\right)}\right\}}{1 - \left\{\frac{2\alpha}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{\theta x^2}{2}\right) + (1-\alpha)\frac{\gamma\left(\left(\frac{\omega+3}{2}\right), \frac{\theta x^2}{2}\right)}{\Gamma\left(\frac{\omega+3}{2}\right)}\right\}}.$$

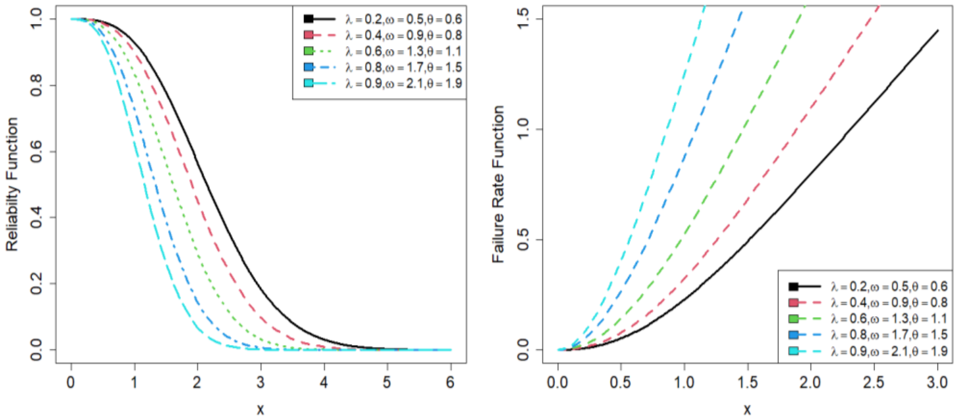

Figure 2: Reliability Function and Failure Rate Function Plots of the $\boldsymbol{MMWM(\alpha,\omega,\theta)}$ Distribution

The reverse hazard rate $r(x;\alpha,\omega,\theta)$ is obtained as

$$r(x;\alpha,\omega,\theta)=\frac{\theta^{\frac{3}{2}}x^2 exp\left(-\frac{\theta x^2}{2}\right)\left\{\alpha\sqrt{\frac{2}{\pi}}+\frac{(1-\alpha)\theta^{\frac{\omega}{2}}x^{\omega}}{2^{\left(\frac{\omega+1}{2}\right)}\Gamma\left(\frac{\omega+3}{2}\right)}\right\}}{\frac{2\alpha}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{\theta x^2}{2}\right)+(1-\alpha)\frac{\gamma\left(\left(\frac{\omega+3}{2}\right),\frac{\theta x^2}{2}\right)}{\Gamma\left(\frac{\omega+3}{2}\right)}}.$$

The cumulative hazard rate $\Delta(x;\alpha,\omega,\theta)$ is expressed as

$$\Delta(x;\alpha,\omega,\theta)=-log\left[1-\left\{\frac{2\alpha}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{\theta x^2}{2}\right)+(1-\alpha)\frac{\gamma\left(\left(\frac{\omega+3}{2}\right),\frac{\theta x^2}{2}\right)}{\Gamma\left(\frac{\omega+3}{2}\right)}\right\}\right].$$

The odds function is expressed as

$$O(x;\alpha,\omega,\theta)=\frac{\frac{2\alpha}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{\theta x^2}{2}\right)+(1-\alpha)\frac{\gamma\left(\left(\frac{\omega+3}{2}\right),\frac{\theta x^2}{2}\right)}{\Gamma\left(\frac{\omega+3}{2}\right)}}{1-\left\{\frac{2\alpha}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{\theta x^2}{2}\right)+(1-\alpha)\frac{\gamma\left(\left(\frac{\omega+3}{2}\right),\frac{\theta x^2}{2}\right)}{\Gamma\left(\frac{\omega+3}{2}\right)}\right\}}.$$

3.2 Ordinary Moments and Related Measures

Moments are very useful to understand the important features of the probability distribution including central tendency, dispersion, skewness and kurtosis. The s^{th} moment about origin is given by

$$\mu_s' = \int_{x=0}^{\infty} x^s\, m(x;\alpha,\omega,\theta)dx,$$

$$\mu_s' = \left(\frac{2}{\theta}\right)^{\frac{s}{2}} \left\{\frac{\alpha\Gamma\left(\frac{s+3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right) + (1-\alpha)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{s+\omega+3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)}\right\}. \qquad (6)$$

The first four non-central moments (moments about origin) are obtained by putting $s = 1, 2, 3$ and 4 in the above equation. If $s = 1$, we get the mean of the proposed distribution as

$$\mu_1' = \left(\frac{2}{\theta}\right)^{\frac{1}{2}} \left\{\frac{\alpha\Gamma\left(\frac{\omega+3}{2}\right) + (1-\alpha)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+4}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)}\right\}$$

For $s = 2, 3$ and 4, we have

$$\mu_2' = \frac{2}{\theta} \left\{\frac{\alpha\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right) + (1-\alpha)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+5}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)}\right\}$$

$$\mu_3' = \left(\frac{2}{\theta}\right)^{\frac{3}{2}} \left\{\frac{2\alpha\Gamma\left(\frac{\omega+3}{2}\right) + (1-\alpha)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+6}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)}\right\}$$

$$\mu_4' = \left(\frac{2}{\theta}\right)^{2} \left\{\frac{\alpha\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right) + (1-\alpha)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+7}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)}\right\}$$

The respective central moments (moments about mean) can be evaluated by using the relationship between moments about origin and moments about mean. The variance of the proposed model is obtained as

$$\mu_2 = \frac{2}{\theta}\left[\left\{\frac{\alpha\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right) + (1-\alpha)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+5}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)}\right\} - \left\{\frac{\alpha\Gamma\left(\frac{\omega+3}{2}\right) + (1-\alpha)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+4}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)}\right\}^2\right]. \qquad (7)$$

3.3 Co-efficient of Skewness and Co-efficient of Kurtosis

The co-efficient of skewness and kurtosis are given by

$$\mu_2 = \frac{2}{\theta}\left[\left\{\frac{\alpha\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)+(1-\alpha)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+5}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)}\right\} - \left\{\frac{\alpha\Gamma\left(\frac{\omega+3}{2}\right)+(1-\alpha)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+4}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)}\right\}^2\right]\mu_2$$

$$= \frac{2}{\theta}\left[\left\{\frac{\alpha\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)+(1-\alpha)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+5}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)}\right\} - \left\{\frac{\alpha\Gamma\left(\frac{\omega+3}{2}\right)+(1-\alpha)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+4}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)}\right\}^2\right]\boldsymbol{\gamma_1}$$

$$= \frac{\{n(\alpha,\omega,3)-3n(\alpha,\omega,2)n(\alpha,\omega,1)+2n^3(\alpha,\omega,1)\}}{N^3}.$$

$$\gamma_2 = \frac{\left\{\begin{array}{c} n(\alpha,\omega,4)-4n(\alpha,\omega,3)n(\alpha,\omega,1) \\ +6n(\alpha,\omega,2)n^2(\alpha,\omega,1)-3n^4(\alpha,\omega,1)\end{array}\right\}}{N^4} - 3.$$

where $N=\sqrt{n(\alpha,\omega,2)-n^2(\alpha,\omega,1)}$.

3.4 Co-efficient of Variation

The co-efficient of variation is given by

$$C.V = \frac{\sqrt{n(\alpha,\omega,2)-n^2(\alpha,\omega,1)}}{n(\alpha,\omega,1)}.$$

where

$$n(\alpha,\omega,i) = \frac{\alpha\Gamma\left(\frac{\omega+3}{2}\right)\Gamma\left(\frac{i+3}{2}\right)+(1-\alpha)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+i+3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)}, i \in I^+.$$

3.5 Harmonic Mean

The Harmonic mean of $MMWM(\alpha,\omega,\theta)$ distribution is given by

$$\frac{1}{H} = \sqrt{\frac{\theta}{2}}\left\{\frac{\alpha\Gamma\left(\frac{\omega+3}{2}\right)+(1-\alpha)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+2}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)}\right\}.$$

3.6 Incomplete Moments

The s^{th} incomplete moment about origin is given by

$$\varphi_s(t) = \int_0^t x^s\, m(x;\alpha,\omega,\theta)dx,$$

Using density function given in (2.2), we obtain

$$\varphi_s(t) = \left(\frac{2}{\theta}\right)^{\frac{s}{2}} \left\{ \frac{\alpha\gamma\left(\left(\frac{s+3}{2}\right),\frac{\theta t^2}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} + \frac{(1-\alpha)\gamma\left(\left(\frac{s+\omega+3}{2}\right),\frac{\theta t^2}{2}\right)}{\Gamma\left(\frac{\omega+3}{2}\right)} \right\}.$$

Putting $s = 1$ in the above equation, we obtain the first order incomplete moment of $MMWM(\alpha,\omega,\theta)$ distribution as given by

$$\varphi_1(t) = \left(\frac{2}{\theta}\right)^{\frac{1}{2}} \left\{ \frac{\alpha\gamma\left(2,\frac{\theta t^2}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} + \frac{(1-\alpha)\gamma\left(\left(\frac{\omega+4}{2}\right),\frac{\theta t^2}{2}\right)}{\Gamma\left(\frac{\omega+3}{2}\right)} \right\}. \quad (8)$$

3.7 Moment Generating Function and Characteristic Function

Theorem 1:

The moment generating function and characteristic function denoted by $M_x(t)$ and $\varphi_x(t)$ are respectively given by:

$$M_x(t) = \sum_{s=0}^{\infty} \frac{t^s}{s!}\left(\frac{2}{\theta}\right)^{\frac{s}{2}} \left\{ \frac{\alpha\Gamma\left(\frac{s+3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right) + (1-\alpha)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+s+3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)} \right\}.$$

$$\varphi_x(t) = \sum_{s=0}^{\infty} \frac{(it)^s}{s!}\left(\frac{2}{\theta}\right)^{\frac{s}{2}} \left\{ \frac{\alpha\Gamma\left(\frac{s+3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right) + (1-\alpha)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+s+3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)} \right\}.$$

Proof:

By the definition, the moment generating function of a random variable X is defined by

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx}\, m(x;\alpha,\omega,\theta)dx,$$

Using Taylor series expansion of $e^{tx} = 1 + tx + \frac{(tx)^2}{2!} + \cdots$, we get

$$M_x(t) = \int_0^{\infty} \left(1 + tx + \frac{(tx)^2}{2!} + \cdots\right) m(x;\alpha,\omega,\theta)dx,$$

$$M_x(t) = \sum_{s=0}^{\infty} \frac{t^s}{s!}\mu_s'$$

$$M_x(t)=\sum_{s=0}^{\infty}\frac{t^s}{s!}\left(\frac{2}{\theta}\right)^{\frac{s}{2}}\left\{\frac{\alpha\Gamma\left(\frac{s+3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)+(1-\alpha)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{s+\omega+3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)}\right\}.$$

Using the relationship between moment generating function and characteristics function, we have

$$\varphi_x(t)=M_x(it).$$

$$\varphi_x(t)=\sum_{s=0}^{\infty}\frac{(it)^s}{s!}\left(\frac{2}{\theta}\right)^{\frac{s}{2}}\left\{\frac{\alpha\Gamma\left(\frac{s+3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)+(1-\alpha)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+s+3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+3}{2}\right)}\right\}.$$

3.8 Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves are not applicable only to measure the income and poverty level but it also gains applicability in reliability, medicine, insurance as well as in demography. The Bonferroni curve initially proposed by Bonferroni (1933), is mathematically stated as

$$B(p)=\frac{1}{p\mu}\int_0^q x\,m(x;\alpha,\omega,\theta)dx,$$

The Lorenz curve (1905) is mathematically defined as

$$L(p)=\frac{1}{\mu}\int_0^q x\,m(x;\alpha,\omega,\theta)dx$$

Using (3.3) and replacing t with q, the above two expressions are evaluated to

$$B(p)=\frac{\alpha\Gamma\left(\frac{\omega+3}{2}\right)\gamma\left(2,\frac{\theta q^2}{2}\right)+(1-\alpha)\Gamma\left(\frac{3}{2}\right)\gamma\left(\left(\frac{\omega+4}{2}\right),\frac{\theta q^2}{2}\right)}{p\left\{\alpha\Gamma\left(\frac{\omega+3}{2}\right)+(1-\alpha)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+4}{2}\right)\right\}}.$$

$$L(p)=\frac{\alpha\Gamma\left(\frac{\omega+3}{2}\right)\gamma\left(2,\frac{\theta q^2}{2}\right)+(1-\alpha)\Gamma\left(\frac{3}{2}\right)\gamma\left(\left(\frac{\omega+4}{2}\right),\frac{\theta q^2}{2}\right)}{\alpha\Gamma\left(\frac{\omega+3}{2}\right)+(1-\alpha)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{\omega+4}{2}\right)}.$$

3.9 Order Statistics

Let $x_1,x_2,x_3,\dots,x_m$ be an ordered sample of size m from $MMWM(\alpha,\omega,\theta)$ distribution. Then, the density function of the r^{th} order statistics is given by

$$m_r(x) = \frac{m!}{(m-r)!\,(r-1)!} \left[\begin{array}{l} \left\{ \frac{2\alpha}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{\theta x^2}{2}\right) + (1-\alpha) \frac{\gamma\left(\left(\frac{\omega+3}{2}\right), \frac{\theta x^2}{2}\right)}{\Gamma\left(\frac{\omega+3}{2}\right)} \right\}^{r-1} \\ \times \left\{ 1 - \frac{2\alpha}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{\theta x^2}{2}\right) - (1-\alpha) \frac{\gamma\left(\left(\frac{\omega+3}{2}\right), \frac{\theta x^2}{2}\right)}{\Gamma\left(\frac{\omega+3}{2}\right)} \right\}^{m-r} \\ \times \theta^{\frac{3}{2}} x^2 exp\left(-\frac{\theta x^2}{2}\right) \left\{ \alpha \sqrt{\frac{2}{\pi}} + \frac{(1-\alpha)\theta^{\frac{\omega}{2}} x^{\omega}}{2^{\left(\frac{\omega+1}{2}\right)} \Gamma\left(\frac{\omega+3}{2}\right)} \right\} \end{array} \right] \tag{9}$$

Substituting $r = 1$ and m in (9), we obtain the minimum and maximum order statistics of the $MMWM(\alpha, \omega, \theta)$ distribution as given by

$$m_1(x) = m \left[\begin{array}{l} \left\{ 1 - \frac{2\alpha}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{\theta x^2}{2}\right) - (1-\alpha) \frac{\gamma\left(\left(\frac{\omega+3}{2}\right), \frac{\theta x^2}{2}\right)}{\Gamma\left(\frac{\omega+3}{2}\right)} \right\}^{m-1} \\ \times \theta^{\frac{3}{2}} x^2 exp\left(-\frac{\theta x^2}{2}\right) \left\{ \alpha \sqrt{\frac{2}{\pi}} + \frac{(1-\alpha)\theta^{\frac{\omega}{2}} x^{\omega}}{2^{\left(\frac{\omega+1}{2}\right)} \Gamma\left(\frac{\omega+3}{2}\right)} \right\} \end{array} \right]$$

$$m_m(x) = m \left[\begin{array}{l} \left\{ \frac{2\alpha}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{\theta x^2}{2}\right) - (1-\alpha) \frac{\gamma\left(\left(\frac{\omega+3}{2}\right), \frac{\theta x^2}{2}\right)}{\Gamma\left(\frac{\omega+3}{2}\right)} \right\}^{m-1} \\ \times \theta^{\frac{3}{2}} x^2 exp\left(-\frac{\theta x^2}{2}\right) \left\{ \alpha \sqrt{\frac{2}{\pi}} + \frac{(1-\alpha)\theta^{\frac{\omega}{2}} x^{\omega}}{2^{\left(\frac{\omega+1}{2}\right)} \Gamma\left(\frac{\omega+3}{2}\right)} \right\} \end{array} \right].$$

4. ENTROPY MEASURES

In information theory, entropy measures are very useful to determine the loss of information or existence of uncertainty associated with a probability model. In this section, we study two measures of entropy, Renyi entropy developed by Renyi (1960) and beta entropy developed by Harvda and Charvat (1967).

4.1 Renyi Entropy

The information measure developed by Alfred Renyi of order γ for a random variable X is mathematically stated as

$$R_E = (1-\gamma)^{-1} log \int_0^\infty m^\beta(x;\alpha,\omega,\theta)dx,\ \gamma > 0 \text{ and } \gamma \neq 1.$$

$$R_E = (1-\gamma)^{-1} log \int_0^\infty \theta^{\frac{3\gamma}{2}} x^{2\gamma} exp\left(-\frac{\gamma\theta x^2}{2}\right)\left\{\alpha\sqrt{\frac{2}{\pi}} + \frac{(1-\alpha)\theta^{\frac{\omega}{2}} x^\omega}{2^{\left(\frac{\omega+1}{2}\right)}\Gamma\left(\frac{\omega+3}{2}\right)}\right\}^\gamma dx$$

Using binomial expansion

$$(a+b)^n = \sum_{k=0}^\infty \binom{n}{k} b^k a^{n-k},$$

We obtain

$$R_E = (1-\gamma)^{-1} log \sum_{k=0}^\infty \binom{\gamma}{k}\left(\alpha\sqrt{\frac{2}{\pi}}\right)^{\gamma-k} \frac{(1-\alpha)^k \theta^{\frac{3\gamma+\omega k}{2}}}{2^{k\left(\frac{\omega+1}{2}\right)}\left\{\Gamma\left(\frac{\omega+3}{2}\right)\right\}^k}\int_0^\infty x^{2\gamma+\omega k} \, exp\left(-\frac{\gamma\theta x^2}{2}\right)dx$$

After solving the integral, we get the required expression for Renyi entropy as

$$R_E = (1-\gamma)^{-1} log\left[\sum_{k=0}^\infty \binom{\gamma}{k}\left(\alpha\sqrt{\frac{2}{\pi}}\right)^{\gamma-k} \frac{(1-\alpha)^k \theta^{\frac{\gamma-1}{2}} 2^{\frac{2\gamma-k+1}{2}}}{\gamma^{\frac{2\gamma+\omega k+1}{2}}} \frac{\Gamma\left(\frac{2\gamma+\omega k+1}{2}\right)}{\left\{\Gamma\left(\frac{\omega+3}{2}\right)\right\}^k}\right].$$

4.2 Beta Entropy

The beta entropy introduced by Harvda and Charvat as one parameter generalization of Shannon entropy (1948) is defined as

$$\beta_E = (1-\beta)^{-1}\left[\int_0^\infty m^\beta(x;\alpha,\omega,\theta)dx - 1\right].$$

Using density function (2.2) and after simplification, the expression for beta entropy is obtained as

$$\beta_E = (1-\beta)^{-1}\left[\sum_{k=0}^\infty \binom{\gamma}{k}\left(\alpha\sqrt{\frac{2}{\pi}}\right)^{\gamma-k} \frac{(1-\alpha)^k \theta^{\frac{\gamma-1}{2}} 2^{\frac{2\gamma-k+1}{2}}}{\gamma^{\frac{2\gamma+\omega k+1}{2}}} \frac{\Gamma\left(\frac{2\gamma+\omega k+1}{2}\right)}{\left\{\Gamma\left(\frac{\omega+3}{2}\right)\right\}^k} - 1\right].$$

5. ESTIMATION OF PARAMETERS

In this section, we describe the maximum likelihood estimation method for estimating three unknown parameters α, ω and θ of $MMWM(\alpha, \omega, \theta)$ distribution. The estimates obtained under maximum likelihood estimation method are not in a closed form. In order to overcome the said hindrance, programming language R has been used to estimate these unknown parameters.

5.1 Maximum Likelihood Estimation

Maximum likelihood estimation is the most popular and efficient method of classical estimation of the unknown parameters. Let $x_1, x_2, x_3, \dots, x_m$ be an iid random sample of size m from $MMWM(\alpha, \omega, \theta)$ distribution. Then, the likelihood function of m observations is given by

$$L = \prod_{k=1}^{m} \theta^{\frac{3}{2}} x_k^2 exp\left(-\frac{\theta x_k^2}{2}\right)\left\{\alpha\sqrt{\frac{2}{\pi}} + \frac{(1-\alpha)\theta^{\frac{\omega}{2}} x_k^{\omega}}{2^{\left(\frac{\omega+1}{2}\right)}\Gamma\left(\frac{\omega+3}{2}\right)}\right\}$$

The log-likelihood function is given by

$$lnL = \ell = \sum_{k=1}^{m} ln\left[\theta^{\frac{3}{2}} x_k^2 exp\left(-\frac{\theta x_k^2}{2}\right)\left\{\alpha\sqrt{\frac{2}{\pi}} + \frac{(1-\alpha)\theta^{\frac{\omega}{2}} x_k^{\omega}}{2^{\left(\frac{\omega+1}{2}\right)}\Gamma\left(\frac{\omega+3}{2}\right)}\right\}\right]$$

For maximum likelihood estimates of α, ω and θ,

$$\frac{\partial \ell}{\partial \alpha} = 0, \frac{\partial \ell}{\partial \omega} = 0 \text{ and } \frac{\partial \ell}{\partial \theta} = 0$$

which yield

$$\sum_{k=1}^{m}\left[\frac{\theta^{\frac{3}{2}} x_k^2 exp\left(-\frac{\theta x_k^2}{2}\right)\left\{\sqrt{\frac{2}{\pi}} - \frac{\theta^{\frac{\omega}{2}} x_k^{\omega}}{2^{\left(\frac{\omega+1}{2}\right)}\Gamma\left(\frac{\omega+3}{2}\right)}\right\}}{\theta^{\frac{3}{2}} x_k^2 exp\left(-\frac{\theta x_k^2}{2}\right)\left\{\alpha\sqrt{\frac{2}{\pi}} + \frac{(1-\alpha)\theta^{\frac{\omega}{2}} x_k^{\omega}}{2^{\left(\frac{\omega+1}{2}\right)}\Gamma\left(\frac{\omega+3}{2}\right)}\right\}}\right] = 0.$$

$$\sum_{k=1}^{m}\left[\frac{\theta^{\frac{3}{2}} x_k^2 exp\left(-\frac{\theta x_k^2}{2}\right)\frac{\partial \ell}{\partial \omega}\left\{\frac{(1-\alpha)\theta^{\frac{\omega}{2}} x_k^{\omega}}{2^{\left(\frac{\omega+1}{2}\right)}\Gamma\left(\frac{\omega+3}{2}\right)}\right\}}{\theta^{\frac{3}{2}} x_k^2 exp\left(-\frac{\theta x_k^2}{2}\right)\left\{\alpha\sqrt{\frac{2}{\pi}} + \frac{(1-\alpha)\theta^{\frac{\omega}{2}} x_k^{\omega}}{2^{\left(\frac{\omega+1}{2}\right)}\Gamma\left(\frac{\omega+3}{2}\right)}\right\}}\right] = 0.$$

$$\sum_{k=1}^{m}\left[\frac{x_k^2\frac{\partial \ell}{\partial \theta}\left[\theta^{\frac{3}{2}}exp\left(-\frac{\theta x_k^2}{2}\right)\left\{\alpha\sqrt{\frac{2}{\pi}}+\frac{(1-\alpha)\theta^{\frac{\omega}{2}}x_k^{\omega}}{2^{\left(\frac{\omega+1}{2}\right)}\Gamma\left(\frac{\omega+3}{2}\right)}\right\}\right]}{\theta^{\frac{3}{2}}x_k^2 exp\left(-\frac{\theta x_k^2}{2}\right)\left\{\alpha\sqrt{\frac{2}{\pi}}+\frac{(1-\alpha)\theta^{\frac{\omega}{2}}x_k^{\omega}}{2^{\left(\frac{\omega+1}{2}\right)}\Gamma\left(\frac{\omega+3}{2}\right)}\right\}}\right]=0.$$

Since, the above system of equations is not in a closed form. Thus, the MLE's of the unknown parameters are difficult to obtain by simple mathematical calculations. Here, we used Newton-Raphson technique, which the most powerful procedure of solving such type of equations numerically.

6. SIMULATION STUDY

This section deals with the simulation study to investigate the consistency of ML estimates in terms of varying sample size. The process is repeated 500 times and the simulation is carried out using the statistical software R. For this purpose, a random sample of size 50, 100, 150 and 300 is generated at each replication for two different combinations of parameters $\alpha = 0.75, \omega = 0.50, \theta = 2.75$ and $\alpha = 0.3, \omega = 1.3, \theta = 3.1$ arbitrarily.

In each case, the average estimates along with their corresponding mean square errors (MSEs), variance and bias are computed. The simulation results are listed in Table 2 and Table 3 respectively.

Table 2
Average Estimate, Bias, Variance and MSEs with Varying Sample Size for the Parameters $\boldsymbol{\alpha = 0.75, \omega = 0.50}$ and $\boldsymbol{\theta = 2.75}$

Sample Size (n)	Parameters	Estimate	Bias	Variance	MSE
50	$\boldsymbol{\alpha}$	0.4485	-0.3015	0.1862	0.2771
	$\boldsymbol{\omega}$	1.6331	1.1332	4.2442	5.5283
	$\boldsymbol{\theta}$	3.5983	0.8483	1.6243	2.3438
100	$\boldsymbol{\alpha}$	0.4885	-0.2615	0.1827	0.2511
	$\boldsymbol{\omega}$	1.2867	0.7867	2.5144	3.1334
	$\boldsymbol{\theta}$	3.2966	0.5466	0.7943	1.0932
150	$\boldsymbol{\alpha}$	0.5140	-0.2359	0.1768	0.2325
	$\boldsymbol{\omega}$	1.2735	0.7735	2.4773	3.0756
	$\boldsymbol{\theta}$	3.2304	0.4804	0.7153	0.9461
300	$\boldsymbol{\alpha}$	0.5363	-0.2137	0.1680	0.2137
	$\boldsymbol{\omega}$	1.1726	0.6726	1.7844	2.2368
	$\boldsymbol{\theta}$	3.0818	0.3318	0.3889	0.4989

Table 3
Average Estimate, Bias, Variance and MSEs with Varying Sample Size for the Parameters $\alpha = 0.3, \omega = 1.3$ and $\theta = 3.1$

Sample Size (n)	Parameters	Estimate	Bias	Variance	MSE
50	α	0.2376	-0.0624	0.0903	0.0942
	ω	2.5937	1.2937	4.2789	5.9525
	θ	3.8948	0.7948	1.5872	2.2190
100	α	0.2477	-0.0523	0.0771	0.0797
	ω	2.2073	0.9073	2.8086	3.6318
	θ	3.5772	0.4772	0.7740	1.0017
150	α	0.2109	-0.0891	0.0479	0.0558
	ω	1.9618	0.6618	1.9736	2.4115
	θ	3.4739	0.3739	0.4779	0.6178
300	α	0.2016	-0.0984	0.0408	0.0504
	ω	1.6516	0.3516	1.2893	1.4129
	θ	3.3156	0.2156	0.2927	0.3392

From Table 2 and Table 3, we observe that the ML estimators are very close to the fixed values of the parameters. Also, we notice that the MSE and bias decreases as we increase the size of sample. Thus, MSE of parameter estimates suggests that estimators are consistent and ML estimation performs quite well.

7. REAL DATA APPLICATIONS

This section demonstrates the practical applicability of the $MMWMD(\alpha, \omega, \theta)$ distribution to two real life data sets taken from the literature in order to show the flexibility of the proposed model. The proposed distribution is compared with other related models such as Maxwell distribution (MD), weighted Maxwell distribution (WMD), length biased Maxwell distribution (LBMD), area biased Maxwell distribution (ABMD) and Marshall Olkin length biased Maxwell distribution (MOLBMD). The model parameters are estimated by employing the maximum likelihood estimation technique. Furthermore, the model selection tools such as log-likelihood, Akaike information criterion (AIC, 1974), Schwarz information criterion (SIC, 1978), corrected Akaike information criterion (AICC, 1987), Hannan-Quinn information criterion (HQIC, 1979) and Kolmogorov Simirnov test has been discussed for model compatibility. In general, the best distribution corresponds to smaller values of these model selection tools.

Data Set I: The data set used by Kotz and Johnson (1983) represents the survival times (in years) of 43 patients suffering from Leukemia has been analysed in this example. The data is given as follows: 0.019, 0.129, 0.159, 0.203, 0.485, 0.636, 0.748, 0.781, 0.869,1.175, 1.206, 1.219, 1.219, 1.282, 1.356, 1.362, 1.458, 1.564, 1.586, 1.592, 1.781, 1.923, 1.959, 2.134, 2.413, 2.466, 2.548, 2.652, 2.951, 3.038, 3.6, 3.655, 3.745, 4.203, 4.690, 4.888, 5.143, 5.167, 5.603, 5.633, 6.192, 6.655, 6.874. This data set was also analysed by Wani et al. (2020).

Data Set II: The second data set also examined by Almongy et al. (2021) to demonstrate the applicability of new extended Rayleigh distribution represents a COVID-19 mortality rate data belongs to Mexico of 108 days that is recorded from 4 March to 20 July 2020. This data formed of rough mortality rate. The data are as follows: 8.826, 6.105, 10.383, 7.267, 13.220, 6.015, 10.855, 6.122, 10.685, 10.035, 5.242, 7.630, 14.604, 7.903, 6.327, 9.391, 14.962, 4.730, 3.215, 16.498, 11.665, 9.284, 12.878, 6.656, 3.440, 5.854, 8.813, 10.043, 7.260, 5.985, 4.424, 4.344, 5.143, 9.935, 7.840, 9.550, 6.968, 6.370, 3.537, 3.286, 10.158, 8.108, 6.697, 7.151, 6.560, 2.988, 3.336, 6.814, 8.325, 7.854, 8.551, 3.228, 3.499, 3.751, 7.486, 6.625, 6.140, 4.909, 4.661, 1.867, 2.838, 5.392, 12.042, 8.696, 6.412, 3.395, 1.815, 3.327, 5.406, 6.182, 4.949, 4.089, 3.359, 2.070, 3.298, 5.317, 5.442, 4.557, 4.292, 2.500, 6.535, 4.648, 4.697, 5.459, 4.120, 3.922, 3.219, 1.402, 2.438, 3.257, 3.632, 3.233, 3.027, 2.352, 1.205, 2.077, 3.778, 3.218, 2.926, 2.601, 2.065, 1.041, 1.800, 3.029, 2.058, 2.326, 2.506, 1.923.

Table 3
ML Estimates of the Model Parameters using Ist Real Life Data Set

Survival Times of Leukemia Patients Data, N=43				
Model	$\hat{\alpha}$	$\hat{\omega}$	$\hat{\theta}$	$\hat{\beta}$
$MMWMD(\alpha,\omega,\theta)$	0.7451	20.6799	0.8230	-
$WMD(\omega,\theta)$	-	0.0010	0.2986	-
$MD(\theta)$	-	-	0.2985	-
$LBMD(\theta)$	-	-	0.3981	-
$ABMD(\theta)$	-	-	0.4976	-
$MOLBMD(\theta,\beta)$	-	-	2.4864	0.0401

Table 4
ML Estimates of the Model Parameters using Real Life Data Sets

Covid-19 Mortality Rate Data of Mexico, N=108				
Model	$\hat{\alpha}$	$\hat{\omega}$	$\hat{\theta}$	$\hat{\beta}$
$MMWMD(\alpha,\omega,\theta)$	0.8418	10.841	0.1080	-
$WMD(\omega,\theta)$	-	0.0011	0.0688	-
$MD(\theta)$	-	-	0.0687	-
$LBMD(\theta)$	-	-	0.0916	-
$ABMD(\theta)$	-	-	0.1146	-
$MOLBMD(\theta,\beta)$	-	-	5.2176	0.0841

Table 5
Goodness of Fit Statistics for Different Models using Data Set I

Survival Times of Leukemia Patients Data, N=43						
Model	$-\ell$	**AIC**	**SIC**	**AICC**	**HQIC**	**K-S**
$MMWMD(\alpha,\omega,\theta)$	92.784	191.568	196.852	192.184	193.571	0.133
$WMD(\omega,\theta)$	109.297	222.594	226.117	222.894	223.893	0.325
$MD(\theta)$	109.277	220.554	222.315	220.651	221.203	0.324
$LBMD(\theta)$	130.671	263.342	265.105	263.440	263.992	0.374
$ABMD(\theta)$	153.681	309.363	311.124	309.460	310.012	0.404
$MOLBMD(\theta,\beta)$	114.261	232.521	236.044	232.821	233.820	0.209

Table 6
Goodness of Fit Statistics for Different Models using Data Set II

Covid-19 Mortality Rate Data of Mexico, N=108						
Model	$-\ell$	**AIC**	**SIC**	**AICC**	**HQIC**	**K-S**
$MMWMD(\alpha,\omega,\theta)$	268.008	542.016	550.062	542.245	545.278	0.1043
$WMD(\omega,\theta)$	276.692	557.384	562.748	557.498	559.558	0.1778
$MD(\theta)$	276.680	555.359	558.041	555.397	556.447	0.1776
$LBMD(\theta)$	291.869	585.738	588.420	585.775	586.825	0.2304
$ABMD(\theta)$	311.115	624.230	626.912	624.268	625.318	0.2673
$MOLBMD(\theta,\beta)$	273.348	550.696	556.061	550.811	552.872	0.1264

Based on the results obtained in the above tables, it can been easily seen that the proposed model $MMWM(\alpha,\omega,\theta)$ corresponds to the lower values of the model selection tools. Thus, we conclude that the $MMWM(\alpha,\omega,\theta)$ distribution provides a better fit as compare to other competing models.

CONCLUDING REMARKS

In this article, we formulate a new mixture model of two continuous probability distributions, Maxwell distribution and weighted Maxwell distribution. The different statistical properties of the proposed model have been unfolded. Maximum likelihood estimation procedure has been implemented to estimate the parameters of the model. Moreover, we compete the proposed model with other sub-models using three real life data sets. Based on different model selection tools, we observe that the $MMWM(\alpha,\omega,\theta)$ distribution provides better fit that its sub-models. We are hopeful that the proposed model will draw wider applications in statistics.

REFERENCES

1. Akaike, H. (1974). A new look at the statistical model identification. *Selected papers of Hirotugu: Springer*, 215-222.
2. Almongy, H.M., Almetwally, E.M., Aljohani, H.M., Alghamdi, A.S. and Hafez, E.H. (2021). A new extended Rayleigh distribution with applications of COVID-19 data. *Results in Physics*, 23, 104012.
3. Amusan, G.E. (2010). *The Beta-Maxwell distribution*. M.A. Mathematics Thesis, Department of Mathematics, Marshall University, West Virginia, USA. (Unpublished).
4. Bazdogan, H. (1987). Model selection and Akaike's information criterion: The general theory and its analytical extensions. *Psychometrika*, 52, 345-370.
5. Bhat, A.A. and Ahmad, S.P. (2018). Mixture of Exponential and Weighted Exponential Distribution: Properties and Applications. *International Journal of Scientific Research in Mathematical and Statistical Sciences*, 5(6), 38-46.
6. Bhat, A.A. and Ahmad, S.P. (2021). Mixture of Gamma and Rayleigh distributions: Mathematical Properties and Applications. *Journal of Applied Probability and Statistics*, 16(2), 55-71.
7. Bonferroni, C.E. (1933). *Elementi di Statistica General*. Seeber, Firenze.
8. Chaturvedi, A. and Vyas, S. (2019). Generalized Gamma-Maxwell distribution: Properties and estimation of reliability functions. *Journal of Statistics and Management Systems*, DOI: 10.1080/09720510.2019.1609553.
9. Chaturvedi, A. and Rani, U. (1998). Classical and Bayesian reliability estimation of the generalized Maxwell failure distribution. *Journal of Statistical Research*, 32(1), 113-120.
10. Cordeiro, G.M., Pescim and Ortega, E.M.M. (2012). The Kumaraswamy generalised half-normal distribution for skewed positive data. *Journal of Data sciences*, 10, 195-202.
11. Dar, A.A., Ahmed, A., and Reshi, J.A. (2018). Characterization and Estimation of Weighted Maxwell-Boltzmann Distribution. *Applied Mathematics & Information Sciences*, 12(1), 193-202.
12. Hannan, E.J. and Quinn, B.G. (1979). The determination of the order of an autoregression. *Journal of the Royal Statistical Society: Series B*, 41, 190-195.
13. Harvda, J. and Charvat, F. (1967). Quantification method in classification processes: Concept of structural entropy. *Kybernetika*. 3, 30-35.
14. Iqbal, T. and Iqbal, M.Z. (2020). On the mixture of weighted Exponential and weighted Gamma distribution. *International Journal of Analysis and Applications*, 18(3),
396-408.
15. Kazmi, S., Aslam, M. and Ali, S. (2012). On the Bayesian estimation for two component mixture of Maxwell distribution, assuming type I censored data. *International Journal of Applied Science & Technology*, 2(1), 197-218.
16. Kotz, S. and Johnson, N.L. (1983). Encyclopedia of statistical sciences. *Wiley New York*. 3.
17. Krishna, H. and Malik, M. (2009). Reliability estimation in Maxwell distribution with Type-II censored data. *International Journal of Quality & Reliability Management*, 26(2), 184-195.

18. Lorenz, M.O. (1905). Methods of measuring the concentration of wealth. *Publications of the American Statistical Association*, 9(70), 209-219.
19. Marco, B., Roberto, B. and Giuseppe, E. (2013). On maximum likelihood estimation of Pareto Mixture. *Computational Statistics*, 28, 161-178.
20. Mathew, J. and Chesneau, C. (2020). Marshall–Olkin Length-Biased Maxwell Distribution and its Applications. *Mathematical and Computational Applications*, 25, 65.
21. Maxwell, J. (1880). On the dynamical theory of gases, presented to the meeting of the British association for the advancement of science. *Scientific Letters*, I, 616.
22. McLachlan, G.J. and Peel, D. (2004). *Finite mixture models*. John Wiley & Sons.
23. Pearson, K. (1894). Contribution to the mathematical theory of evolution. *Philosophical Transition Royal Society*, 185, 71-110.
24. Renyi, A. (1960). On measures of entropy and information. *Berkeley Symposium on Mathematical Statistics and Probability*, 1(1), 547-561.
25. Schwarz, G. (1978). Estimating the dimensions of a model. *The Annals of Statistics*, 6, 461-464.
26. Shannon, E. (1948). A mathematical theory of communication. *Bell System Technical Journal*, 27(3), 379-423.
27. Sharma, V.K., Bakouch, H.S. and Suthar, K. (2016a). An extended Maxwell distribution: Properties and applications. *Communication in Statistics Simulation and Computation*, 46(9), 1-26. DOI: 10.1080/03610918.2016.1222422.
28. Singh, A., Bakouch, H., Kumar, S. and Singh, U. (2018). *Power Maxwell distribution: Statistical properties, estimation and application.* Arxiv: 1807.01200v1.
29. Suksaengrakcharoen, S. and Bodhiswan W. (2014). A new family of generalised Gamma distribution and its application. *Journal of Mathematics and Statistics*, 10(2), 211-220.
30. Tyagi, R.K. and Bhattacharya, S.K. (1989). A note on the MVU estimation of reliability for the Maxwell failure distribution. *Estadistica*, 41(137), 73-79.
31. Alzaatreh, A., Lee, C. and Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, 71(1), 63-79.
32. Wani, S.A., Hassan. A., Shafi, S. and Shafi, S. (2020). Pranav Quasi Gamma Distribution: Properties and Applications. *Journal of Statistical Theory and Applications*, 19(4), 506-517.