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AILAMUJIA INVERTED WEIBULL DISTRIBUTION WITH APPLICATIONS TO LIFETIME DATA

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ABSTRACT

Ailamujia distribution is a newly proposed lifetime model that has many engineering applications. This work proposes an extension of Ailamujia distribution which is called Ailamujia inverted Weibull distribution (AIWD) with some applications. Statistical properties such as reliability function, hazard function, mean residual function, moments, moment generating function, Shannon's entropy, and order statistics were derived for this purpose. The parameters of the proposed distribution have been estimated by the maximum likelihood method of estimation. Three real data sets were analyzed; they demonstrated better fitting than all other extensions of the Ailamujia distribution as well as some other distributions.

KEYWORDS

Exponential distribution, Transformed transformer, Estimation, Survival analysis.

1. INTRODUCTION

Probability distributions have been extensively used as models to describe and predict real world phenomena in different fields such as engineering, medical sciences, biological studies among other fields. The statistical analysis depends heavily on the assumed model. In recent years, there has been much interests among statisticians in proposing generalizations and extensions of well-known probability distribution models to achieve more flexibility in data modeling. These extensions are based on transformations and compounding by introducing one or more additional parameters to the baseline distribution.

Different generated families of distributions are proposed in the literature. The most common of these includes: the exponentiated distributions (Gupta and Kundu (1998); Gupta and Kundu (1999)); Kumaraswamy generalized family of distributions (Kumaraswamy, (1980)); Alpha power transformation method (Mahdavi and Kundu, (2017)); and Marshall-Olkin-G family distributions (Marshal and Oklin, (1997)). Also, Alzaatreh *et al*. (2013) have proposed a general method of extending probability distributions known as T-X family distributions which used *CDF* of any random variable.

Different lifetime data can be represented by several well-known continuous probability distributions as well as their generalizations. Ailamujia distribution is a newly proposed lifetime model that has many engineering applications (Kotz and Pensky, (2003)). Ailamujia distribution is considered a convenient one compared to other models for such applications as the repair time, which guarantees the distribution delay time. Lv *et al*. (2002) studied the different properties of this distribution including mean, variance, median and maximum likelihood estimators. This distribution has also been investigated for the interval estimation and the hypothesis testing (Pan, (2009)). The minimax estimation of the Ailamujia model parameter has been discussed under a non-informative prior using the three loss functions (Li, (2016)).

Different extensions and modifications of Ailamujia distribution were considered in the literature. For example, Uzma et al. (2017) presented the weighted analogue of Ailamujia distribution and explored its various properties. JayaKumar *et al*. (2019) suggested area biased distribution and explored its various properties; they also applied the area biased distribution to bladder cancer data. Aijaz *et al*. (2020) proposed the inverse analogue of Ailamujia distribution with its statistical properties and applications. Rather *et al*. (2018) developed the size biased Ailamujia distribution and implemented it in the analysis of data from engineering and medical science.

In this paper, an extension of Ailamujia distribution called Ailamujia inverted Weibull distribution (AIWD) is considered. Statistical properties such as reliability function, hazard function, mean residual function, moments, moment generating function, Shannon's entropy, and order statistics are also derived and presented. Parameter estimation of the distribution parameters using the method of maximum likelihood is discussed. Three real data sets are analyzed to illustrate the applicability of the proposed extension.

2. AILAMUJIA INVERTED WEIBULL DISTRIBUTION

This section presents the fundamentals of Ailamujia inverted Weibull distribution (AIWD). The probability density function of the Ailamujia distribution is given by:

$$f(x,\theta) = 4\theta^2 x e^{-2\theta x};\ x \geq 0, \theta > 0 \tag{1}$$

The cumulative distribution function is given by:

$$F(x,\theta) = 1 - (1 + 2\theta x)e^{-2\theta x};\ x \geq 0, \theta > 0 \tag{2}$$

where θ is an unknown parameter.

It can be concluded that

$$E(X) = \frac{1}{\theta} \text{ and } \sigma^2 = \frac{1}{2\theta}$$

The maximum likelihood estimator for θ is given by

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} X_i}$$

The survival function r(x) and failure rate $h(x)$ are given by:

$$r(x) = (1 + 2\theta x)e^{-2\theta x}$$

$$h(x) = \frac{4\theta^2 x}{1 + 2\theta x}.$$

Alzaatreh *et al.* (2013) proposed a transformed transformer method for generating families of continuous distributions. The following is among these new families:

$$1 - F(x) = \int_0^{-\log(G(x))} f(t)dt \tag{3}$$

where $f(t)$ is the baseline distribution, $G(x)$ is the cumulative distribution function of some continuous distribution.

Generation of new distribution members has been considered by many authors; for example, the Weibull-Raleigh and Rayleigh Pareto distributions were developed by Ganji *et al.* (2016) and Al-Kadim and Bnadher (2018), respectively.

In equation (3), the baseline distribution $f(t)$ is the Ailamujia distribution given in (1). Thus substituting (1) in (3) results in the following expression:

$$1 - F(x) = \int_0^{-\log(G(x))} 4\theta^2 t e^{-2\theta t} dt \tag{4}$$

A one parameter $G(x)$ inverted Weibull distribution is considered in this work, in which its cumulative distribution function (CDF) is given by:

$$G(x) = e^{-x-\alpha}, x \geq 0, \alpha > 0$$

Based on equation (4), the *CDF* of the new extension can be expressed as:

$$F(x) = (1 + 2\theta x^{-\alpha})e^{-2\theta x^{-\alpha}}, x > 0, \alpha > 0 \tag{5}$$

Accordingly, the probability density function (*PDF*) is given by:

$$f(x) = 4\alpha\theta^2 x^{-2\alpha-1}\, e^{-2\theta x^{-\alpha}}, x > 0, \theta > 0, \alpha > 0 \tag{6}$$

The behavior of the PDF of Ailamujia inverted Weibull distribution for varying values of parameters have been shown in Figures 1 and 2.

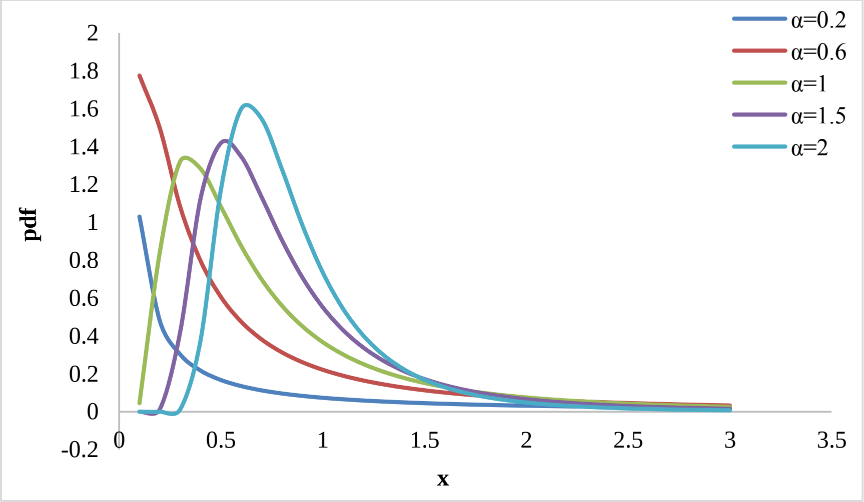

Figure (1): The pdf of AIWD with α=0.2, 0.6, 1, 1.5, 2 and θ=0.5

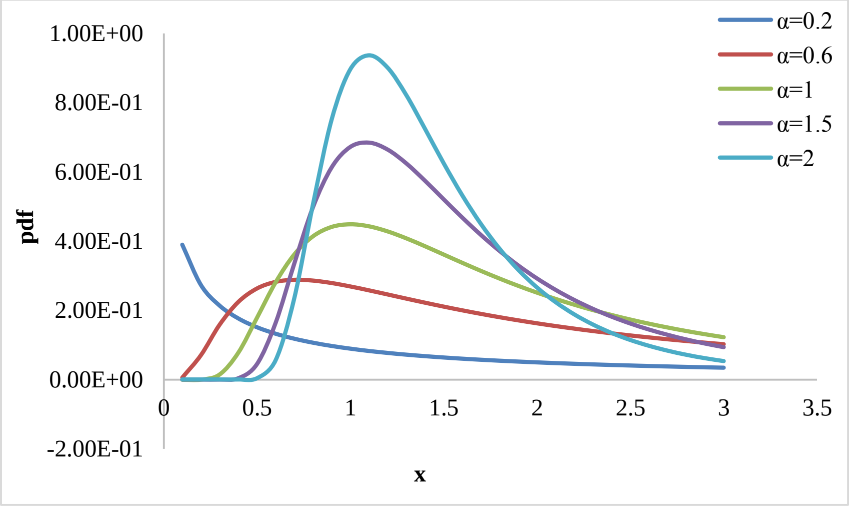

Figure (2): The pdf of AIWD with α=0.2, 0.6, 1, 1.5, 2 and θ = 1.5

3. RELIABILITY MEASURES

In this section, we discuss the reliability measures of Ailamujia inverted Weibull distribution (AIWD) such as; survival function, hazard (or failure) function, reverse hazard function, and mean residual life function.

Survival Function

The survival function is given by

$$R(x) = 1 - F(x)$$

$$R(x) = 1 - \left\{(1 + 2\theta x^{-\alpha})e^{-2\theta x^{-\alpha}}\right\}.$$

Hazard Function

The hazard function is given by

$$h(x) = \frac{f(x)}{R(x)}$$

$$h(x) = \frac{4\theta^2 \alpha x^{-2\alpha-1} e^{-2\theta x^{-\alpha}}}{1 - \{(1 + 2\theta x^{-\alpha})e^{-2\theta x^{-\alpha}}\}}$$

The shapes of the hazard rate function of Ailamujia inverted Weibull distribution for varying values of the parameters have been shown in Figures 3 and 4.

Reverse Hazard Function

The reverse hazard function is given by

$$r(x) = \frac{f(x)}{F(x)}$$

$$r(x) = \frac{4\theta^2 \alpha x^{-2\alpha-1} e^{-2\theta x^{-\alpha}}}{(1 + 2\theta x^{-\alpha})e^{-2\theta x^{-\alpha}}}$$

$$r(x) = \frac{4\theta^2 \alpha x^{-2\alpha-1}}{(1 + 2\theta x^{-\alpha})}.$$

Mean Residual Life Function

Proposition 1

The mean residual life function of AIWD with probability density function given in (6) is given by

$$m(x) = \frac{(2\theta)^{\frac{1}{\alpha}} \Gamma\left(\frac{1}{\alpha} - 2, u\right)}{1 - \{(1 + 2\theta x^{-\alpha})e^{-2\theta x^{-\alpha}}\}} - x$$

Proof:

The mean residual life function is given by

$$\begin{aligned} m(x) &= \frac{1}{R(x)} \int_x^\infty t\, f(t)\, dt - x \\ &= \frac{1}{R(x)} \int_x^\infty t\, 4\theta^2 \alpha t^{-2\alpha-1} e^{-2\theta t^{-\alpha}}\, dt - x \\ &= \frac{4\theta^2 \alpha}{R(x)} \int_x^\infty t^{-2\alpha} e^{-2\theta t^{-\alpha}}\, dt - x \end{aligned}$$

$$= \frac{4\theta^2\alpha}{R(x)} \times \frac{(2\theta)^{\frac{1}{\alpha}-2}}{\alpha} \Gamma\left(\frac{1}{\alpha} - 2, u\right) - x$$

$$= \frac{(2\theta)^{\frac{1}{\alpha}} \Gamma\left(\frac{1}{\alpha} - 2, x\right)}{1 - \{(1 + 2\theta x^{-\alpha}) e^{-2\theta x^{-\alpha}}\}} - x.$$

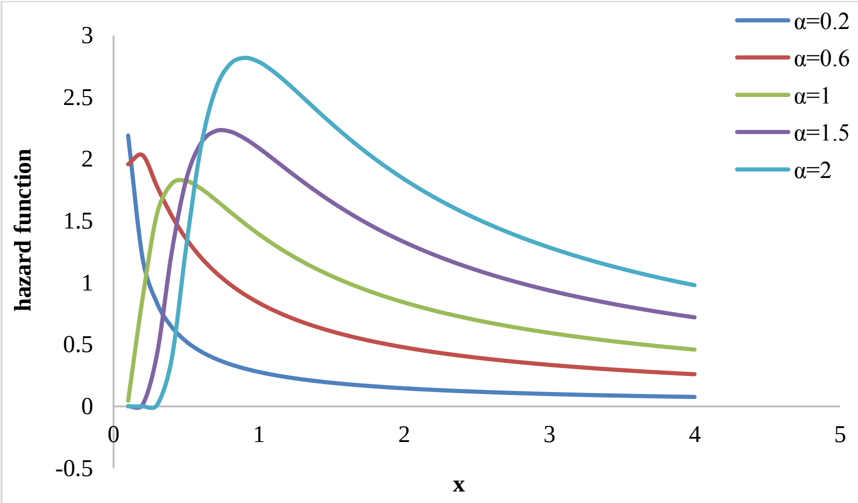

Figure (3): The Hazard Function of AIWD with α= 0.2, 0.6, 1, 1.5, 2 and θ=0.5

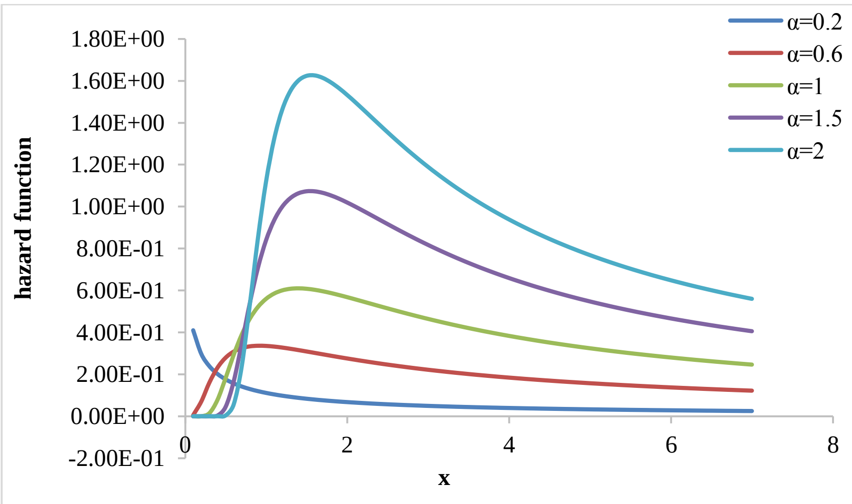

Figure (3): The Hazard Function of AIWD with α=0.2, 0.6, 1, 1.5, 2 and θ=1.5

4. STATISTICAL CHARACTERISTICS

In this section, we discuss some statistical characteristics of the proposed Ailamujia inverted Weibull distribution such as; moments, moment generating function, harmonic mean, mode, median, Shannon's entropy, and order statistics. This section presents the main statistical properties of Ailamujia inverted Weibull distribution.

4.1 Moments

Proposition 2

If X is a random variable having the AIWD distribution then the r^{th} moment about the origin is

$$E(X^r) = (2\theta)^{\frac{r}{\alpha}}\,\Gamma\left(-\frac{r}{\alpha}+2\right)$$

Furthermore, the mean and variance of the distribution are given as

$$E(X) = (2\theta)^{\frac{1}{\alpha}}\,\Gamma\left(-\frac{1}{\alpha}+2\right)$$

$$Var(X) = (2\theta)^{\frac{2}{\alpha}}\left[\Gamma\left(-\frac{2}{\alpha}+2\right)-\left\{\Gamma\left(-\frac{1}{\alpha}+2\right)\right\}^2\right].$$

Proof:

The r^{th} mean about the origin is given by

$$E(X^r) = \int_{-\infty}^{\infty} x^r\, f(x)\, dx$$

$$E(X^r) = \int_{0}^{\infty} x^r\, 4\theta^2\alpha x^{-2\alpha-1}e^{-2\theta x^{-\alpha}}\, dx$$

[Let $2\theta x^{-\alpha} = u \Longrightarrow 2\theta(-\alpha)x^{-\alpha-1}dx = du \Rightarrow 2\theta\alpha x^{-\alpha-1}dx = -du$]

Therefore

$$E(X^r) = \int_{0}^{\infty} (2\theta u^{-1})^{\frac{r}{\alpha}} 2\theta\,\frac{u}{2\theta}\,e^{-u}\,e^{-u}\,dx$$

$$= (2\theta)^{\frac{r}{\alpha}}\int_{0}^{\infty} u^{-\frac{r}{\alpha}+1}e^{-u}\,dx$$

$$= (2\theta)^{\frac{r}{\alpha}}\int_{0}^{\infty} u^{\left(-\frac{r}{\alpha}+2\right)-1}e^{-u}\,dx$$

$$= (2\theta)^{\frac{r}{\alpha}}\,\Gamma\left(-\frac{r}{\alpha}+2\right)$$

Put $r = 1, 2$ we get

$$Mean = E(X) = (2\theta)^{\frac{1}{\alpha}}\,\Gamma\left(-\frac{1}{\alpha}+2\right)$$

$$E(X^2) = (2\theta)^{\frac{2}{\alpha}}\,\Gamma\left(-\frac{2}{\alpha}+2\right)$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$Var(X) = (2\theta)^{\frac{2}{\alpha}}\left[\Gamma\left(-\frac{2}{\alpha}+2\right) - \left\{\Gamma\left(-\frac{1}{\alpha}+2\right)\right\}^2\right].$$

4.2 Moment Generating Function (MGF):

Proposition 3

If X is a random variable has the AIWD distribution, then the moment generating function is

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}(2\theta)^{\frac{r}{\alpha}}\,\Gamma\left(-\frac{r}{\alpha}+2\right).$$

Proof:

The moment generating function is given by

$$\begin{aligned} M_X(t) = E[e^{tX}] &= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \\ &= \int_0^{\infty}\left(1+\frac{tx}{1!}+\frac{(tx)^2}{2!}+\frac{(tx)^3}{3!}+\cdots\right) f(x)\,dx \\ &= \int_0^{\infty}\left(\sum_{r=0}^{\infty}\frac{(tx)^r}{r!}\right) f(x)\,dx \\ &= \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty} x^r f(x)\,dx \\ &= \sum_{r=0}^{\infty}\frac{t^r}{r!}(2\theta)^{\frac{r}{\alpha}}\,\Gamma\left(-\frac{r}{\alpha}+2\right). \end{aligned}$$

4.3 Harmonic Mean

Proposition 4

If X is the random variable has the AIWD distribution, then the harmonic mean is

$$H = \frac{1}{(2\theta)^{-\frac{1}{\alpha}}\,\Gamma\left(\frac{1}{\alpha}+2\right)} = \frac{(2\theta)^{1/\alpha}}{\Gamma\left(\frac{1}{\alpha}+2\right)}.$$

Proof:

$$\begin{aligned} \frac{1}{H} = E\left[\frac{1}{X}\right] &= \int_0^{\infty}\frac{1}{x} f(x)\,dx \\ &= \int_0^{\infty}\frac{1}{x}4\theta^2\alpha x^{-2\alpha-1}e^{-2\theta x^{-\alpha}}\,dx \end{aligned}$$

$$= \int_0^\infty 4\theta^2 \alpha x^{-2\alpha-2} e^{-2\theta x^{-\alpha}}\ dx$$

[Let $2\theta x^{-\alpha} = u \Longrightarrow 2\theta(-\alpha)x^{-\alpha-1}dx = du \Rightarrow 2\theta\alpha x^{-\alpha-1}dx = -du$]

Therefore

$$\frac{1}{H} = \int_0^\infty 2\theta\ (2\theta u^{-1})^{-\frac{1}{\alpha}} \frac{u}{2\theta} e^{-u}\ du$$

$$= (2\theta)^{-\frac{1}{\alpha}} \int_0^\infty\ u^{\left(\frac{1}{\alpha}+2\right)-1} e^{-u}\ du$$

$$= (2\theta)^{-\frac{1}{\alpha}}\ \Gamma\left(\frac{1}{\alpha}+2\right)$$

Hence

$$H = \frac{1}{(2\theta)^{-\frac{1}{\alpha}}\,\Gamma\left(\frac{1}{\alpha}+2\right)} = \frac{(2\theta)^{1/\alpha}}{\Gamma\left(\frac{1}{\alpha}+2\right)}.$$

4.4 Mode

Proposition 5

If X is a random variable having the AIWD distribution then the mode of the distribution is

$$Mode = \left(\frac{2\theta\alpha}{2\alpha-1}\right)^{\frac{1}{\alpha}}.$$

Proof

$$f(x) = 4\theta^2 \alpha x^{-2\alpha-1} e^{-2\theta x^{-\alpha}}$$

$$f'(x) = 4\theta^2\alpha\Big[\big\{x^{-2\alpha-1}e^{-2\theta x^{-\alpha}}(2\theta(-\alpha)x^{-\alpha-1})\big\} + \big\{e^{-2\theta x^{-\alpha}}(-2\alpha-1)x^{-2\alpha-2}\big\}\Big]x^{-2\alpha-1}e^{-2\theta x^{-\alpha}}$$

$$f'(x) = 4\theta^2 \alpha x^{-2\alpha-2} e^{-2\theta x^{-\alpha}}[2\theta\alpha x^{-\alpha} - (2\alpha+1)]$$

Equate $f'(x) = 0$, we have $2\theta\alpha x^{-\alpha} - (2\alpha+1) = 0$ which gives $x = \left(\frac{2\theta\alpha}{2\alpha-1}\right)^{\frac{1}{\alpha}}$.

$$f''(x) = 4\theta^2\alpha\Big[x^{-2\alpha-2}e^{-2\theta x^{-\alpha}}\{(2\theta\alpha(-\alpha)x^{-\alpha-1} - 0)\}\Big] + [2\theta\alpha x^{-\alpha} - (2\alpha+1)]\Big[\big\{x^{-2\alpha-2}e^{-2\theta x^{-\alpha}}(-2\theta(-\alpha)x^{-\alpha-1})\big\} + e^{-2\theta x^{-\alpha}}(-2\alpha-2)x^{-2\alpha-3}\Big]$$

$$= 4\theta^2\alpha\Big[\big\{x^{-2\alpha-2}e^{-2\theta x^{-\alpha}}(-2\theta\alpha^2)x^{-\alpha-1}\big\} + \{2\theta\alpha x^{-\alpha} - (2\alpha+1)\}x^{-2\alpha-3}e^{-2\theta x^{-\alpha}}\{(2\theta\alpha x^{-\alpha}) - (2\alpha+2)\}\Big]$$

$$= 4\theta^2\alpha x^{-2\alpha-3}e^{-2\theta x^{-\alpha}}[-2\theta\alpha^2 x^{-\alpha} + \{2\theta\alpha x^{-\alpha} - (2\alpha+1)\}\{(2\theta\alpha x^{-\alpha}) - (2\alpha+2)\}]$$

Put $x = \left(\frac{2\theta\alpha}{2\alpha-1}\right)^{\frac{1}{\alpha}}$ in the above we get

$$f''(x)\Big|_{x=\left(\frac{2\theta\alpha}{2\alpha-1}\right)^{\frac{1}{\alpha}}} = 4\theta^2\alpha\left(\left(\frac{2\theta\alpha}{2\alpha-1}\right)^{\frac{1}{\alpha}}\right)^{-2\alpha-3} + \times\left[\begin{array}{c}-2\theta\alpha^2\left(\left(\frac{2\theta\alpha}{2\alpha-1}\right)^{\frac{1}{\alpha}}\right)^{-\alpha} + \left\{2\theta\alpha\left(\left(\frac{2\theta\alpha}{2\alpha-1}\right)^{\frac{1}{\alpha}}\right)^{-\alpha} - (2\alpha+1)\right\} \\ \left\{\left(2\theta\alpha\left(\left(\frac{2\theta\alpha}{2\alpha-1}\right)^{\frac{1}{\alpha}}\right)^{-\alpha}\right) - (2\alpha+2)\right\}\end{array}\right]$$

$$= \underbrace{4\theta^2\alpha^3(2\alpha+1)}_{+Ve}\underbrace{\left(\frac{(2\alpha+1)}{2\theta\alpha}\right)^{\overline{(2\alpha+3)}^{\alpha}}}_{+Ve}\underbrace{e^{-\frac{2\alpha+1}{\alpha}}}_{+Ve}$$

which gives $f''(x)\Big|_{x=\left(\frac{2\theta\alpha}{2\alpha-1}\right)^{\frac{1}{\alpha}}} < 0 \Rightarrow x = \left(\frac{2\theta\alpha}{2\alpha-1}\right)^{\frac{1}{\alpha}}$ is the mode.

4.5 Median

Using the following formula

$$Median = \frac{1}{3}Mode - \frac{2}{3}Mean$$

Then, the median equals

$$Median = \frac{(2\theta\alpha)^{\frac{1}{\alpha}}}{3}\left(\frac{\alpha}{2\alpha-1}\right)^{\frac{1}{\alpha}} - \frac{2}{3}(2\theta)^{\frac{1}{\alpha}}\Gamma\left(-\frac{1}{\alpha}+2\right).$$

4.6 Shannon's Entropy

Shannon entropy is a widely used measure of uncertainty obtained in a probability distribution.

Proposition 6

If X is the random variable has the AIWD distribution then the harmonic mean is

$$H = \frac{1}{\alpha}(log\theta + 0.12175) - log(\alpha) + 2.3586.$$

Proof:

The Shannon's entropy is given by

$$\begin{aligned} H &= -E[log\{f(x)\}] \\ &= -E\left[log\left\{4\theta^2\alpha x^{-2\alpha-1}e^{-2\theta x^{-\alpha}}\right\}\right] \\ &= -E\left[log(4\theta^2\alpha) + log(x^{-2\alpha-1}) + log\left(e^{-2\theta x^{-\alpha}}\right)\right] \end{aligned}$$

$$= -E[log(4\theta^2\alpha) - (2\alpha+1)logx - 2\theta x^{-\alpha}]$$
$$= -E[log(4\theta^2\alpha)] + (2\alpha+1)E[logx] + 2\theta E[x^{-\alpha}]$$
$$= -log(4\theta^2\alpha) + (2\alpha+1)A_1 + A_2$$

$$A_1 = E[logx] = \int_{-\infty}^{\infty}(logx)f(x)\ dx$$
$$= \int_0^{\infty}(logx)4\theta^2\alpha x^{-2\alpha-1}e^{-2\theta x^{-\alpha}}\ dx$$

[Let $2\theta x^{-\alpha} = u \Longrightarrow 2\theta(-\alpha)x^{-\alpha-1}dx = du \Rightarrow 2\theta\alpha x^{-\alpha-1}dx = -du$]

$$= \int_0^{\infty}\frac{1}{\alpha}log(2\theta u^{-1})2\theta\frac{u}{2\theta}e^{-u}\,e^{-u}\,dx$$
$$= \frac{1}{\alpha}\int_0^{\infty}u(log2\theta - logu)\,e^{-u}\,dx$$
$$= \frac{log2\theta}{\alpha}\int_0^{\infty}u\,e^{-u}\,dx - \frac{1}{\alpha}\int_0^{\infty}u(logu)\,e^{-u}\,dx$$
$$= \left(\frac{log2\theta}{\alpha}\times 1\right) - \left(\frac{1}{\alpha}\times 0.42278\right)$$
$$= \frac{1}{\alpha}[(log2\theta) - 0.42278]$$
$$= \frac{1}{\alpha}[(log2 + log\theta) - 0.42278]$$
$$= \frac{1}{\alpha}[(0.30103 + log\theta) - 0.42278]$$
$$= \frac{1}{\alpha}[(log\theta) - 0.12175]$$

$$A_2 = E[x^{-\alpha}] = \int_{-\infty}^{\infty}x^{-\alpha}f(x)\,dx$$
$$= \int_0^{\infty}4\theta^2\alpha x^{-3\alpha-1}e^{-2\theta x^{-\alpha}}\ dx$$

[Let $2\theta x^{-\alpha} = u \Longrightarrow 2\theta(-\alpha)x^{-\alpha-1}dx = du \Rightarrow 2\theta\alpha x^{-\alpha-1}dx = -du$]

$$= \int_0^{\infty}2\theta\left(\frac{u}{2\theta}\right)^2 e^{-u}\ du$$
$$= \frac{1}{2\theta}\int_0^{\infty}u^2e^{-u}\ du$$
$$= \frac{1}{2\theta}\times 2 = \frac{1}{\theta}.$$

Therefore,

$$H = -log(4\theta^2\alpha) + (2\alpha+1)\frac{1}{\alpha}[(log\theta) - 0.12175] + 2\theta\times\frac{1}{\theta}$$
$$= -log(4) - log(\theta^2) - log(\alpha) + \frac{(2\alpha+1)}{\alpha}[(log\theta) - 0.12175] + 2$$

$$= -2log(2) - 2log(\theta) - log(\alpha) + 2(log\theta) + \frac{1}{\alpha}(log\theta) - 2(0.12175) - \left(\frac{1}{\alpha}0.12175\right) + 2$$

$$= (-2 \times 0.30103) - \left(2log(\theta)\right) - log(\alpha) + \left(2(log\theta)\right) + \left(\frac{1}{\alpha}(log\theta)\right) - \left(2(0.12175)\right) - \left(\frac{1}{\alpha}0.12175\right) + 2$$

$$= (0.60206) - log(\alpha) + \left(\frac{1}{\alpha}(log\theta)\right) - (0.24350) - \left(\frac{1}{\alpha}0.12175\right) + 2$$

$$= \frac{1}{\alpha}(log\theta + 0.12175) - log(\alpha) + 2.3586.$$

4.7 Order Statistics

Let X_1, X_2,…, X_n be a random sample from a distribution with PDF $f(x)$ and CDF $F(x)$. Then the pdf of the j^{th} order statistic is

$$f_{X_{(j)}}(x) = n\binom{n-1}{j-1}f(x)[F(x)]^{j-1}[1-F(x)]^{n-j}$$

First order statistic (smallest OS) is $(j = 1)$

$$f_{X_{(1)}}(x) = nf(x)[1-F(x)]^{n-1}$$

For AIWD, the PDF of the minimum order statistics is obtained by

$$f_{X_{(1)}}(x) = n\left\{4\theta^2\alpha x^{-2\alpha-1}e^{-2\theta x^{-\alpha}}\right\}\left[1-\left\{(1+2\theta x^{-\alpha})e^{-2\theta x^{-\alpha}}\right\}\right]^{n-1}$$

The n^{th} order statistic (smallest OS) is $(j = 1)$

$$f_{X_{(n)}}(x) = nf(x)[F(x)]^{n-1}$$

For AIWD, the PDF of the maximum order statistics is obtained by

$$f_{X_{(n)}}(x) = n\left\{4\theta^2\alpha e^{-2\theta x^{-\alpha}}\right\}\left[(1+2\theta x^{-\alpha})e^{-2\theta x^{-\alpha}}\right]^{n-1}$$

$$f_{X_{(n)}}(x) = 4\theta^2\alpha n x^{-2\alpha-1}(1+2\theta x^{-\alpha})e^{-n\theta x^{-\alpha}}.$$

5. MAXIMUM LIKELIHOOD ESTIMATION

The maximum likelihood estimates of the parameters of AIWD are derived in this section. Let X_1, X_2,…, X_n be a random sample from AIWD with the probability density function given in (6). Then the likelihood function is given by:

$$L = \prod_{i=1}^{n} 4\alpha\theta^2 x_i^{-2\alpha-1}e^{-2\theta x_i^{-\alpha}}$$

$$L = 4^n \alpha^n \theta^{2n} \left[\prod_{i=1}^{n} x_i^{-2\alpha-1} \right] e^{-2\theta \sum_{i=1}^{n} x_i^{-\alpha}}$$

The log-likelihood function can be written as:

$$L = n \log 4 + n \log \alpha + 2n \log \theta - (2\alpha + 1) \sum_{i=1}^{n} \log x_i - 2\theta \sum_{i=1}^{n} x_i^{-\alpha}$$

The normal equations become:

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} - 2 \sum_{i=1}^{n} \ln x_i + 4\theta \sum_{i=1}^{n} x_i^{-\alpha} \ln x_i = 0$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{2n}{\theta} - 2 \sum_{i=1}^{n} x_i^{-2\alpha} = 0.$$

Obviously, the two normal equations cannot be solved analytically for α and θ. In order to find the values of α and θ it is imperative to apply numerical iterative techniques. The MLE of the parameters can be obtained by using the Newton-Raphson method, bisection method, secant method, and other methods. In the applications section, the maxlogL function in R software will be used to obtain the maximum likelihood estimates.

The asymptotic normality results can be used to obtain the approximate confidence intervals for the maximum likelihood estimators.

The Fisher information matrix **V** can be expressed as follows:

$$V = E \begin{bmatrix} -\dfrac{\partial^2 \log L(\theta, \alpha)}{\partial \theta^2} & -\dfrac{\partial^2 \log L(\theta, \alpha)}{\partial \theta \partial \alpha} \\ -\dfrac{\partial^2 \log L(\theta, \alpha)}{\partial \alpha \partial \theta} & -\dfrac{\partial^2 \log L(\theta, \alpha)}{\partial \alpha^2} \end{bmatrix}$$

where,

$$\frac{\partial^2 \log L(\theta, \alpha)}{\partial \theta^2} = -\frac{2n}{\theta^2}$$

$$\frac{\partial^2 \log L(\theta, \alpha)}{\partial \theta \partial \alpha} = 4 \sum_{i=1}^{n} x_i^{-\alpha} \ln x_i$$

$$\frac{\partial^2 \log L(\theta, \alpha)}{\partial \alpha \partial \theta} = 4 \sum_{i=1}^{n} x_i^{-\alpha} \ln x_i$$

$$\frac{\partial^2 \log L(\theta, \alpha)}{\partial \alpha^2} = -\frac{n}{\alpha^2} - 8\theta \sum_{i=1}^{n} x_i^{-\alpha} (\ln x_i)^2.$$

The asymptotic distribution of the maximum likelihood estimators is a bivariate normal as

$$\begin{pmatrix}\hat{\theta}\\ \hat{\alpha}\end{pmatrix} \approx BVN\left(\begin{pmatrix}\theta\\ \alpha\end{pmatrix}, V^{-1}\right).$$

Here an estimate of V can be obtained by using the observed Fisher information matrix where the parameters θ and α are replaced by the corresponding maximum likelihood estimates

Therefore, the two sided 100(1-α)% asymptotic approximate confidence intervals for θ and α can be expressed as:

$$\hat{\theta} \pm z_{\frac{\alpha}{2}}\sqrt{var(\hat{\theta})}\,, \hat{\alpha} \pm z_{\frac{\alpha}{2}}\sqrt{var(\hat{\alpha})}.$$

6. APPLICATIONS

To verify its applicability and flexibility, the Ailamujia inverted Weibull distribution was compared with the baseline distribution, namely, the Ailamujia distribution and other extensions of the Ailamujia distribution. Three real lifetime data sets are analyzed. The maxlogL function in R will be used to obtain the maximum likelihood estimates, AIC, and BIC statistics. Comparisons among the distributions were accomplished using the criteria AIC (Akaike information criterion), AICC (corrected Akaike information criterion), and BIC (Bayesian information criterion). The distribution corresponds to the lower values of AIC, AICC, and BIC statistics is considered the best one. The generic formulas for finding AIC, AICC, and BIC are given as:

$$AIC = 2k - 2logL$$

$$AICC = AIC + \frac{2k(k+1)}{n-k-1}$$

$$BIC = k \log n - 2\log L$$

where *k* is the number of parameters in the statistical model.

Data Set 1:

This set of data represents the survival times, in weeks of 33 patients suffering from acute myelogenous leukemia. The observations are shown as follows:

65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43

Ul Ain (2020) used this set of data to fit his proposed power Ailamujia distribution (PAD) model and compared the results with the Weighted Ailamujia distribution (WAD), Area Biased Ailamujia Distribution (ABAD), Length Biased Ailamujia Distribution (LBAD), and Ailamujia Distribution (AD) models. The -2logL, AIC, AICC and BIC of the fitted distributions are shown in Table 1. The results of fitting the proposed AIWD of the current work are also presented in Table 1. It can clearly be seen that the proposed Ailamujia inverted Weibull distribution (AIWD) has the lower values of AIC, AICC, -2logL, and BIC as compared to the other models. Hence the Ailamujia inverted distribution shows better fitting of the data than all the other distributions. Following R command is executed to obtain the maximum likelihood estimates and AIC and BIC statistics.

```
#R Code to obtain the maximum likelihood estimates, AIC, and BIC statistics
# for Ailumujia inverted Weibull distribution
install.packages("EstimationTools")
library("EstimationTools")
# data: survival times in weeks for 33 patients suffering from acute
# myelogenous leukemia
xx<-c(65,156,100,134,16,108,121,4,39,143,56,26,22,1,1,5,65,56,65,17,7,
+16,22,3,4,2,3,8,4,30,4,43)
data=xx
ALI <- function(x, alpha, theta,log=FALSE)
{
loglik <- log(4)+2*log(theta)+log(alpha)-(2*alpha+1)*log(x)-2*theta*x^(-alpha)
}
par.hat <- maxlogL(x = data, dist = "ALI",link = list(over = c("alpha", "theta"),
fun = c("log_link", "log_link")))
summary(par.hat)
```

Table 1
Performance of Different Distribution Models for Data Set 1

Model	**Parameter Estimates**	**-2logL**	**AIC**	**AICC**	**BIC**
PAD	$\hat{\alpha} = 0.719$, $\hat{\beta} = 0.512$	307.01	311.01	311.41	314.01
WAD	$\hat{\alpha} = 0.024$, $\hat{\beta} = 0.001$	343.58	347.58	347.58	347.58
ABAD	$\hat{\alpha} = 0.049$, $\hat{\beta} = 0.004$	433.58	437.58	437.71	439.08
LBAD	$\hat{\alpha} = 0.024$, $\hat{\beta} = 0.003$	341.54	345.54	345.67	347.03
AD	$\hat{\alpha} = 0.024$, $\hat{\beta} = 0.003$	341.54	345.54	345.67	347.03
AIWD (This work)	$\hat{\alpha} = 0.471$, $\hat{\beta} = 3.012$	304.64	308.64	309.04	311.57

Data Set 2:

This set of data has been reported by Gross and Clark (1975), which signifies the relief times of 20 patients getting an analgesic. The data are shown as follows:

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0.

Aijaz (2020) used this set of data to fit his proposed Inverse Ailamujia Distribution (IAD) model and compared the results with Ailamujia distribution (AD), exponential distribution (ED), Inverse Exponential distribution (IED), Lindley distribution (LD), and

Inverse Lindley distribution (ILD). The -2logL, AIC, AICC, and BIC of these fitted models are presented in Table 2. The results of the proposed distribution (AIWD) of the current work are also presented in Table 2. Again, the Ailamujia inverted Weibull distribution (AIWD) reveals lower values of AIC, AICC, -2logL, and BIC compared to other distribution models indicating that it fits such set of data better than the other distribution models presented in Table 2.

Table 2
Performance of Different Distribution Models for Data Set 2

Model	Parameter Estimates	-2logL	AIC	AICC	BIC
IAD	1.725	51.65	53.65	53.99	54.65
AD	0.526	52.33	54.33	64.66	55.32
ED	0.526	65.67	67.67	68.01	68.67
IED	1.724	65.34	67.34	67.67	68.33
LD	0.816	60.50	62.50	62.83	63.50
ILD	2.255	63.51	65.51	65.85	66.51
AIWD (This work)	$\hat{\alpha} = 2.670, \hat{\theta} = 3.658$	30.95	34.95	35.55	36.94

Data Set 3:

The third set of data are reported by Bader and Priest (1982), which is related to the failure stresses (in GPa) and composed of 65 single carbon fibers of lengths 50 mm, respectively. The data set is given as follows:

1.339, 1.434, 1.549, 1.574, 1.589, 1.613, 1.746, 1.753, 1.764,1.807, 1.812, 1.84, 1.852, 1.852, 1.862, 1.864, 1.931, 1.952, 1.974, 2.019, 2.051, 2.055, 2.058,2.088, 2.125, 2.162, 2.171, 2.172, 2.18, 2.194, 2.211, 2.27, 2.272, 2.28, 2.299, 2.308, 2.335,2.349, 2.356, 2.386, 2.39, 2.41, 2.43, 2.431, 2.458, 2.471, 2.497, 2.514, 2.558, 2.577, 2.593,2.601, 2.604, 2.62, 2.633, 2.67, 2.682, 2.699, 2.705, 2.735, 2.785, 3.02, 3.042, 3.116, 3.174

Uzma *et al.* (2017) used this set of data to fit their proposed weighted Ailamujia distribution (WAD) model and compared the results with the Ailamujia distribution (AD), length biased Ailamujia distribution (LBAD), and area biased Ailamujia distribution (ABAD) models. The -2logL, AIC, AICC and BIC of these distribution models are presented in Table 3. The results of the proposed distribution of this work are also presented in Table 3. Again, the Ailamujia inverted distribution (AIWD) showed superiority in terms of the lowest values of AIC, AICC, -2logL, and BIC as compared to all other distribution models. This also indicates that the Ailamujia inverted distribution proposed in this work fits this set of data better than other distribution models considered in Table 3.

Table 3
Performance of different distribution models for data set 3

Model	Parameter Estimates	-2logL	AIC	AICC	BIC
WAD	$c = 8.288\,, \hat{\theta} = 2.288$	97.19	99.19	99.39	103.54
AD	0.446	187.15	189.15	189.21	191.32
LBAD	0.668	161.31	163.31	163.37	165.48
ABAD	0.891	144.00	146.00	146.06	148.18
AIWD (This work)	$\hat{\alpha} = 3.543\,, \hat{\theta} = 12.892$	40.83	85.65	85.83	90.00

6. CONCLUSION

A new extension of Ailamujia distribution, namely the Ailamujia inverted Weibull (AIWD) is proposed in this work. Several mathematical properties, such as reliability functions, failure rate, mean residual function, moments, mean, variance, mode, median, moment generating function, Shannon's entropy, and order statistics were derived. Parameter estimation using the method of maximum likelihood and asymptotic confidence intervals was presented. Three sets of data from the literature were analyzed to illustrate the applicability of the proposed distribution model. The proposed extension of AIWD showed superiority in fitting the three sets of data compared to other extensions of Ailamujia distribution models as well as some other distribution models.

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