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EXPONENTIATED INVERTED TOPP-LEONE DISTRIBUTION

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ABSTRACT

In this article, a new generalized distribution named exponentiated inverted Topp-Leone distribution is introduced and some of its properties are presented. Some models of stress strength, quantile, moments, Rényi entropy and order statistics are discussed. The relation between exponentiated inverted Topp-Leone distribution and other distributions are derived. The maximum likelihood and Bayes estimators for the parameters, the reliability and hazard rate functions of the exponentiated inverted Topp-Leone distribution based on Type II censored samples are obtained. A simulation study is performed to investigate the effectiveness of the proposed distribution. Finally, a real data set is analyzed to illustrate its flexibility for real-life application.

KEYWORDS

Topp-Leone distribution, maximum likelihood estimation, Type II censored, Monte Carlo simulation.

1. INTRODUCTION

Topp-Leone (TL) distribution is one of the continuous unimodal distributions with bounded support. Hence, it is appropriate for modeling lifetime of distributions with finite support such as limited power supply, maintenance repair resource, or design life of the system. It was proposed by Topp and Leone (1955). It provides closed forms of the *cumulative distribution function* (cdf) and *probability distribution function* (pdf).

Considering that a random variable X has TL distribution with parameter α denoted by $X \sim TL\ (\alpha)$, with the pdf

$$g_{TL}(x) = 2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1}, 0 < x < 1, \alpha > 0. \quad (1)$$

The corresponding cdf is

$$G_{TL}(x) = x^{\alpha}(2-x)^{\alpha}, 0 < x < 1, \alpha > 0. \quad (2)$$

The inverted distributions are important in a wide range of applications; for example, in the problems which are related to econometrics, biological sciences, survey sampling, engineering sciences, medical research and life testing problems. In addition, it is employed in financial literature, environmental studies, survival and reliability theory. *Inverted Topp-Leone* (ITL) distribution was introduced by Hassan *et al.* (2020) and some of its properties were presented. The ITL distribution can be derived from TL distribution using the transformation

$$T = \frac{1}{X} - 1.$$

The pdf and cdf of the ITL distribution are as follows:

$$g_{ITL}(t;\alpha) = 2\alpha t(1+t)^{-2\alpha-1}(1+2t)^{\alpha-1}, t > 0, \alpha > 0, \quad (3)$$

and

$$G_{ITL}(t;\alpha) = 1 - \frac{(1+2t)^{\alpha}}{(1+t)^{2\alpha}}, t > 0, \alpha > 0. \quad (4)$$

Many authors focused on generalizing distributions; for example, a general class of distributions generated from the logit of the beta random variable which was introduced by Eugene *et al.* (2002), Cordeiro and de Castro (2011) proposed the Kumaraswamy generalized distributions. Alexander *et al.* (2012) presented generalized beta-generated distributions. Korkmaz and Genç (2017) defined the two-sided generalized class of distributions as beta-generated and Kumaraswamy-generated distributions. Korkmaz *et al.* (2018) derived and studied a family of distributions called the exponential Lindely odd log-logistic-G family.

There are several ways of adding one or more parameters to a distribution function. Such an addition of parameters makes the resulting distribution richer and more flexible for modeling and analyzing data. Adding a parameter by exponentiation is one of the methods for generalizing distributions; exponentiation goes back to Gompertz (1825) and Verhulst (1838, 1845 and 1847). Lehman (1953) defined the exponentiated distribution as a non-parametric class of alternatives. Ahuja and Nash (1967) introduced exponentiated Gompertz-Verhulst family. Gupta *et al.* (1998) considered a new family of distributions termed as exponentiated exponential distribution. Gupta and Kundu (2001) studied some properties of the distribution and they observed that many properties of the new family are similar to those of the Weibull or gamma family. Exponentiated distributions have received much attention in recent years for example, Ali *et al.* (2007) suggested some exponentiated distributions. Also, the exponentiated Lomax distribution was proposed by Abdul-Moniem and Abdel-Hameed (2012). Ebraheim (2014) introduced exponentiated transmuted Weibull distribution. Pourdarvish *et al.* (2015) presented a generalization of the TL distribution referred to as the exponentiated TL distribution. Rao and Mbwambo (2019) derived the exponentiated inverse Rayleigh distribution.

Applications of exponentiated models have been carried out by some authors. In particular, the exponentiated Weibull (EW) family was applied in analyzing bathtub failure data by Mudholkar and Srivastava (1993), flood data by Mudholkar and Hutson (1996), bus-motor-failure data in Davis (1952), head–neck cancer clinical trial data in Efron (1988) by Mudholkar *et al.* (1995) and Rao and Mbwambo (2019) presented an application to coating weights of iron sheets data.

There are three methods to obtain exponentiated distribution:

- Powering a positive real number θ to the cdf distribution, [See Gupta *et al.* (1998) and Gupta and Kundu (2001)], where

$$F(t) = [G(t)]^{\theta}, \theta > 0.$$

- Nadarajah and Kotz (2006) suggested a method for generating a new distribution by using the reliability function as follows:

$$F(t) = 1 - [1 - G(t)]^{\theta}, \theta > 0. \tag{5}$$

- Nadarajah (2005) introduced exponentiation by using the transformation: $t = \log(x)$, where x is a non-negative random variable for the base distribution.

In this paper *Exponentiated Inverted Topp-Leone* (EITL) distribution is derived by applying the method suggested by Nadarajah and Kotz (2006).

The cdf and pdf of EITL distribution are, respectively, given by

$$F(t;\alpha,\theta) = 1 - [2(t+1)^{-1} - (t+1)^{-2}]^{\alpha\theta}, t > 0;\ \alpha,\theta > 0. \tag{6}$$

and

$$f(t;\alpha,\theta) = 2\theta\alpha[2(t+1)^{-1} - (t+1)^{-2}]^{\alpha\theta-1}[(t+1)^{-2} - (t+1)^{-3}], \quad t > 0;\ \alpha,\theta > 0. \tag{7}$$

The plots of the pdf are provided in Figure 1.

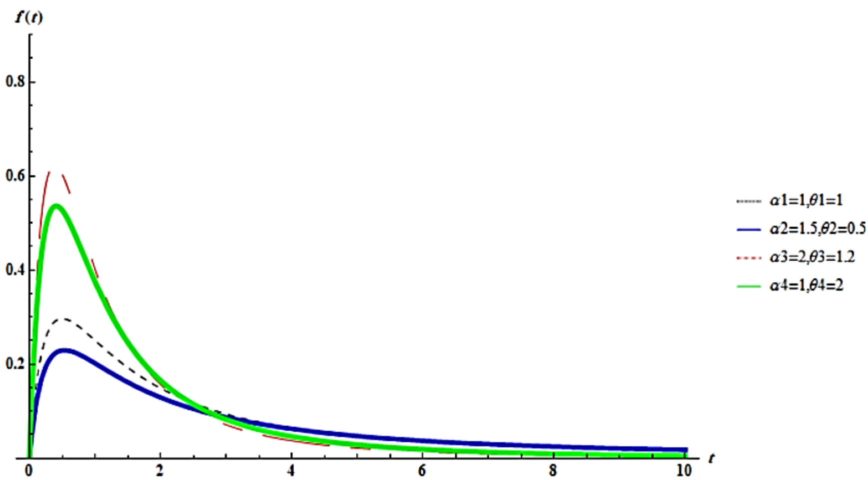


Figure 1: Plots of the Probability Density Function at Different Values of α and θ

From Figure 1 one can observe that the pdf of EITL distribution is unimodal and the curves are right skewed with long tail.

This paper is organized as follows: in Section 2 the main properties of the EITL distribution are presented. In Section 3, the *maximum likelihood* (ML) estimation for the unknown parameters is derived, also the *reliability function* (rf), *hazard rate function* (hrf) and the *reversed hazard rate function* (rhf) of the EITL distribution based on Type II censored samples are obtained. In Section 4, a numerical illustration is presented to investigate the precision of the ML estimates and a real data set is discussed to demonstrate how the theoretical results can be used in practice.

2. SOME PROPERTIES OF THE EXPONENTIATED INVERTED TOPP-LEONE DISTRIBUTION

a. Reliability Function

$$R_1(t) = [2(t+1)^{-1} - (t+1)^{-2}]^{\alpha\theta}, t > 0, \alpha > 0. \quad (8)$$

b. Some Stress-Strength Reliability Model

Two formulae of the stress-strength reliability model are given below

i. First Formula

Let T and Y are independent and have EITL distribution, and assuming that T is the strength of a component which is subjected to the stress Y. Then the pdf of T and Y are given, respectively, by

$$f(t;\alpha,\theta_1) = 2\theta_1\alpha[2(t+1)^{-1} - (t+1)^{-2}]^{\alpha\theta_1 - 1}[(t+1)^{-2} - (t+1)^{-3}], t > 0, \alpha, \theta_1 > 0, \quad (9)$$

and

$$f(y;\alpha,\theta_2) = 2\theta_2\alpha[2(y+1)^{-1} - (y+1)^{-2}]^{\alpha\theta_2 - 1}[(y+1)^{-2} - (y+1)^{-3}], y > 0, \alpha, \theta_2 > 0. \quad (10)$$

Then the rf; $R_2(t)$, is given by

$$R_2(t) = P(Y < T) = \int_0^\infty \int_0^t f(t;\alpha,\theta_1)\, f(y;\alpha,\theta_2) dy\, dt$$
$$= \int_0^\infty f(t;\alpha,\theta_1) \mathrm{F_y}(y;\alpha,\theta_2) dt,$$

where $F_y(y;\alpha,\theta_2)$ is the cdf of Y at the point t.

Using the generalized binomial expansion, where θ is real non integer, $\theta > 0$ and $|z| < 1$ as

$$(1+z)^{\theta-1} = \sum_{j=0}^{\infty} \binom{\theta - 1}{j} z^j,$$

hence

$$R_2(t) = 1 - \sum_{j=0}^{\infty} 2\theta_1\alpha \binom{\alpha\theta_1 + \alpha\theta_2 - 1}{j} \boldsymbol{IB}(j+2, 2(\alpha\theta_1 + \alpha\theta_2) - 1). \quad (11)$$

where $\boldsymbol{IB}(j+2, 2(\alpha\theta_1 + \alpha\theta_2) - 1)$ is incomplete beta, $j > 0$ and $\alpha\theta_1 + \alpha\theta_2 > 0$.

ii. Second Formula

Let Y and Z be two independent random stress variables with known cdfs $F_Y(y)$ and $F_Z(z)$ which have EITL (α, θ_2) and EITL (α, θ_3), respectively, and let T be independent of Y and Z is a random strength variable with known cdf $F_T(t)$ and has EITL (α, θ_1). Then

$$R_3(t) = P(Y < T < Z) = P(Y < T) - P[Y < T, Z < T]$$
$$= \int_0^\infty F_Y(t) f_T(t)\, dt - \int_0^\infty F_Y(t) F_Z(t) f_t(t)\, dt,$$

hence

$$R_3(t) = \sum_{k=0}^{\infty} 2\theta_1\alpha \binom{\alpha\theta_1 + \alpha\theta_3 - 1}{k} \boldsymbol{IB}(k + 2,2(\alpha\theta_1 + \alpha\theta_3) - 1) + \sum_{l=0}^{\infty} 2\theta_1\alpha \binom{\alpha\theta_1 + \alpha\theta_2 - 1}{l} \boldsymbol{IB}(l + 2,2(\alpha\theta_1 + \alpha\theta_2) - 1) - \sum_{j=0}^{\infty} 2\theta_1\alpha \binom{\alpha\theta_1 + \alpha\theta_2 - 1}{j} \boldsymbol{IB}(j + 2,2(\alpha\theta_1 + \alpha\theta_2) - 1) - \sum_{m=0}^{\infty} 2\theta_1\alpha \binom{\alpha\theta_1 + \alpha\theta_2 + \alpha\theta_3 - 1}{m} \times \boldsymbol{IB}(m + 2,2(\alpha\theta_1 + \alpha\theta_2 + \alpha\theta_3) - 1). \quad (12)$$

where $\boldsymbol{IB}(.,.)$ is incomplete beta, $k, l, j, m > 0$ and $\alpha, \theta_1, \theta_2, \theta_3 > 0$.

c. Hazard Rate Function

$$h_{EITL}(t) = \frac{2\theta\alpha[(t+1)^{-2} - (t+1)^{-3}]}{[2(t+1)^{-1} - (t+1)^{-2}]}. \quad (13)$$

d. Reversed Hazard Rate Function

$$rhf_{EITL}(t) = \frac{2\theta\alpha[2(t+1)^{-1} - (t+1)^{-2}]^{\alpha\theta-1}[(t+1)^{-2} - (t+1)^{-3}]}{1 - [2(t+1)^{-1} - (t+1)^{-2}]^{\alpha\theta}}. \quad (14)$$

The plots of hrf and rhf are provided in Figures 2 and 3, respectively.

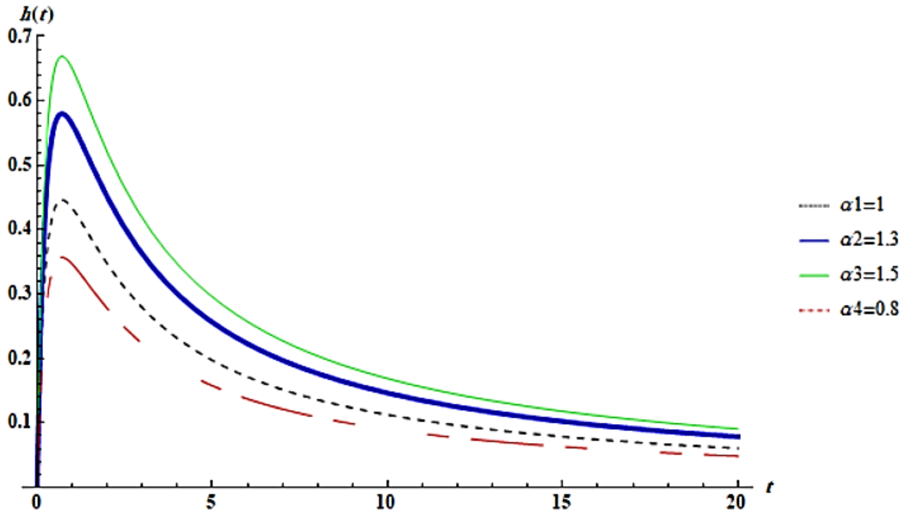


Figure 2: Plots of the Hazard Rate Function at Different Values of α and $\theta = 1.3$

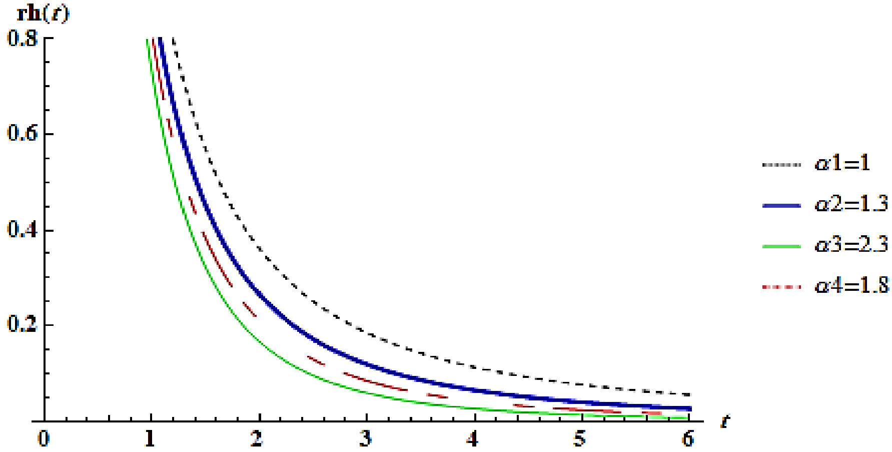


Figure 3: Plots of the Reversed Hazard Rate Function at Different Values of α and $\theta = 1.3$

From Figure 2, one can observe that the hrf of the EITL distribution is unimodal and right skewed. In Figure 3, the rhf of EITL distribution is monotone decreasing function.

e. Quantile

Using (6), the quantile function of the EITL distribution is

$$t_q = \frac{\left(1-(1-q)^{\frac{1}{\alpha\theta}}\right)+\sqrt{1-(1-q)^{\frac{1}{\alpha\theta}}}}{(1-q)^{\frac{1}{\alpha\theta}}}, 0 < q < 1. \quad (15)$$

Special cases can be obtained from (15) such as the first quartile Q_1(if $q = 0.25$), the second quartile Q_2 (the median if $q = 0.5$), and the third quartile Q_3(if $q = 0.75$).

f. Moments

If T~EITL (α, θ) distribution, then the r^{th} moment is given by

$$\mu'_r = E(t^r) = \int_0^\infty t^r f(t;\alpha,\theta)\,dt = \sum_{s=0}^{\infty} 2\alpha\theta \binom{\alpha\theta-1}{s} \boldsymbol{IB}(r+s+2, 2\alpha\theta - r), \quad (16)$$

where $\boldsymbol{IB}(.,.)$ is incomplete beta, $r, s > 0$ and $\alpha\theta > r$.

The central moments of EITL (α, θ) can be obtained by using the relationship between the central moments and the non-central moments in (16) as follows:

$$\mu_r = \sum_{j=0}^{r} \binom{r}{j} (-1)^j \mu^j \mu'_{r-j}.$$

Some numerical values for the mean, variance, median, skewness and kurtosis of EITL distribution at different values of θ and α are given in Tables 1 and 2.

Table 1
The Mean, Variance, Median, Skewness and Kurtosis of EITL at Different Values of θ and $\alpha = 5$

θ	Mean	Variance	Median	Skewness	Kurtosis
1	0.8221	0.2823	0.6614	1.4210	4.4563
3	0.2823	0.0273	0.0262	0.7584	3.3176
5	0.2356	0.0270	0.1987	0.5149	1.9765
6	0.2328	0.0282	0.1792	1.2028	3.8300
10	0.1475	0.0087	0.1132	1.2722	4.6791

Table 2
The Mean, Variance, Median, Skewness and Kurtosis of EITL at Different Values of α and $\theta = 5$

α	Mean	Variance	Median	Skewness	Kurtosis
1	0.9249	0.6703	0.6688	1.4922	4.3327
3	0.3050	0.0453	0.2578	1.0326	3.7425
5	0.2356	0.0270	0.1987	0.5149	1.9765
6	0.1904	0.0157	0.1774	0.8173	3.3701
10	0.1747	0.0091	0.1650	1.0954	4.7529

From the values of the mean, variance and median in Tables 1 and 2 one can notice that there is a reverse relation between α and θ, also the values of the skewness are positive which ensures that the EITL distribution is a right skewed distribution, and almost the values of the kurtosis show that the EITL distribution is leptokurtic.

g. Moment Generating Function

The moment generating function is

$$E(e^{it}) = \int_0^\infty e^{izt} f(t;\alpha,\theta)dt\,,$$

Using the series $e^{it} = \sum_{z=0}^{\infty} \frac{(it)^z}{z!}$, hence

$$E(e^{it}) = \sum_{y=0}^{\infty}\sum_{z=0}^{\infty} 2\alpha\theta \frac{(i)^z}{z!}\binom{\alpha\theta-1}{y}\boldsymbol{IB}(z+y+2,2\alpha\theta-z). \tag{17}$$

where $\boldsymbol{IB}(.,.)$ is incomplete beta, $z, y > 0$ and $\alpha,\theta > z$.

Rényi Entropy

An entropy of a random variable T with the pdf $f(t;\alpha,\theta)$ is a measure of variation of the uncertainty and is denoted by $H_R(\rho)$. It is defined by

$$H_R(\rho) = (1-\rho)^{-1}\log\left\{\int_0^\infty f(t;\alpha,\theta)^\rho dt\right\}$$

$$= (1-\rho)^{-1}\,log\left\{\sum_{p=0}^{\infty}(2\theta\alpha)^\rho\binom{\alpha\theta\rho-\rho}{p}\boldsymbol{IB}(p+\rho+1,2\alpha\theta\rho-1)\right\}, \tag{18}$$

$\rho > 0\,, \rho \neq 1.$

h. Mean Residual Life

The mean residual life is the expected remaining life, $T - t_0$, given that the item has survived to time t_0. It is denoted by $m(t_0)$ and is given by

$$m(t_0) = E(T - t_0|T \geq t_0) = \frac{1}{R(t_0)} \int_{t_0}^{\infty} R(t_0)\, dt$$
$$= \left[\frac{[2(t_0+1)^{-1} - (t_0+1)^{-2}]}{2(\alpha\theta+1)[(t_0+1)^{-2} - (t_0+1)^{-3}]}\right]. \quad (19)$$

i. Mean Past Lifetime

In a real-life situation, where systems often are not monitored continuously, one might be interested in getting inference more about the history of the system, e.g. when the individual components have failed. Assume that a component with lifetime T has failed at or some time before t_0, $t_0 \geq 0$. Consider the conditional random variable $T - t_0|t_0 \leq T$. This conditional random variable shows in fact, the time elapsed from the failure of the component given that its lifetime is less than or equal to t_0. Hence, the mean past lifetime of the component can be defined by

$$\mu(t_0) = E(T - t_0|t_0 \leq T) = \frac{\int_0^{t_0} F(t)dt}{F(t_0)}$$
$$= \frac{1}{F(t_0)}\left[t_0 - \frac{[2(t_0+1)^{-1} - (t_0+1)^{-2}]^{\alpha\theta+1}}{2(\alpha\theta+1)[(t_0+1)^{-2} - (t_0+1)^{-3}]}\right]. \quad (20)$$

j. Order Statistics

Suppose that $T_1, T_2, \ldots, T_r, \ldots, T_n$ is a random sample with pdf; $f(t)$ and cdf; $F(t)$. Let $t_1 < t_2 < \cdots t_r < \ \ldots < t_n$ denote the corresponding order statistics, hence

The rth Order Statistic

$$f\left(t_{(r)}\right) = \frac{n!}{(r-1)!\,(n-r)!}\left[F\left(t_{(r)}\right)\right]^{r-1}\left[1 - F\left(t_{(r)}\right)\right]^{n-r} f\left(t_{(r)}\right)$$
$$= \frac{n!}{(r-1)!\,(n-r)!} 2\alpha\theta$$
$$\times \left[1 - \left[2\left(t_{(r)}+1\right)^{-1} - \left(t_{(r)}+1\right)^{-2}\right]^{\alpha\theta}\right]^{r-1}$$
$$\times \left[2\left(t_{(r)}+1\right)^{-1} - \left(t_{(r)}+1\right)^{-2}\right]^{\alpha\theta(n-r+1)-1}$$
$$\times \left[\left(t_{(r)}+1\right)^{-2} - \left(t_{(r)}+1\right)^{-3}\right]. \quad (21)$$

The Smallest Order Statistic at $r = 1$

$$f\left(t_{(1)}\right) = n\left[1 - F\left(t_{(1)}\right)\right]^{n-1} f\left(t_{(1)}\right)$$
$$= 2n\theta\alpha\left[2\left(t_{(1)}+1\right)^{-1} - \left(t_{(1)}+1\right)^{-2}\right]^{\alpha\theta n-1}$$
$$\times \left[\left(t_{(1)}+1\right)^{-2} - \left(t_{(1)}+1\right)^{-3}\right]. \quad (22)$$

The Largest Order Statistic at $r = n$

$$f\left(t_{(n)}\right) = n\left[F\left(t_{(n)}\right)\right]^{n-1} f\left(t_{(n)}\right)$$
$$= 2n\theta\alpha\left[1-\left[2\left(t_{(r)}+1\right)^{-1}-\left(t_{(r)}+1\right)^{-2}\right]^{\alpha\theta}\right]^{n-1}$$
$$\times\left[2\left(t_{(r)}+1\right)^{-1}-\left(t_{(r)}+1\right)^{-2}\right]^{\alpha\theta-1}$$
$$\times\left[\left(t_{(r)}+1\right)^{-2}-\left(t_{(r)}+1\right)^{-3}\right]. \tag{23}$$

The Joint Probability Function of x and y

Let $x = t_r, y = t_s$, where $s = r + 1$,

$$f_{r,s}(x,y) = \frac{n!}{(r-1)!\,(s-r)!\,(n-s)!} f(x)f(y)[F(x)]^{r-1}$$
$$\times [F(y)-F(x)]^{s-r-1}\,[1-F(y)]^{n-s}$$
$$= \frac{n!}{(r-1)!\,(s-r)!\,(n-s)!} 4\theta^2\alpha^2[2(x+1)^{-1}$$
$$-(x+1)^{-2}]^{\alpha\theta-1}[(x+1)^{-2}$$
$$-(x+1)^{-3}][2(y+1)^{-1}-(y+1)^{-2}]^{\alpha\theta(n-s+1)-1}$$
$$\times[(y+1)^{-2}-(y+1)^{-3}]$$
$$\left\{1-[2(x+1)^{-1}-(x+1)^{-2}]^{\alpha\theta}\right\}^{r-1}$$
$$\times\left[[2(x+1)^{-1}-(x+1)^{-2}]^{\alpha\theta}\right.$$
$$\left.-[2(y+1)^{-1}-(y+1)^{-2}]^{\alpha\theta}\right]^{s-r-1}, 0 < \mathrm{x} < \mathrm{y} < \infty. \tag{24}$$

3. MAXIMUM LIKELIHOOD ESTIMATION

The ML approach is applied to estimate the parameters of EITL based on Type II censored samples.

Suppose that $T_{(1)} \le T_{(2)} \le \cdots \le T_{(r)}$ is a censored sample of size r obtained from a life test on n items, the likelihood function based on Type II censored samples is given by

$$L\left(\alpha,\theta;\underline{t}\right) \propto \prod_{i=1}^{r} f(t_i,\alpha,\theta)[R(t_r,\alpha,\theta)]^{n-r}$$
$$\propto (2\theta\alpha)^r \prod_{i=1}^{r}[2(t+1)^{-1}-(t+1)^{-2}]^{\alpha\theta-1}$$
$$\times[(t+1)^{-2}-(t+1)^{-3}]$$
$$\times[2(t+1)^{-1}-(t+1)^{-2}]^{\alpha\theta}]^{n-r}. \tag{25}$$

The natural logarithm of the likelihood function is given by

$$\ell \equiv \ln L\left(\alpha,\theta;\underline{t}\right)$$
$$\propto r\ln(\theta) + r\ln(\alpha) + (\theta\alpha-1)\textstyle\sum_{i=1}^{r} \ln(2(t_i+1)^{-1}-(t_i+1)^{-2})$$
$$+\sum_{i=1}^{r} \ln((t_i+1)^{-2}-(t_i+1)^{-3})$$
$$+(n-r)(\theta\alpha)\ln(2(t_r+1)^{-1}-(t_r+1)^{-2}). \tag{26}$$

Suppose that the two parameters α and θ are unknown, differentiating ℓ in (26) with respect to α and θ, one obtains

$$\frac{\partial \ell}{\partial \theta} = \frac{r}{\theta} + (n-r)\alpha \ln(2(t_r+1)^{-1} - (t_r+1)^{-2}) + \alpha \sum_{i=1}^{r} \ln(2(t_i+1)^{-1} - (t_i+1)^{-2}), \tag{27}$$

and

$$\frac{\partial \ell}{\partial \alpha} = \frac{r}{\alpha} + (n-r)\theta \ln(2(t_r+1)^{-1} - (t_r+1)^{-2}) + \sum_{i=1}^{r} \ln(2(t_i+1)^{-1} - (t_i+1)^{-2}). \tag{28}$$

Equating (27) and (28) to zeroes, and solving the system of non-linear equations numerically, then the ML estimates for α and θ can be evaluated.

The ML estimates of $R_1(t), R_2(t), R_3(t), h(t), and\ rh(t)$ can be derived, using the invariance property of the ML estimators by replacing the parameters α and θ in (8), (11), (12), (13), and (14) by their ML estimators. Then

$$\hat{R}_1(t) = [2(t+1)^{-1} - (t+1)^{-2}]^{\hat{\alpha}\hat{\theta}}, t > 0, \hat{\alpha}, \hat{\theta} > 0. \tag{29}$$

$$\hat{R}_2(t) = 1 - \sum_{j=0}^{\infty} 2\hat{\theta}_1 \hat{\alpha} \binom{\hat{\alpha}\hat{\theta}_1 + \hat{\alpha}\hat{\theta}_2 - 1}{j} \boldsymbol{IB}(j+2, 2(\hat{\alpha}\hat{\theta}_1 + \hat{\alpha}\hat{\theta}_2) - 1). \tag{30}$$

$$\begin{aligned}\hat{R}_3(t) = &\sum_{k=0}^{\infty} 2\hat{\alpha}\hat{\theta}_1 \binom{\hat{\alpha}\hat{\theta}_1 + \hat{\alpha}\hat{\theta}_3 - 1}{k} \boldsymbol{IB}\big(k+2, 2(\hat{\alpha}\hat{\theta}_1 + \hat{\alpha}\hat{\theta}_3) - 1\big) \\ &+ \sum_{l=0}^{\infty} 2\hat{\alpha}\hat{\theta}_1 \binom{\hat{\alpha}\hat{\theta}_1 + \hat{\alpha}\hat{\theta}_2 - 1}{l} \boldsymbol{IB}\big(l+2, 2(\hat{\alpha}\hat{\theta}_1 + \hat{\alpha}\hat{\theta}_2) - 1\big) \\ &- \sum_{j=0}^{\infty} 2\hat{\theta}_1\hat{\alpha} \binom{\hat{\alpha}\hat{\theta}_1 + \hat{\alpha}\hat{\theta}_2 - 1}{j} \boldsymbol{IB}\big(j+2, 2(\hat{\alpha}\hat{\theta}_1 + \hat{\alpha}\hat{\theta}_2) - 1\big) \\ &- \sum_{m=0}^{\infty} 2\hat{\alpha}\hat{\theta}_1 \binom{\hat{\alpha}\hat{\theta}_1 + \hat{\alpha}\hat{\theta}_2 + \alpha\theta_3 - 1}{m} \\ &\times \boldsymbol{IB}\big(m+2, 2(\hat{\alpha}\hat{\theta}_1 + \hat{\alpha}\hat{\theta}_2 + \hat{\alpha}\hat{\theta}_3) - 1\big),\end{aligned} \tag{31}$$

$$\hat{h}(t) = \frac{2\hat{\theta}\hat{\alpha}[(t+1)^{-2} - (t+1)^{-3}]}{[2(t+1)^{-1} - (t+1)^{-2}]}, \tag{32}$$

and

$$\widehat{rh}(t) = \frac{2\hat{\theta}\hat{\alpha}[2(t+1)^{-1} - (t+1)^{-2}]^{\hat{\theta}\hat{\alpha}-1}[(t+1)^{-2} - (t+1)^{-3}]}{1 - [2(t+1)^{-1} - (t+1)^{-2}]^{\hat{\theta}\hat{\alpha}}}. \tag{33}$$

4. NUMERICAL ILLUSTRATION

This section aims to investigate the precision of the theoretical results of estimation based on the simulated and real data.

4.1 Simulation Study

1. A simulation study is performed to investigate the precision of the maximum likelihood estimates and to ensure the effectiveness of the proposed distribution, based on generated data from EITL distribution, for different sample sizes (*n* = 30, 50, 70, 90, 200 and 500). The computations are performed using Mathematica 9.
2. The *estimated risks* (ERs) of the ML estimates for the shape parameters, rf, hrf and rhf are computed as follows:

$$\mathrm{ER}(\mathrm{estimator}) = \frac{\sum_{p=1}^{\mathrm{sim}}(estimator - true\ value)^2}{\mathrm{sim}}. \tag{33}$$

3. Tables 4 and 5 display the ML averages, ERs of the ML estimates and 95% confidence intervals for the parameters. The population parameter values of α and θ from EITL distribution; used in this simulation study, are α = 0.3, 0.2 and $\theta = 0.5, 0.8$. Different samples of size *n*=30, 50, 90, 200 and 500 are drawn under two levels of censoring (30% and 10%).
4. Tables 6 and 7 present the ML averages, ERs of the ML estimates and 95% confidence intervals of the rf, hrf and rhf at α = 0.3, 0.2 and $\theta = 0.5, 0.8$ from EITL distribution for different samples of size *n*=30, 50, 90, 200 and 500 under two levels of censoring (30% and 10%).

Concluding Remarks

- Tables 4 and 5 show that the ML averages, ERs of the ML estimates for the two parameters α and θ perform better when the sample size n increases. One can observe that as the level of censoring decreases, the ERs of the ML estimate decrease. Also, the lengths of the CIs become narrower as the sample size increases.
- It is noticed from Tables 6 and 7 that the ERs of the ML estimates for rf, hrf and rhf decrease when the sample size n increases. It is observed that as the level of censoring decreases the ERs and ML estimate decrease. The lengths of the CIs become narrower as the sample size increases.

4.2 Application

This subsection aims to demonstrate how the proposed EITL distribution; based on Type-II censoring with level of censoring 10%, can be used in practice.

Using the data given by Singh and Maddala (1976) the EITL and ITL distribution is fitted to the real data using Kolmogorov-Smirnov goodness of fit test through Mathematica 9. The data represents the strength of 1.5 cm glass fibers for 60 devices. The data are:

0.636, 0.252, 0.157, 0.187, 2.771, 0.209, 0.617, 2.078, 1.013, 0.499, 0.431, 0.642, 0.46, 0.749, 0.205, 0.576, 0.439, 0.471, 0.262, 0.387, 0.324, 0.424, 0.548, 1.794, 1.233, 0.915, 0.702, 0.417, 0.337, 0.435, 0.359, 0.293, 0.147, 0.87, 0.608, 0.153, 0.098, 0.557, 0.415, 0.122, 0.912, 0.341, 0.725, 0.364, 0.24, 0.594, 0.325, 0.416, 0.08, 0.582, 1.257, 1.575, 0.48, 0.909, 0.17, 0.319, 0.09, 0.154, 2.248, 0.292.

- Kolmogorov-Smirnov goodness of fit test is applied to check the validity of the EITL and ITL distributions. The p-values are given, respectively, by 0.2671 and 0.080. The p value given in each case showed that the EITL and ITL distributions fits the data very well.

- The EITL distribution is compared to ITL distribution. To verify which distribution fits better to the real data set, the Kolmogorov-Smirnov goodness of fit test is employed. Criterion including -2*log-likelihood* (-2LL), *Akaike information criterion* (AIC), *Bayesian information criterion* (BIC), and *corrected AIC* (CAIC), are provided for model selection.

- The best distribution corresponds the lowest value of -2LL, AIC, AICC and BIC, which are calculated from (35).

- ML estimates, *standard errors* (SE), -2LL, AIC, BIC, and CAIC are presented in Table 3.

$$AIC = 2K - 2logL, BIC = Klogn - 2logL, CAIC = -2LL + \frac{2Kn}{n-K-1} \quad (35)$$

where K is the number of the parameters in the statistical model, n is the sample size and L is the maximized value of the likelihood function for the estimated model.

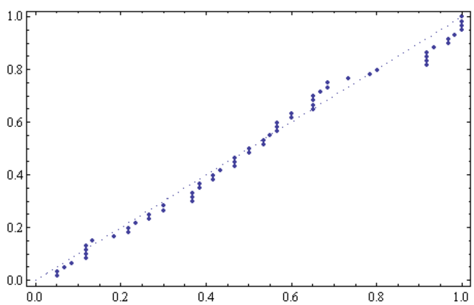
Table 3

Model	MLE	SE	-2LL	AIC	BIC	CAIC
EITL	$\hat{\alpha} = 4.7215$ $\hat{\theta} = 1.0554$	0.1965 0.0439	115.5592	61.7796	65.9683	61.9901
ITL	$\hat{\theta} = 1.4112$	0.6722	593.9700	298.985	301.079	299.054

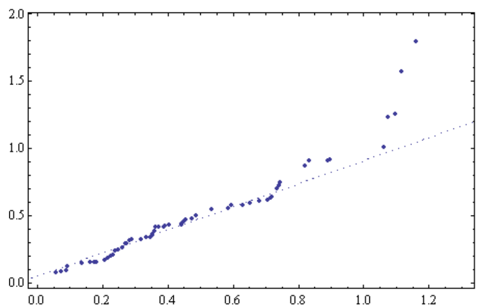
- From Table 3, one can observe that EITL distribution has the lowest value in SE, -2LL, AIC, BIC and CAIC.

- The plots of the fitted pdf, PP-plot, QQ-plot and TTT-plot for the EITL distribution for the real data set are provided in Figure 4.

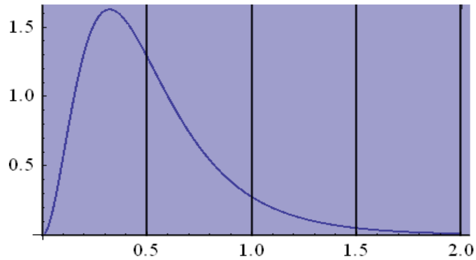
- Figure 4 indicates that the EITL distribution provides better fits to the real data set.



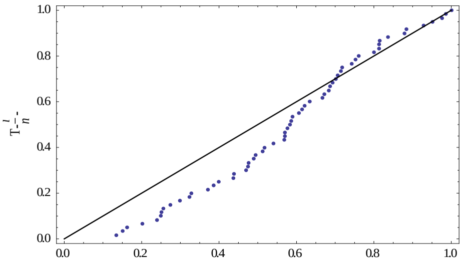
PP-Plot



QQ-Plot



Plot of the Fitted pdf



TTT-Plot

Figure 5: The Fitted pdf, PP-Plot, QQ-Plot and TTT-Plot for the Real Data Set

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APPENDIX

Table 4
Maximum Likelihood Averages, Estimated Risks, Variances and 95% Confidence Intervals of the Parameters α = 0.3 and $\theta = 0.5$ from EITL for Samples of Size n

N	*r*	Estimator	Average	ER	UI	LI	Length
30	21	$\hat{\alpha}$ $\hat{\theta}$	0.3082 0.5136	0.0012 0.0033	0.3745 0.6242	0.2418 0.4030	0.1327 0.2212
	27	$\hat{\alpha}$ $\hat{\theta}$	*0.3149* *0.5249*	0.0009 0.0025	0.3659 0.6098	0.2640 0.4400	0.1019 0.1698
50	35	$\hat{\alpha}$ $\hat{\theta}$	0.3085 0.5142	0.0007 0.0019	0.3568 0.5946	0.2603 0.4338	0.0965 0.1608
	45	$\hat{\alpha}$ $\hat{\theta}$	*0.3112* *0.5186*	0.0005 0.0014	0.3499 0.5831	0.2725 0.4541	0.0774 0.1290
70	49	$\hat{\alpha}$ $\hat{\theta}$	0.3074 0.5124	0.0005 0.0013	0.3479 0.5798	0.2670 0.4451	0.0809 0.1348
	63	$\hat{\alpha}$ $\hat{\theta}$	*0.3112* *0.5186*	0.0004 0.0010	0.3421 0.5702	0.2802 0.4670	0.0620 0.1033
90	63	$\hat{\alpha}$ $\hat{\theta}$	0.3075 0.5126	0.0004 0.0011	0.3439 0.5731	0.2712 0.4520	0.0726 0.1211
	81	$\hat{\alpha}$ $\hat{\theta}$	0.3115 0.5192	0.0003 0.0010	0.3403 0.5672	0.2827 0.4712	0.0576 0.0959
200	140	$\hat{\alpha}$ $\hat{\theta}$	0.3071 0.5119	0.0002 0.0006	0.3315 0.5524	0.2828 0.4714	0.0486 0.0810
	180	$\hat{\alpha}$ $\hat{\theta}$	0.3108 0.5180	0.0002 0.0006	0.3298 0.5497	0.2918 0.4863	0.0381 0.0634
500	350	$\hat{\alpha}$ $\hat{\theta}$	0.3060 0.5100	0.0001 0.0003	0.3216 0.5359	0.2904 0.4841	0.0311 0.0519
	450	$\hat{\alpha}$ $\hat{\theta}$	0.3107 0.5178	0.0002 0.0004	0.3227 0.5379	0.2968 0.4977	0.0241 0.0401

Table 5
Maximum Likelihood Averages, Estimated Risks, Variances and 95% Confidence Intervals of the Parameters $\alpha = 0.2$ and $\theta = 0.8$ from EITL for Samples of Size n

n	r	Estimator	Average	ER	UI	LI	Length
30	21	$\hat{\alpha}$ $\hat{\theta}$	0.2009 0.8036	0.0005 0.0072	0.2426 0.9703	0.1592 0.6368	0.0833 0.3334
	27	$\hat{\alpha}$ $\hat{\theta}$	0.2050 0.8198	0.0003 0.0053	0.2393 0.9573	0.1706 0.6823	0.0688 0.2751
50	35	$\hat{\alpha}$ $\hat{\theta}$	0.2027 0.8108	0.0003 0.0046	0.2355 0.9421	0.1699 0.6795	0.0657 0.2627
	45	$\hat{\alpha}$ $\hat{\theta}$	0.2030 0.8123	0.0002 0.0031	0.2297 0.9190	0.1764 0.7056	0.0533 0.2134
70	49	$\hat{\alpha}$ $\hat{\theta}$	0.2006 0.8026	0.0002 0.0032	0.2282 0.9128	0.1731 0.6924	0.0551 0.2204
	63	$\hat{\alpha}$ $\hat{\theta}$	0.2031 0.8124	0.0002 0.0025	0.2267 0.9067	0.1795 0.7181	0.0472 0.1887
90	63	$\hat{\alpha}$ $\hat{\theta}$	0.2004 0.8014	0.0002 0.0027	0.2257 0.9029	0.1750 0.7000	0.0507 0.2030
	81	$\hat{\alpha}$ $\hat{\theta}$	0.2023 0.8093	0.0001 0.0017	0.2220 0.8881	0.1826 0.7305	0.0394 0.1575
200	140	$\hat{\alpha}$ $\hat{\theta}$	0.1998 0.7990	0.0001 0.0011	0.2163 0.8651	0.1832 0.7330	0.0330 0.1321
	180	$\hat{\alpha}$ $\hat{\theta}$	0.2018 0.8070	0.0001 0.0009	0.2156 0.8626	0.1879 0.7515	0.0278 0.1111
500	350	$\hat{\alpha}$ $\hat{\theta}$	0.2000 0.8001	0.0000 0.0005	0.2110 0.8438	0.1891 0.7564	0.0218 0.0874
	450	$\hat{\alpha}$ $\hat{\theta}$	0.2001 0.8003	0.0000 0.0004	0.2103 0.8412	0.1898 0.7593	0.0205 0.0819

Table 6
Maximum Likelihood Averages, Estimated Risks, Variances and 95% Confidence Intervals of the Reliability, Hazard Rate and Reversed Hazard Rate Functions at $\alpha = 0.3$ and $\theta = 0.5$ from EITL for Samples of Size n

n	*r*	Estimator	Average	ER	UI	LI	Length
30	21	$\hat{R}_1(t_0)$	0.4847	0.0027	0.5830	0.3865	0.1966
		$\hat{h}(t_0)$	0.0550	0.0001	0.0788	0.0311	0.0469
		$\widehat{rh}(t_0)$	1.8120	0.000	1.8314	1.8078	0.0236
	27	$\hat{R}_1(t_0)$	0.4949	0.0019	0.5701	0.4196	0.1505
		$\hat{h}(t_0)$	0.0571	0.0001	0.0758	0.0384	0.0374
		$\widehat{rh}(t_0)$	1.8185	0.0000	1.8278	1.8093	0.0185
50	35	$\hat{R}_1(t_0)$	0.4854	0.0015	0.5569	0.4139	0.1430
		$\hat{h}(t_0)$	0.0548	0.0001	0.0720	0.0376	0.0343
		$\widehat{rh}(t_0)$	1.8197	0.0000	1.8282	1.8112	0.0170
	45	$\hat{R}_1(t_0)$	0.4894	0.0011	0.5467	0.4321	0.1146
		$\hat{h}(t_0)$	0.0556	0.0001	0.0695	0.0417	0.0278
		$\widehat{rh}(t_0)$	1.8193	0.0000	1.8262	1.8124	0.0137
70	49	$\hat{R}_1(t_0)$	0.4839	0.0008	0.5439	0.4239	0.1200
		$\hat{h}(t_0)$	0.0543	0.0001	0.0685	0.0401	0.0285
		$\widehat{rh}(t_0)$	1.8199	0.0000	1.8270	1.8129	0.0141
	63	$\hat{R}_1(t_0)$	0.4894	0.0008	0.5353	0.4435	0.0918
		$\hat{h}(t_0)$	0.0555	0.0001	0.0666	0.0444	0.0222
		$\widehat{rh}(t_0)$	1.8193	0.0000	1.8248	1.8138	0.0110
90	63	$\hat{R}_1(t_0)$	0.4840	0.0009	0.5379	0.4302	0.1077
		$\hat{h}(t_0)$	0.0543	0.0001	0.0672	0.0414	0.0258
		$\widehat{rh}(t_0)$	1.8199	0.0000	1.8263	1.8135	0.0128
	81	$\hat{R}_1(t_0)$	0.4900	0.0008	0.5326	0.4473	0.0853
		$\hat{h}(t_0)$	0.0556	0.0000	0.0659	0.0453	0.0206
		$\widehat{rh}(t_0)$	1.8193	0.0000	1.8244	1.8142	0.0102
200	140	$\hat{R}_1(t_0)$	0.4835	0.0005	0.5196	0.4474	0.0722
		$\hat{h}(t_0)$	0.0540	0.0000	0.0626	0.0455	0.0171
		$\widehat{rh}(t_0)$	1.8200	0.0000	1.8243	1.8158	0.0085
	180	$\hat{R}_1(t_0)$	0.4889	0.0005	0.5171	0.4607	0.0564
		$\hat{h}(t_0)$	0.0553	0.0000	0.0621	0.0485	0.0136
		$\widehat{rh}(t_0)$	1.8194	0.0000	1.8228	1.8161	0.0067
500	350	$\hat{R}_1(t_0)$	0.4818	0.0002	0.5049	0.4587	0.0462
		$\hat{h}(t_0)$	0.0536	0.0000	0.0590	0.0481	0.0109
		$\widehat{rh}(t_0)$	1.8203	0.0000	1.8230	1.8176	0.0054
	450	$\hat{R}_1(t_0)$	0.4887	0.0003	0.5066	0.4709	0.0357
		$\hat{h}(t_0)$	0.0552	0.0000	0.0595	0.0509	0.0086
		$\widehat{rh}(t_0)$	1.8195	0.0000	1.8216	1.8173	0.0042

Table 7
Maximum Likelihood Averages, Estimated Risks, Variances and 95% Confidence Intervals of the Reliability, Hazard Rate and Reversed Hazard Rate Functions at $\alpha = 0.2$ and $\theta = 0.8$ from EITL for samples of size n

n	*r*	Estimator	Average	ER	UI	LI	Length
30	21	$\hat{R}_1(t_0)$	0.7738	0.0062	0.9284	0.6193	0.3091
		$\hat{h}(t_0)$	0.0560	0.0001	0.0792	0.0328	0.0463
		$\widehat{rh}(t_0)$	1.8191	0.0000	1.8305	1.8067	0.0229
	27	$\hat{R}_1(t_0)$	0.7890	0.0045	0.9162	0.6617	0.2544
		$\hat{h}(t_0)$	0.0581	0.0001	0.0777	0.0384	0.0393
		$\widehat{rh}(t_0)$	1.8181	0.0000	1.8278	1.8083	0.0194
50	35	$\hat{R}_1(t_0)$	0.7807	0.0040	0.9024	0.6590	0.2434
		$\hat{h}(t_0)$	0.0568	0.0001	0.0751	0.0385	0.0366
		$\widehat{rh}(t_0)$	1.8187	0.0000	1.8278	1.8096	0.0181
	45	$\hat{R}_1(t_0)$	0.7821	0.0027	0.8809	0.6832	0.1976
		$\hat{h}(t_0)$	0.0569	0.0001	0.0718	0.0419	0.0300
		$\widehat{rh}(t_0)$	1.8187	0.0000	1.8261	1.8112	0.01484
70	49	$\hat{R}_1(t_0)$	0.7731	0.0027	0.8753	0.6709	0.2043
		$\hat{h}(t_0)$	0.0555	0.0001	0.0708	0.0403	0.0305
		$\widehat{rh}(t_0)$	1.8193	0.0000	1.8269	1.8118	0.0151
	63	$\hat{R}_1(t_0)$	0.7822	0.0021	0.8696	0.6948	0.1748
		$\hat{h}(t_0)$	0.0568	0.0001	0.0700	0.0436	0.0264
		$\widehat{rh}(t_0)$	1.8187	0.0000	1.8252	1.8122	0.0130
90	63	$\hat{R}_1(t_0)$	0.7720	0.0023	0.8662	0.6779	0.1883
		$\hat{h}(t_0)$	0.0553	0.0001	0.0693	0.0413	0.0280
		$\widehat{rh}(t_0)$	1.8194	0.0000	1.8263	1.8125	0.0139
	81	$\hat{R}_1(t_0)$	0.7794	0.0015	0.8524	0.7064	0.1460
		$\hat{h}(t_0)$	0.0563	0.0000	0.0673	0.0453	0.0220
		$\widehat{rh}(t_0)$	1.8189	0.0000	1.8244	1.8135	0.1090
200	140	$\hat{R}_1(t_0)$	0.7699	0.0010	0.8311	0.7086	0.1225
		$\hat{h}(t_0)$	0.0549	0.0000	0.0640	0.0458	0.0182
		$\widehat{rh}(t_0)$	1.8196	0.0000	1.8241	1.8151	0.0090
	180	$\hat{R}_1(t_0)$	0.7773	0.0007	0.8288	0.7258	0.1030
		$\hat{h}(t_0)$	0.0559	0.0000	0.0636	0.0482	0.0154
		$\widehat{rh}(t_0)$	1.8191	0.0000	1.8229	1.8153	0.0076
500	350	$\hat{R}_1(t_0)$	0.7709	0.0004	0.8114	0.7304	0.0811
		$\hat{h}(t_0)$	0.0550	0.0000	0.0610	0.0490	0.0120
		$\widehat{rh}(t_0)$	1.8196	0.0000	1.8226	1.8166	0.0059
	450	$\hat{R}_1(t_0)$	0.7711	0.0004	0.8090	0.7331	0.0759
		$\hat{h}(t_0)$	0.0550	0.0000	0.0606	0.0493	0.0113
		$\widehat{rh}(t_0)$	1.8196	0.0000	1.8224	1.8168	0.0056