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ON WEIGHTED GENERALIZED PAST MEASURE OF INACCURACY

M. Sultan Shah[1§], Showkat Ahmad Lone[2] and **M.A.K. Baig[1]**

[1] Department of Statistics, University of Kashmir
Srinagar, J&K (India)
Email: msultanshahh01@gmail.com, baigmak@gmail.com

[2] Department of Basic Science, Saudi Electronic University
Saudi Arabia. Email: s.lone@seu.edu.sa

[§] Corresponding Author: msultanshahh01@gmail.com

ABSTRACT

In this paper, a new length-biased generalized inaccuracy measure of order γ and type δ between the two past lifetime distributions for the interval $(0,t)$ is introduced. On account of the idea of proportional reversed hazard model (PRHM), some significant characterization results of the proposed inaccuracy measure have been studied. Further, some bounds to the length-biased generalized inaccuracy measure have also been derived. Finally, based on two independent exponential distributions, we study the monotonic behavior of the proposed length-biased generalized past inaccuracy measure.

KEYWORDS

Shannon's entropy, Kerridge's Inaccuracy, Lifetime distributions, Proportional reversed hazard model, Characterization results.

1. INTRODUCTION

In the area of information theory, a paramount contribution to the well-known and fundamental measure of uncertainty commonly called as Shannon entropy was provided by Claude Shannon (1948). It has been developed and extended in both parametric as well as non-parametric directions and one of the prominent and relevant extensions in this direction is the inaccuracy measure, which was first introduced by Kerridge (1961) for discrete random variables and later on by Nath (1968) in the case of continuous random variables. It is worthwhile to note that the continuous form of the measure is also called as Fraser information, refer to Fraser (1965), which has been greatly used by Kent (1982), Kent (1983), Ebrahimi et al. (2010) and several others in terms of the information gain about a parameter. In addition, it is applied in numerous fields like, estimation, statistical inference and coding theory.

Let the two absolutely continuous non-negative r.v's X and Y with common support $(0,\infty)$, cumulative distribution functions $F(x)=P(X\le x)$ and $G(x)=P(X\le x)$, Survival functions $\overline{F}(x)=1-F(x)$ and $\overline{G}(x)=1-G(x)$, and probability density functions $f(x)$ and $g(x)$, respectively, describing the failure times of two units of a machine or system, then Kerridge's (1961) measure of inaccuracy (also called as relative distance or cross entropy) is given by

$$KI_{(X,Y)} = -\int_0^\infty f(x)\log g(x)\,dx = E\left[-\log g(X)\right], \tag{1}$$

where, "log" represents the natural logarithm. The measure given in (1) is considered as non-parametric generalization of the leading Shannon's entropy defined as

$$SE_{(X)} = -\int_0^\infty f(x)\log f(x)\,dx = E\left[-\log f(X)\right]. \tag{2}$$

The two r.v's X and Y, throughout this paper, are taken as absolutely continues non-negative r.v's.

The measures defined in (1) and (2) play an important role in the field of information theory. But they have a common drawback of assigning equal weight or importance to the occurrence of every event. However, in certain practical situations such as reliability or neurobiology, it is necessary to deal with the shift-dependent measures.

In agreement with Belis and Guiasu (1968), Di Crescenzo and Longobardi (2006), Kumar et al. (2010), introduced the weighted form of inaccuracy as

$$KI^{w}_{(X,Y)} = -\int_0^\infty x\, f(x)\log g(x)\,dx = E\left[-X\log g(X)\right]. \tag{3}$$

Sometimes, in survival analysis and in life testing, it is very important to ascertain the uncertainty of a system related to the past lifetime rather than the residual lifetime. For example, if a system is checked or inspected at time t, and is found to be failed, then its life uncertainty depends on past, that is, we need to know at which particular point of the time $(0,t)$, the system has failed. So taking the situation in to consideration, the measures given in (1) and (2) cannot applied and hence Di Crescenzo and Longobardi (2002), and Taneja et al. (2009) introduced the concept of past entropy and past inaccuracy, respectively defined as

$$SPE_{(X;t)} = -\int_0^t \frac{f(x)}{F(t)}\log\frac{f(x)}{F(t)}\,dx, \tag{4}$$

and

$$KPI_{(X,Y;t)} = -\int_0^t \frac{f(x)}{F(t)}\log\frac{g(x)}{G(t)}\,dx. \tag{5}$$

Further, Di Crescenzo and Longobardi (2006), extended the concept of length-biased shift-dependent or weighted version of (4) which is known as weighted past entropy, defined as

$$SPE^{w}_{(X;t)} = -\int_0^t x\frac{f(x)}{F(t)}\log\frac{f(x)}{F(t)}\,dx. \tag{6}$$

Similarly, Kumar and Taneja (2012) developed the weighted version of (5) which is known as weighted past inaccuracy defined as

$$KPI^{w}_{(X,Y;t)} = -\int_{0}^{t} x \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} dx . \tag{7}$$

In the recent past, there has been profound interest among the researchers to generalize the measure of inaccuracy in a number of ways and have been very useful in different fields. For detailed information see Thapliyal and Taneja (2015), Kundu et al. (2016), Ghosh and Kundu (2018), Daneshi et al. (2019), Tahmasebi et al. (2019), Cali et al. (2020).

In order to manipulate the complex and dynamic situations of real life data, it is very important for an experimenter to develop and use the generalized measures rather than the classical one. Moreover, due to presence of additional parameters, the generalized measures have been found more flexible. So, various attempts have been made by different authors to generalize (1) in different ways and consequently in this paper, in agreement with Renyi (1961), Verma (1966), Nath (1968) and Bilal et al. (2019), we make an effort to introduce and develop the parametric generalization of (1) given as

$$KI^{(\gamma,\delta)}_{(X,Y)} = \frac{\delta}{\delta-\gamma} \log \int_{0}^{\infty} f(x)\left(g(x)\right)^{\gamma-\delta} dx \; ; \delta-1<\gamma<\delta, \delta\geq 1 . \tag{8}$$

Some Particular Cases for (8)

1. When $\delta = 1, \gamma \to 1$, then (8) reduces to Kerridge inaccuracy measure.
2. When $\delta = 1$, then (8) reduces to generalized inaccuracy measure given by Nath (1968).
3. When $f(x) = g(x), \delta = 1$, then (8) reduces to Renyi's entropy (1961).
4. When $f(x) = g(x), \delta = 1, \gamma \to 1$, then (8) reduces to Shannon's entropy.

The generalized measure given in (8) is much more flexible because of the presence of additional parameters γ and δ, and one can make it more or less sensitive to the shape of different types of probability distributions.

Based on the concept given in (5), the parametric generalized past inaccuracy (GPI) is given by

$$KPI^{(\gamma,\delta)}_{(X,Y;t)} = \frac{\delta}{\delta-\gamma} \log \int_{0}^{t} \frac{f(x)}{F(t)} \left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta} dx \; ; \delta-1<\gamma<\delta, \delta\geq 1 . \tag{9}$$

The main motive in this paper is to propose a new weighted shift dependent parametric generalized inaccuracy measure between the two past lifetime distributions for the interval $(0,t)$. The remaining sections of the paper are organized as follows:

In section 2, we propose the weighted generalized inaccuracy (WGI). Under section 3, we introduce the weighted generalized past inaccuracy and by using the concept of proportional reversed hazard model (PRHM), some characterization results are proved. In section 4, some bounds to the length-biased generalized inaccuracy measure have been derived. In the last section 5, we briefly discuss the conclusion.

2. WEIGHTED GENERALIZED INACCURACY (WGI)

In this section, we propose the weighted form of (8), more specifically dubbed as weighted generalized inaccuracy (WGI). In addition, we calculate the WGI for some well-known lifetime distributions.

Analogous to (3) and in view of (8), the WGI is described as

$$KI_{(X,Y)}^{w(\gamma,\delta)} = \frac{\delta}{\delta-\gamma} \log \int_0^\infty x\, f(x) \left(g(x)\right)^{\gamma-\delta} dx \; ; \delta-1<\gamma<\delta, \delta\geq 1. \tag{10}$$

Example 2.1:

Let X_1, Y_1 and Y_2 denote the lifetimes of three components of a system with probability density functions, respectively, as

$$f_1(x)=1, 0<x<1 \; ; \; g_1(x)=m x^{m-1}, m>0, \; 0<x<1$$

and $g_2(x)=m(1-x)^{m-1}, m>0, \; 0<x<1,$

then by using the simple calculations, we obtain

$$KI_{(X_1,Y_1)}^{(\gamma,\delta)} = KI_{(X_1,Y_2)}^{(\gamma,\delta)} = \frac{\delta}{\delta-\gamma} \log\left(\frac{m^{\gamma-\delta}}{(m-1)(\gamma-\delta)+1} \right),$$

$$KI_{(X_1,Y_1)}^{w(\gamma,\delta)} = \frac{\delta}{\delta-\gamma} \log\left(\frac{m^{\gamma-\delta}}{(m-1)(\gamma-\delta)+2} \right),$$

and

$$KI_{(X_1,Y_2)}^{w(\gamma,\delta)} = \frac{\delta}{\delta-\gamma} \log\left(\frac{m^{\gamma-\delta}}{\left((m-1)(\gamma-\delta)+1\right)\left((m-1)(\gamma-\delta)+2\right)} \right).$$

Thus, in general we conclude that, different distributions with common support $(0,1)$ can have the same generalized inaccuracies but may not have the same weighted generalized inaccuracies.

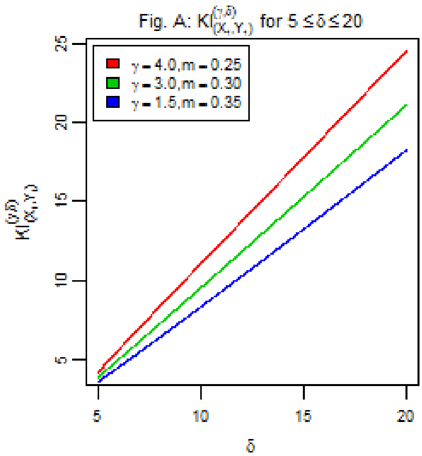
In particular, the below given table 1 provides the values of $KI_{(X_1,Y_1)}^{(\gamma,\delta)} = KI_{(X_1,Y_2)}^{(\gamma,\delta)}$, $KI_{(X_1,Y_1)}^{w(\gamma,\delta)}$ and $KI_{(X_1,Y_2)}^{w(\gamma,\delta)}$ for different values of $\gamma, \delta \,\&\, m$.

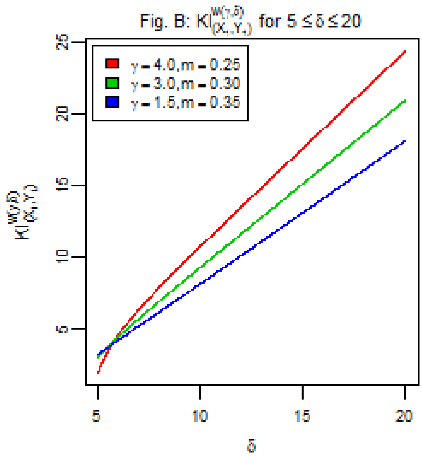
Table 1
Comparison between GI and WGI

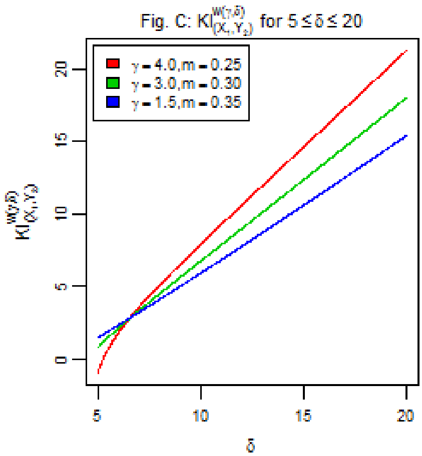
γ	δ	m	$KI_{(X_1,Y_1)}^{(\gamma,\delta)}$	$KI_{(X_1,Y_1)}^{w(\gamma,\delta)}$	$KI_{(X_1,Y_2)}^{w(\gamma,\delta)}$
0.2	2.0	0.2	2.2278	1.8461	0.8550
0.4	2.2		2.4505	2.0307	0.9405
0.6	2.4		2.6733	2.2154	1.0260
0.8	2.6		2.8961	2.3999	1.1115
1.2	3.6	0.4	1.9607	1.4454	0.1074
1.4	3.8		2.0696	1.5257	0.1134
1.6	4.0		2.1785	1.6060	0.1194
1.8	4.2		2.2875	1.6863	0.1254
2.2	7.2	0.6	2.0959	1.6817	0.0997
2.4	7.4		2.1542	1.7284	0.1024
2.6	7.6		2.2124	1.7751	0.1052
2.8	7.8		2.2706	1.8218	0.1080
3.2	16.0	0.8	1.9831	1.6736	0.0864
3.4	16.2		2.0079	1.6946	0.0875
3.6	16.4		2.0327	1.7155	0.0886
3.8	16.6		2.0575	1.7364	0.0897

Thus, from the above calculation, we infer that the generalized inaccuracy between (X_1,Y_1) and (X_1,Y_2) is same, but the weighed generalized inaccuracy between them is not same. i.e., $KI_{(X_1,Y_1)}^{(\gamma,\delta)} = KI_{(X_1,Y_2)}^{(\gamma,\delta)}$, but $KI_{(X_1,Y_1)}^{w(\gamma,\delta)} \neq KI_{(X_1,Y_2)}^{w(\gamma,\delta)}$. Further, from Table 1, it can be seen that for different values of $\gamma,\delta \& m$, the generalized inaccuracy values between (X_1,Y_1) and (X_1,Y_2) are greater than their respective weighted generalized inaccuracy values. i.e., $KI_{(X_1,Y_1)}^{(\gamma,\delta)} > KI_{(X_1,Y_1)}^{w(\gamma,\delta)}$ and $KI_{(X_1,Y_2)}^{(\gamma,\delta)} > KI_{(X_1,Y_2)}^{w(\gamma,\delta)}$.

In the below given figures such as, Fig. A, Fig. B, & Fig. C, we study monotonic behavior of $KI_{(X_1,Y_1)}^{(\gamma,\delta)}, KI_{(X_1,Y_1)}^{w(\gamma,\delta)} \& KI_{(X_1,Y_2)}^{w(\gamma,\delta)}$, respectively.







Thus, from the above figures, Fig. A, Fig. B and Fig. C, we observe that for different values of $\gamma, \delta \,\&\, m$, the measures $KI^{(\gamma,\delta)}_{(X_1,Y_1)}, KI^{w(\gamma,\delta)}_{(X_1,Y_1)} \,\&\, KI^{w(\gamma,\delta)}_{(X_1,Y_2)}$, respectively, show increasing behavior.

Definition 2.1:

For the two r.v's U and V, if the mathematical relationship existing in between them is in the form of

$$\mu_G(u) = \theta \mu_F(u),$$

or

$$G(u) = \left[F(u)\right]^{\theta}, \tag{11}$$

then, it is called as proportional reversed hazard model (PRHM).

Where, θ is a constant of proportionality, $F(u)$ is the true distribution function and $G(u)$ is the reference distribution function. This model was proposed by Gupta et al. (1998). It helps in developing the better statistical models and has a key role in various practical fields of medicine, reliability, economics etc. For more information about this model see, Di Crescenzo (2000), Kundu and Gupta (2004), Gupta and Gupta (2007), Sankaran and Gleeja (2008), Li and Xu (2008), Kundu and Gupta (2010), Sengupta and Nanda (2011), Asokan and Sankaran (2014), Unnikrishnan et al. (2018), Di Crescenzo et al. (2019).

In the following Table 2, corresponding to some lifetime distributions, we present the general expressions of GPI.

Table 2
The Expressions of GPI for Some Well-Known Lifetime Distributions

Distribution	$f(x)$	$g(x)$	x	$KPI_{(X,Y;t)}^{(\gamma,\delta)}$
Power	$\frac{b\,x^{b-1}}{a^b}$	$\frac{b\theta\,x^{b\theta-1}}{a^{b\theta}}$	$0<x<a, b>0, \theta>0$	$R\log\left(\frac{b^{\gamma-\delta+1}\,\theta^{\gamma-\delta}}{\left((b\theta-1)(\gamma-\delta)+b\right)t^{\gamma-\delta}}\right)$
Extreme type III	$k\,e^{k(x-m)}$	$k\theta\,e^{k\theta(x-m)}$	$x<m, k>0, \theta>0$	$R\log\left(\frac{k^{\gamma-\delta+1}\,\theta^{\gamma-\delta}\left(1-e^{-t\,p}\right)}{p}\right)$
Uniform	$\frac{1}{n-m}$	$\frac{\theta(x-m)^{\theta-1}}{(n-m)^{\theta}}$	$m<x<n, \theta>0$	$R\log\left(\frac{(t-m)^{\delta-\gamma}\,\theta^{\gamma-\delta}}{\left((\theta-1)(\gamma-\delta)+1\right)}\right)$
Beta	$c\,x^{c-1}$	$c\theta\,x^{c\theta-1}$	$0\le x<1, c>0, \theta>0$	$R\log\left(\frac{c^{\gamma-\delta+1}\,\theta^{\gamma-\delta}t^{\delta-\gamma}}{\left((c\theta-1)(\gamma-\delta)+c\right)}\right)$

where,

$$R=\frac{\delta}{\delta-\gamma},\ p=k\left(\theta(\gamma-\delta)+1\right).$$

3. WEIGHTED GENERALIZED PAST INACCURACY (WGPI)

In this section, we introduce the weighted form of GPI (9), known as weighted generalized past inaccuracy (WGPI). We also infer the general expressions of WGPI for some well-known lifetime distributions and prove some characterization results.

Corresponding to (7) and using the idea given in (8), the weighted generalized past inaccuracy can be described as

$$KPI^{w(\gamma,\delta)}_{(X,Y;t)} = \frac{\delta}{\delta-\gamma}\log\int_0^t x\frac{f(x)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta}dx\ ;\delta-1<\gamma<\delta,\delta\geq 1\,. \tag{12}$$

Example 3.1:

Let the two r.v's X and Y be distributed as follows

$$f(x)=\frac{x}{2} \text{ and } g(x)=\frac{2-x}{2};\ 0\leq x\leq 2\,.$$

Using the values in (9) and (10), we obtain

$$KPI^{(\gamma,\,\delta)}_{(X,\,Y;t)} = \frac{\delta}{\delta-\gamma}\log\frac{2^r}{t^{r+1}(4-t)^{\gamma-\delta}}\left[\frac{2^{r+1}-s^{r+1}}{r(r+1)}-\frac{t\,s^r}{r}\right],$$

$$KPI^{w(\gamma,\,\delta)}_{(X,\,Y;t)} = \frac{\delta}{\delta-\gamma}\log\frac{2^r}{t^{r+1}(4-t)^{\gamma-\delta}}\left[\frac{2}{r}\left(\frac{2^{r+2}-s^{r+2}}{(r+1)(r+2)}-\frac{t\,s^{r+1}}{(r+1)}\right)-\frac{t^2s^r}{r}\right]$$

Again,

$$KPI^{(\gamma,\,\delta)}_{(Y,\,X;t)} = \frac{\delta}{\delta-\gamma}\log\frac{2^r}{t^{2(\gamma-\delta)+1}(4-t)}\left[\frac{2t^r}{r}-\frac{t^{r+1}}{(r+1)}\right],$$

$$KPI^{w(\gamma,\,\delta)}_{(Y,\,X;t)} = \frac{\delta}{\delta-\gamma}\log\frac{2^r}{t^{2(\gamma-\delta)+1}(4-t)}\left[\frac{2t^{r+1}}{(r+1)}-\frac{t^{r+2}}{(r+2)}\right].$$

where,

$$r=\gamma-\delta+1, s=(2-t)\,.$$

The below given Fig. 3.1 provides the monotonic behaviors of example 3.1 for $KPI^{(\gamma,\,\delta)}_{(X,\,Y;t)}$, $KPI^{w(\gamma,\,\delta)}_{(X,\,Y;t)}$, $KPI^{(\gamma,\,\delta)}_{(Y,\,X;t)}$ and $KPI^{w(\gamma,\,\delta)}_{(Y,\,X;t)}$ against $1\leq t\leq 2$.

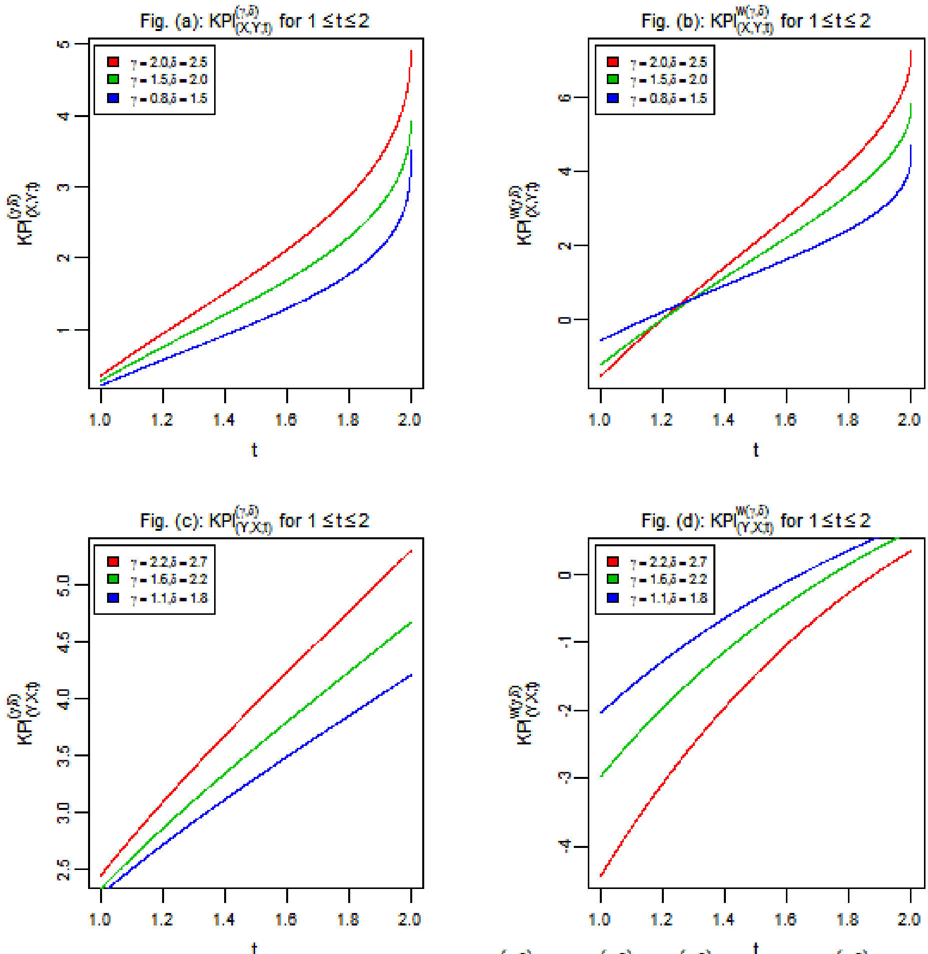


Fig.3.1: plots of $KPI^{(\gamma,\delta)}_{(X,Y;t)}$, $KPI^{w(\gamma,\delta)}_{(X,Y;t)}$, $KPI^{(\gamma,\delta)}_{(Y,X;t)}$ and $KPI^{w(\gamma,\delta)}_{(Y,X;t)}$.

Thus, from the above Fig. 3.1, corresponding to different values of parameters, we observe that $KPI^{(\gamma,\delta)}_{(X,Y;t)}$, $KPI^{w(\gamma,\delta)}_{(X,Y;t)}$, $KPI^{(\gamma,\delta)}_{(Y,X;t)}$ and $KPI^{w(\gamma,\delta)}_{(Y,X;t)}$ show increasing behavior.

In the following Table 3, we present the general expressions of WGPI for some notable lifetime distributions.

Table 3
The Expressions of (WGPI) for Some Well-Known Lifetime Distributions

Distribution	$f(x)$	$g(x)$	x	$KPI_{(X,Y;t)}^{w(\gamma,\delta)}$
Beta	$c\,x^{c-1}$	$c\theta\,x^{c\theta-1}$	$0 \le x < 1,$ $c > 0, \theta > 0$	$R\log\left(\dfrac{c^{\gamma-\delta+1}\,\theta^{\gamma-\delta}t^{\delta-\gamma+1}}{\left((c\theta-1)(\gamma-\delta)+c+1\right)}\right)$
Uniform	$\dfrac{1}{n-m}$	$\dfrac{\theta(x-m)^{\theta-1}}{(n-m)^{\theta}}$	$m < x < n,$ $\theta > 0$	$R\log\left(\dfrac{\theta^{\gamma-\delta}\left(t(S+1)-(t-m)\right)}{S(S+1)}\right)$
Power	$\dfrac{b\,x^{b-1}}{a^{b}}$	$\dfrac{b\theta\,x^{b\theta-1}}{a^{b\theta}}$	$0 < x < a,$ $b > 0, \theta > 0$	$R\log\left(\dfrac{t^{\delta-\gamma+1}b^{\gamma-\delta+1}\,\theta^{\gamma-\delta}}{\left((b\theta-1)(\gamma-\delta)+b+1\right)}\right)$
Extreme type III	$k\,e^{k(x-m)}$	$k\theta e^{k\theta(x-m)}$	$x < m,$ $k > 0, \theta > 0$	$R\log\left(\dfrac{k^{\gamma-\delta+1}\,\theta^{\gamma-\delta}\left(tp+1-e^{-tp}\right)}{p^{2}}\right)$

where,

$$p = k\left(\theta(\gamma-\delta)+1\right), R = \frac{\delta}{\delta-\gamma}\ and\ \ S = (\theta-1)(\gamma-\delta)+1\,.$$

In order to characterize (12), we prove the following theorems.

Theorem 3.1:

For all $t > 0$ and under proportional reversed hazard model (11), the following equality holds

$$KPI_{(X,Y;t)}^{w(\gamma,\delta)} = \frac{\delta}{\delta-\gamma}\log\Bigg[t\exp\left\{\left(\frac{\delta-\gamma}{\delta}\right)KPI_{(X,Y;t)}^{(\gamma,\delta)}\right\}$$
$$+\int_0^t\left(\frac{F(v)}{F(t)}\right)^{\theta(\gamma-\delta)+1}\exp\left\{\left(\frac{\delta-\gamma}{\delta}\right)KPI_{(X,Y;v)}^{(\gamma,\delta)}\right\}dv\Bigg]. \tag{13}$$

Proof:

$$\int_0^t x\left(\frac{f(x)}{F(t)}\right)\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta}dx = \int_0^t\left(\int_0^x v^{\,0}dv\right)\left(\frac{f(x)}{F(t)}\right)\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta}dx$$

$$=\int_0^t\left(\int_0^t v^{\,0}dv+\int_t^x v^{\,0}dv\right)\left(\frac{f(x)}{F(t)}\right)\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta}dx$$

$$= t\int_0^t \left(\frac{f(x)}{F(t)}\right)\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta} dx + \int_{v=0}^{t}\left(\int_{x=v}^{t}\left(\frac{f(x)}{F(t)}\right)\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta} dx\right)dv \; . \tag{14}$$

Using (9), we get

$$\int_0^t \left(\frac{f(x)}{F(t)}\right)\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta} dx = \exp\left[\left(\frac{\delta-\gamma}{\delta}\right) KPI_{(X,Y;t)}^{(\gamma,\delta)}\right]. \tag{15}$$

and

$$\int_0^t f(x)\left(g(x)\right)^{\gamma-\delta} dx = F(t)\left(G(t)\right)^{\gamma-\delta} \exp\left[\left(\frac{\delta-\gamma}{\delta}\right) KPI_{(X,Y;t)}^{(\gamma,\delta)}\right]$$

Using proportional reversed hazard model (PRHM) (11), we obtain

$$\int_0^t f(x)\left(g(x)\right)^{\gamma-\delta} dx = \left(F(t)\right)^{\theta(\gamma-\delta)+1} \exp\left[\left(\frac{\delta-\gamma}{\delta}\right) KPI_{(X,Y;t)}^{w(\gamma,\delta)}\right]. \tag{16}$$

Using the values of (14), (15) and (16) in (12), the stated result is satisfied.

In the next theorem 3.2, we prove that $KPI_{(X,Y;t)}^{w(\gamma,\delta)}$ uniquely determines $\overline{F}(t)$.

Theorem 3.2:

If the two r.v's X and Y satisfy the proportional reversed hazard model (PRHM) with proportionality constant $\theta > 0$ and $KPI_{(X,Y;t)}^{w(\gamma,\delta)}$ is decreasing for all $t > 0$, then $KPI_{(X,Y;t)}^{w(\gamma,\delta)}$ uniquely determines $\overline{F}(t)$.

Proof:

Rewriting (12) as

$$\exp\left[\left(\frac{\delta-\gamma}{\gamma}\right) KPI_{(X,Y;t)}^{w(\gamma,\delta)}\right] = \int_0^t x\frac{f(x)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta} dx \; . \tag{17}$$

Differentiating (17) w.r.t t, we obtain

$$\frac{\partial}{\partial t}\exp\left[\left(\frac{\delta-\gamma}{\gamma}\right) KPI_{(X,Y;t)}^{w(\gamma,\delta)}\right] = t\mu_F(t)\left(\mu_G(t)\right)^{\gamma-\delta}$$

$$-\left(\mu_F(t) + (\gamma-\delta)\mu_G(t)\right)\int_0^t x\frac{f(x)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta} dx \; . \tag{18}$$

where, $\mu_F(t)=\dfrac{f(t)}{F(t)}$ and $\mu_G(t)=\dfrac{g(t)}{G(t)}$ are the reversed hazard rate functions of X and Y respectively.

Under the proportional reversed hazard model $\mu_G(t)=\theta\mu_F(t)$. Using this in (18), we get

$$\frac{\partial}{\partial t}\exp\left[\left(\frac{\delta-\gamma}{\gamma}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)}\right]=t\,\theta^{\gamma-\delta}\left(\mu_F(t)\right)^{\gamma-\delta+1}$$

$$-\left(\theta\left(\gamma-\delta\right)+1\right)\mu_F(t)\int_0^t x\frac{f(x)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta}dx. \tag{19}$$

Using (17), we can write (19) as

$$t\,\theta^{\gamma-\delta}\left(\mu_F(t)\right)^{\gamma-\delta+1}-\left(\theta\left(\gamma-\delta\right)+1\right)\exp\left[\left(\frac{\delta-\gamma}{\gamma}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)}\right]\mu_F(t)$$

$$-\frac{\partial}{\partial t}\exp\left[\left(\frac{\delta-\gamma}{\gamma}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)}\right]=0\,. \tag{20}$$

Hence for fixed $t>0$, $\mu_F(t)$ is a solution of the equation $\varphi(x_t)=0$, where

$$\varphi(x_t)=t\,\theta^{\gamma-\delta}\left(x_t\right)^{\gamma-\delta+1}-\left(\theta\left(\gamma-\delta\right)+1\right)\exp\left[\left(\frac{\delta-\gamma}{\gamma}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)}\right]x_t$$

$$-\frac{\partial}{\partial t}\exp\left[\left(\frac{\delta-\gamma}{\gamma}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)}\right]. \tag{21}$$

Here, $\varphi(0)=-\dfrac{\partial}{\partial t}\exp\left[\left(\dfrac{\delta-\gamma}{\gamma}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)}\right]\le 0$, since we have assumed that $KPI_{(X,Y;t)}^{w(\gamma,\delta)}$ is decreasing in t, and also as $x_t\to\infty, \varphi(x_t)=\infty$.

Differentiating (21) w.r.t x_t, we have

$$\frac{\partial}{\partial x_t}\varphi(x_t)=t\left(\gamma-\delta+1\right)\theta^{\gamma-\delta}\left(x_t\right)^{\gamma-\delta}-\left(\theta\left(\gamma-\delta\right)+1\right)\exp\left[\left(\frac{\delta-\gamma}{\gamma}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)}\right]$$

and

$$\frac{\partial^2}{\partial x_t^2}\varphi(x_t)=t\left(\gamma-\delta\right)\left(\gamma-\delta+1\right)\theta^{\gamma-\delta}\left(x_t\right)^{\gamma-\delta-1}.$$

Now, $\frac{\partial}{\partial x_t}\varphi(x_t)=0$ gives

$$x_t=\left[\frac{\left(\theta(\gamma-\delta)+1\right)\exp\left(\left(\frac{\delta-\gamma}{\gamma}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)}\right)}{t(\gamma-\delta+1)\theta^{\gamma-\delta}}\right]^{\frac{1}{\gamma-\delta}}=x_0\,(say)$$

Therefore, $\varphi(x_t)=0$ has a unique solution, so $\mu_F(t)$, and hence $\overline{F}(t)$ is uniquely determined by the weighted generalized past inaccuracy (WGPI) $KPI_{(X,Y;t)}^{w(\gamma,\delta)}$ under the assumption that $\frac{\partial}{\partial x_t}KPI_{(X,Y;t)}^{w(\gamma,\delta)}\le 0$. This completes the proof.

4. BOUNDS AND INEQUALITIES OF $KPI_{(X,Y;t)}^{w(\gamma,\delta)}$

In this section, we obtain some bounds and inequalities of WGPI.

Definition 4.1:

The distribution functions F and G may have increasing (decreasing) WGPI of order γ and δ written as IWGPI or DWGPI, if $KPI_{(X,Y;t)}^{w(\gamma,\delta)}$ is increasing (decreasing) in $t,\ t>0$.

In other words, we can say, F and G have IWGPI or DWGPI, if $\frac{\partial}{\partial t}KPI_{(X,Y;t)}^{w(\gamma,\delta)}\ge(\le)0.$

Theorem 4.1:

Let the two r.v's X and Y have DWGPI, then the lower bound of $KPI_{(X,Y;t)}^{w(\gamma,\delta)}$ is designated as

$$KPI_{(X,Y;t)}^{w(\gamma,\delta)}\ge\left(\frac{\delta}{\delta-\gamma}\right)\log\left(\left(\frac{\theta(\gamma-\delta)+1}{t\,\theta^{\gamma-\delta}}\right)^{-1}\left(\frac{1-\frac{\partial}{\partial t}\delta_F(t)}{\delta_F(t)}\right)^{\gamma-\delta}\right).$$

Proof:

Since, from (12), we have

$$\left(\frac{\delta-\gamma}{\delta}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)}=\log\int_0^t x\frac{f(x)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta}dx \tag{22}$$

Differentiating (22) both sides w.t.t t, we get

$$\frac{\partial}{\partial t}\left(\frac{\delta-\gamma}{\delta}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)}=t\mu_F(t)(\mu_G(t))^{\gamma-\delta}$$
$$\exp\left(-\left(\frac{\delta-\gamma}{\delta}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)}\right)-(\theta(\gamma-\delta)+1)\mu_F(t).$$

Now, using PRHM, we get

$$\frac{\partial}{\partial t}\left(\frac{\delta-\gamma}{\delta}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)}=t\theta^{\gamma-\delta}(\mu_F(t))^{\gamma-\delta+1}$$
$$\exp\left(-\left(\frac{\delta-\gamma}{\delta}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)}\right)-(\theta(\gamma-\delta)+1)\mu_F(t). \tag{23}$$

Using $\mu_F(t)=\frac{1-\frac{\partial}{\partial t}\delta_F(t)}{\delta_F(t)}$

$$\frac{\partial}{\partial t}\left(\frac{\delta-\gamma}{\delta}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)}=t\theta^{\gamma-\delta}\left(\frac{1-\frac{\partial}{\partial t}\delta_F(t)}{\delta_F(t)}\right)^{\gamma-\delta+1}\exp\left(-\left(\frac{\delta-\gamma}{\delta}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)}\right)$$
$$-(\theta(\gamma-\delta)+1)\left(\frac{1-\frac{\partial}{\partial t}\delta_F(t)}{\delta_F(t)}\right).$$

Since, $KPI_{(X,Y;t)}^{w(\gamma,\delta)}$ is decreasing w.r.t t and thus we obtain

$$KPI_{(X,Y;t)}^{w(\gamma,\delta)}\geq\left(\frac{\delta}{\delta-\gamma}\right)\log\left(\left(\frac{t\theta^{\gamma-\delta}}{\theta(\gamma-\delta)+1}\right)\left(\frac{1-\frac{\partial}{\partial t}\delta_F(t)}{\delta_F(t)}\right)^{\gamma-\delta}\right).$$

Theorem 4.2:

Let F and G have IWGPI (DWGPI) and $\gamma<\delta$ then

$$\mu_F(t)\geq(\leq)\left[(\theta(\gamma-\delta)+1)\exp\left(\frac{\delta-\gamma}{\delta}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)}\right]^{\frac{1}{\gamma-\delta}}.$$

Proof:

From (23), we have

$$\frac{\partial}{\partial t}\left(\frac{\delta-\gamma}{\delta}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)} = t\,\theta^{\gamma-\delta}\left(\mu_F(t)\right)^{\gamma-\delta+1}$$
$$\exp\left(-\left(\frac{\delta-\gamma}{\delta}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)}\right)-\left(\theta(\gamma-\delta)+1\right)\mu_F(t)$$

Since, F and G have IWGPI (DWGPI) and $\gamma<\delta$, therefore, we have

$$\mu_F(t)\left[t\,\theta^{\gamma-\delta}\left(\mu_F(t)\right)^{\gamma-\delta}\exp\left(-\left(\frac{\delta-\gamma}{\delta}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)}\right)-\left(\theta(\gamma-\delta)+1\right)\right]\geq(\leq)0.$$

Which results in

$$\mu_F(t)\geq(\leq)\left[\left(\theta(\gamma-\delta)+1\right)\exp\left(\frac{\delta-\gamma}{\delta}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)}\right]^{\frac{1}{\gamma-\delta}}.$$

Theorem 4.3:

Let X and Y be the two r.v's with absolutely continuous distribution functions $F(t)$ and $G(t)$ respectively, $t>0$, then for $\gamma<\delta<1$ $(\gamma>\delta)$, the following inequality holds

$$KPI_{(X,Y;t)}^{w(\gamma,\delta)}\geq(\leq)\delta\,KPI_{(X,Y;t)}+\frac{\delta}{\delta-\gamma}E\left[\log X\mid X\leq t\right]. \qquad (24)$$

Proof:

On considering log-sum inequality, we have

$$\int_0^t f(x)\log\frac{f(x)}{x\frac{f(x)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta}}dx\geq\int_0^t f(x)dx\,\log\frac{\int_0^t f(x)dx}{\int_0^t x\frac{f(x)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta}dx}.$$

$$=F(t)\left[\log F(t)-\left(\frac{\delta-\gamma}{\delta}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)}\right]. \qquad (25)$$

where, (25) is derived from (12).

The left side of (25) reduces to

$$(\delta-\gamma)\int_0^t f(x)\log\frac{g(x)}{G(t)}dx-\int_0^t f(x)\log x\,dx+F(t)\log F(t) \qquad (26)$$

Using (26) in (25), we get (24).

Theorem 4.4:

Suppose that assumption of Theorem A holds and let $X\, and\, Y$ have support $(0,b]$, then for $\gamma<\delta<1(\gamma>\delta)$, the following upper bound of $KPI_{(X,Y;t)}^{w(\gamma,\delta)}$ holds

$$KPI_{(X,Y;t)}^{w(\gamma,\delta)} \le (\ge) \frac{\delta}{\delta-\gamma}\left[\frac{\int_0^t x\frac{f(x)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta}\log\left(x\frac{f(x)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta}\right)dx}{\int_0^t x\frac{f(x)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta}dx}+\log t\right].$$

Proof:

In view of log-sum inequality and from (12), we get

$$\int_0^t x\frac{f(x)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta}\log\left(x\frac{f(x)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta}\right)dx$$

$$\ge \int_0^t x\frac{f(x)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta}dx\log\frac{\int_0^t x\, f(x)\left(g(x)\right)^{\gamma-\delta}dx}{\int_0^t F(t)\left(G(t)\right)^{\gamma-\delta}dx}$$

$$=\int_0^t x\frac{f(x)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta}dx\left[\left(\frac{\delta-\gamma}{\delta}\right)KPI_{(X,Y;t)}^{w(\gamma,\delta)}-\log t\right].$$

On solving the above, we get the proof.

Proposition 4.1:

For the two random variables X and Y having WGPI $KPI_{(X,Y;t)}^{w(\gamma,\delta)}$ and $\gamma>\delta$, we have

$$KPI_{(X,Y;t)}^{w(\gamma,\delta)} \ge \frac{\delta}{\gamma-\delta}\left[1-\int_0^t x\frac{f(x)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta}dx\right].$$

Proof:

On considering, $-\log v \ge 1-v$, we can easily verify the result.

Theorem 4.5:

If $\mu_G(t)=\frac{g(t)}{G(t)}$, the reversed hazard rate is decreasing in t, then

$$KPI_{(X,Y;t)}^{w(\gamma,\delta)} \le \frac{\delta}{\delta-\gamma}\left(\int_0^t x\frac{f(x)}{F(t)}\left(\frac{G(x)}{G(t)}\right)^{\gamma-\delta} dx - 1\right) - \delta\log\mu_G(t).$$

Proof:

Rewriting equation (12) as

$$KPI_{(X,Y;t)}^{w(\gamma,\delta)} = \frac{\delta}{\delta-\gamma}\log\int_0^t x\frac{f(x)}{F(t)}\left(\mu_G(x)\right)^{\gamma-\delta}\left(\frac{G(x)}{G(t)}\right)^{\gamma-\delta} dx .$$

Since, $\mu_G(t) = \dfrac{g(t)}{G(t)}$, reversed hazard rate is decreasing in t, $\mu_G(x) \ge \mu_G(t)$, for $0 < x < t$. Moreover, $\log x \ge x - 1$. Therefore

$$KPI_{(X,Y;t)}^{w(\gamma,\delta)} \le \frac{\delta}{\delta-\gamma}\left[\log\left(\mu_G(t)\right)^{\gamma-\delta} + \log\int_0^t x\frac{f(x)}{F(t)}\left(\frac{G(x)}{G(t)}\right)^{\gamma-\delta} dx\right]$$

$$= \frac{\delta}{\delta-\gamma}\left[\log\left(\mu_G(t)\right)^{\gamma-\delta} + \int_0^t x\frac{f(x)}{F(t)}\left(\frac{G(x)}{G(t)}\right)^{\gamma-\delta} dx - 1\right].$$

which gives the required result.

Theorem 4.6:

Let X and Y be the two absolutely continuous r.v's with density functions $f(x)$ & $g(x)$ and cumulative distribution functions $F(x)$ & $G(x)$, respectively. If $f(x)$ is increasing (decreasing) in $x > 0$, then

$$KPI_{(X,Y;t)}^{w(\gamma,\delta)} \le (\ge) \frac{\delta}{\delta-\gamma}\log\mu_F(t) + \frac{\delta}{\delta-\gamma}\log\int_0^t x\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta} dx .$$

Proof:

Let $f(x)$ be increasing (decreasing) in $x > 0$, then for $x < t$, we have

$$\frac{f(x)}{F(t)} \le (\ge) \frac{f(t)}{F(t)} .$$

Moreover, $\dfrac{g(x)}{G(t)}$ is positive. Therefore

$$\frac{f(x)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta} \le (\ge)\ \frac{f(t)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta}$$

or

$$x\frac{f(x)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta} \le (\ge)\ x\frac{f(t)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta}. \tag{27}$$

Integrating (27) from 0 to t with respect to x, taking log and then multiply $\frac{\delta}{\delta-\gamma}$ on both sides, the result can be easily derived.

Theorem 4.7:

Let X and Y be the two r.v's as described in theorem 4.6. If $g(x)$ is increasing (decreasing) in $x>0$, then

$$KPI_{(X,Y;t)}^{w(\gamma,\delta)} \le (\ge)\ \frac{\delta}{\delta-\gamma}\log\mu_F^*(t) - \delta\log\mu_F(t).$$

where, $\mu_F^*(t) = \int_0^t x\frac{f(x)}{F(t)}dx = E[X \mid X \le t]$, is the mean past lifetime of r.v X.

Proof:

Let $g(x)$ be increasing (decreasing) in $x>0$, then for $x<t$, we have

$$\frac{g(x)}{G(t)} \le (\ge)\ \frac{g(t)}{G(t)}.$$

or

$$\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta} \le (\ge) \left(\frac{g(t)}{G(t)}\right)^{\gamma-\delta}. \tag{28}$$

Moreover, $\frac{f(x)}{F(t)}$ is positive. Therefore from (28), we get

$$\frac{f(x)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta} \le (\ge)\ \frac{f(t)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta}$$

or

$$x\frac{f(x)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta} \le (\ge)\ x\frac{f(t)}{F(t)}\left(\frac{g(x)}{G(t)}\right)^{\gamma-\delta}. \tag{29}$$

Integrating (29) from 0 to t with respect to x and then multiplying by $\frac{\delta}{\delta-\gamma}$, the required result is obtained.

Definition 4.2:

If X and Y are two r.v's with density function $f(x)$ and $g(x)$ respectively, then X is said to be less than or equal to Y in the likelihood ratio ordering, denoted by $X \overset{lr}{\leq} Y$, if $\frac{f(x)}{g(x)}$ is decreasing in x. i.e., if $\frac{f(x)}{g(x)} \leq \frac{f(t)}{g(t)}$, for all $x > t$. For more see, Shaked and Shantikumar (2007), Maryam et al. (2020).

Theorem 4.8:

If $X \overset{lr}{\leq} Y$, then for $\gamma > \delta$

$$KPI_{(X,Y;t)}^{w(\gamma,\delta)} \leq \frac{\delta}{\delta-\gamma} \log \int_0^t x \left(\frac{f(x)}{F(t)} \right)^{\gamma-\delta+1} dx + \delta \log \left(\frac{\mu_F(t)}{\mu_G(t)} \right)$$

Proof:

From equation (12), we have

$$KPI_{(X,Y;t)}^{w(\gamma,\delta)} = \frac{\delta}{\delta-\gamma} \log \int_0^t x \frac{f(x)}{F(t)} \left(\frac{g(x)}{G(t)} \right)^{\gamma-\delta} dx$$

$$= \frac{\delta}{\delta-\gamma} \log \int_0^t x \left(\frac{f(x)}{F(t)} \right)^{\gamma-\delta+1} \left(\frac{g(x)}{f(x)} \frac{F(t)}{G(t)} \right)^{\gamma-\delta} dx . \tag{30}$$

Since, $\frac{f(x)}{g(x)} \geq \frac{f(t)}{g(t)}$, for all $0 \leq x \leq t$. Therefore equation (30) reduces to

$$KPI_{(X,Y;t)}^{w(\gamma,\delta)} \leq \frac{\delta}{\delta-\gamma} \log \int_0^t x \left(\frac{f(x)}{F(t)} \right)^{\gamma-\delta+1} \left(\frac{g(t)}{f(t)} \frac{F(t)}{G(t)} \right)^{\gamma-\delta} dx ,$$

which leads to

$$KPI_{(X,Y;t)}^{w(\gamma,\delta)} \leq \frac{\delta}{\delta-\gamma} \log \int_0^t x \left(\frac{f(x)}{F(t)} \right)^{\gamma-\delta+1} dx + \delta \log \left(\frac{\mu_F(t)}{\mu_G(t)} \right).$$

In the following theorems, we consider the three non-negative random variables and obtain the bounds.

Theorem 4.9:

Let X_1, X_2 & X_3 be the three non-negative r.v's with density functions f_1, f_2 & f_3, cumulative distribution functions F_1, F_2 & F_3 and reversed hazard rate functions μ_{F_1}, μ_{F_2} & μ_{F_3} respectively. Further, if $X_1 \overset{lr}{\le} X_2$, then

(I) $KPI^{w(\gamma,\delta)}_{(X_1,X_3;t)} \le KPI^{w(\gamma,\delta)}_{(X_2,X_3;t)} + \frac{\delta}{\delta-\gamma}\log\left(\frac{\mu_{F_1}(t)}{\mu_{F_2}(t)}\right)$, *if* $0<\gamma<\delta$.

(II) $KPI^{w(\gamma,\delta)}_{(X_1,X_3;t)} \ge KPI^{w(\gamma,\delta)}_{(X_2,X_3;t)} + \frac{\delta}{\delta-\gamma}\log\left(\frac{\mu_{F_1}(t)}{\mu_{F_2}(t)}\right)$, *if* $\gamma>\delta$.

Proof (I):

Since, by the given condition, we have

$$\frac{f_1(x)}{f_2(x)} \le \frac{f_1(t)}{f_2(t)}.$$

Therefore, for $0<\gamma<\delta$, from (12), we obtain

$$KPI^{w(\gamma,\delta)}_{(X_1,X_3;t)} \le \frac{\delta}{\delta-\gamma}\log\int_0^t x\frac{f_1(t)\,f_2(x)}{f_2(t)F_1(t)}\left(\frac{f_3(x)}{F_3(t)}\right)^{\gamma-\delta}dx$$

$$= \frac{\delta}{\delta-\gamma}\log\int_0^t x\left(\frac{\mu_{F_1}(t)}{\mu_{F_2}(t)}\right)\frac{f_2(x)}{F_2(t)}\left(\frac{f_3(x)}{F_3(t)}\right)^{\gamma-\delta}dx.$$

This completes the proof of (I). Similarly we can prove (II).

Theorem 4.10:

Consider the three r.v's as mentioned in the theorem 4.9. Further, assume $X_2 \overset{lr}{\le} X_3$, that is $\frac{f_3(x)}{f_2(x)}$ is increasing in $x>0$, then

$$KPI^{w(\gamma,\delta)}_{(X_1,X_2;t)} \le KPI^{w(\gamma,\delta)}_{(X_1,X_3;t)} - \delta\log\left(\frac{\mu_{F_2}(t)}{\mu_{F_1}(t)}\right).$$

Proof:

Since, we have $\frac{f_2(x)}{f_3(x)} \le \frac{f_2(t)}{f_3(t)}$, therefore, from (12), we have

$$KPI^{w(\gamma,\delta)}_{(X_1,X_2;t)} \le \frac{\delta}{\delta-\gamma}\log\int_0^t x\frac{f_1(x)}{F_1(t)}\left(\frac{f_2(t)f_3(x)}{f_3(t)F_2(t)}\right)^{\gamma-\delta}dx$$

$$=\frac{\delta}{\delta-\gamma}\log\int_0^t x\left(\frac{\mu_{F_2}(t)}{\mu_{F_3}(t)}\right)^{\gamma-\delta}\frac{f_1(x)}{F_1(t)}\left(\frac{f_3(x)}{F_3(t)}\right)^{\gamma-\delta}dx\,.$$

Hence we get the desired result.

Theorem 4.11:

Consider the three r.v's as mentioned in the theorem 4.9. Further, assume $X_1 \overset{lr}{\le} X_3$, that is $\frac{f_3(x)}{f_1(x)}$ is increasing in $x>0$, then

$$KPI^{w(\gamma,\delta)}_{(X_2,X_3;t)} \le KPI^{w(\gamma,\delta)}_{(X_2,X_1;t)} - \delta\log\left(\frac{\mu_{F_3}(t)}{\mu_{F_1}(t)}\right).$$

Proof:

Since, we have $\frac{f_1(x)}{f_3(x)} \le \frac{f_1(t)}{f_3(t)}$, therefore, from (12), we have

$$KPI^{w(\gamma,\delta)}_{(X_2,X_3;t)} \le \frac{\delta}{\delta-\gamma}\log\int_0^t x\frac{f_2(x)}{F_2(t)}\left(\frac{f_3(t)f_1(x)}{f_1(t)F_1(t)}\right)^{\gamma-\delta}dx$$

$$=\frac{\delta}{\delta-\gamma}\log\int_0^t x\left(\frac{\mu_{F_3}(t)}{\mu_{F_1}(t)}\right)^{\gamma-\delta}\frac{f_2(x)}{F_2(t)}\left(\frac{f_1(x)}{F_1(t)}\right)^{\gamma-\delta}dx\,.$$

Proposition 4.2:

Let X and Y be the two r.v's with finite support $[0,m]$ and symmetric with respect to $\frac{m}{2}$, i.e., $F(x)=F(m-x)$ and $G(x)=G(m-x)$ for $0<x<m$, then

$$KPI^{w(\gamma,\delta)}_{(X,Y;t)} = \frac{\delta}{\delta-\gamma}\log\left[m\exp\left(\left(\frac{\delta-\gamma}{\delta}\right)KPI^{(\gamma,\delta)}_{(X,Y;m-t)}\right)-\exp\left(\left(\frac{\delta-\gamma}{\delta}\right)KPI^{w(\gamma,\delta)}_{(X,Y;m-t)}\right)\right].$$

Proof:

Using the symmetric property in (12), we get

$$KPI^{w(\gamma,\delta)}_{(X,Y;t)} = \frac{\delta}{\delta-\gamma}\log\int_0^t x\frac{f(m-x)}{F(m-t)}\left(\frac{g(m-x)}{G(m-t)}\right)^{\gamma-\delta}dx\,.$$

Suppose $v=m-x$, Then

$$KPI_{(X,Y;t)}^{w(\gamma,\delta)} = \frac{\delta}{\delta-\gamma}\log\int_{m-t}^{m}(m-v)\frac{f(v)}{F(m-t)}\left(\frac{g(v)}{G(m-t)}\right)^{\gamma-\delta}dv .$$

$$= \frac{\delta}{\delta-\gamma}\log\left[\int_{m-t}^{m} m\frac{f(v)}{F(m-t)}\left(\frac{g(v)}{G(m-t)}\right)^{\gamma-\delta}dv - \int_{m-t}^{m} v\frac{f(v)}{F(m-t)}\left(\frac{g(v)}{G(m-t)}\right)^{\gamma-\delta}dv\right].$$

On further simplifications, we get the proof.

Example 4.1:

If the actual distribution function $F(x)$ and predicted distribution function $G(x)$ are exponentially distributed with parameters $\lambda_1 > 0$ and $\lambda_2 > 0$ respectively, then

$$f(x) = \lambda_1 e^{-\lambda_1 x},\ \overline{F}(x) = 1 - F(x) = e^{-\lambda_1 x},\ x > 0$$

and

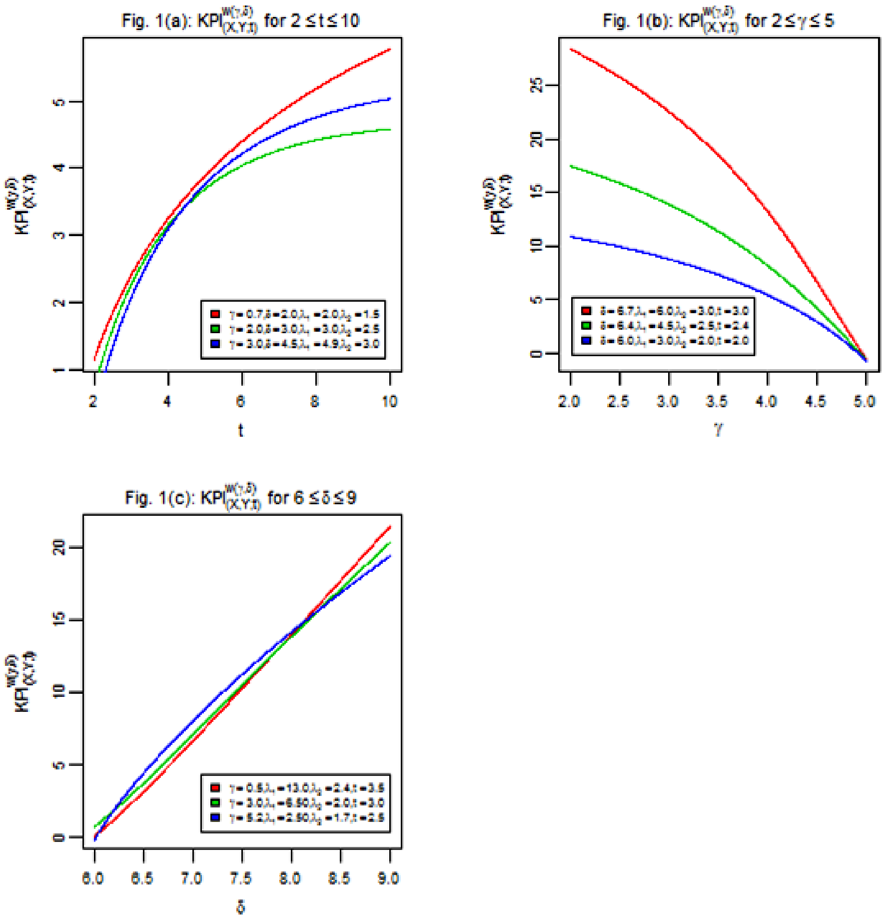
$$g(x) = \lambda_2 e^{-\lambda_2 x},\ \overline{G}(x) = 1 - G(x) = e^{-\lambda_2 x},\ x > 0 .$$

Using the values in (12), we obtain the weighted generalized past inaccuracy measure as

$$KPI_{(X,Y;t)}^{w(\gamma,\delta)} = \frac{\delta}{\delta-\gamma}\log\left[\frac{\lambda_1(\lambda_2)^{\gamma-\delta}\left(1-e^{-At}(1+At)\right)}{A^2\left(1-e^{-\lambda_1 t}\right)\left(1-e^{-\lambda_2 t}\right)^{\gamma-\delta}}\right], \tag{31}$$

where, $A = \lambda_1 + (\gamma-\delta)\lambda_2$.

The graphical behavior of the expression given in (31) for different values of $\gamma, \delta, \lambda_1, \lambda_2, t$ and k is given by



Thus, from the above, we observe that $KPI^{w(\gamma,\delta)}_{(X,Y;t)}$ Shows increasing behavior with respect to t & δ and decreasing behavior on γ .

Further to the general case given in (31), we obtain the following two particular cases.

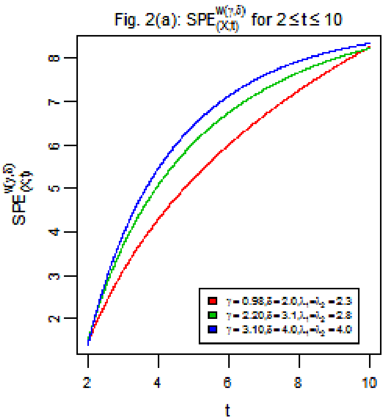
Case I:

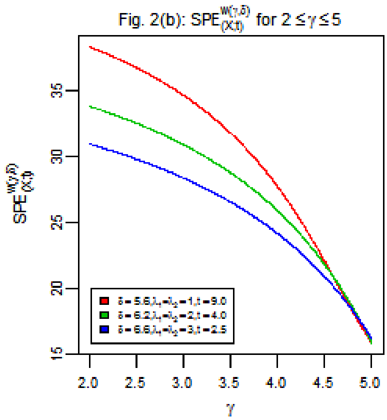
When $\lambda_1 = \lambda_2$ that is $G(x) = F(x)$. In this case the weighted generalized past inaccuracy $KPI^{w(\gamma,\delta)}_{(X,Y;t)}$ reduces to the weighted generalized past entropy $SPE^{w(\gamma,\delta)}_{(X;t)}$, given as

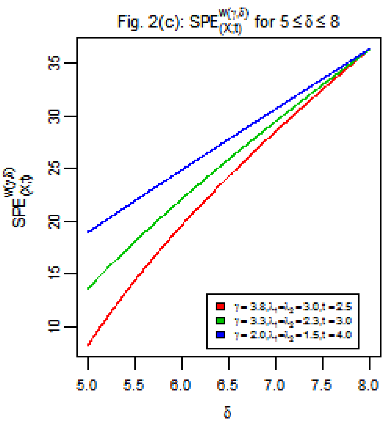
$$SPE^{w(\gamma,\delta)}_{(X,Y;t)} = \frac{\delta}{\delta-\gamma} \log \left[\frac{(\lambda_1)^{\gamma-\delta+1} \left(1-e^{-Bt}\left(1+Bt\right)\right)}{B^2 \left(1-e^{-\lambda_1 t}\right)^{\gamma-\delta+1}} \right], \tag{32}$$

where, $B = \lambda_1(\gamma - \delta + 1)$.

The plots of the expression given in (32) for different values of $\gamma, \delta, \lambda_1, \lambda_2, t\ and\ k$ are given by







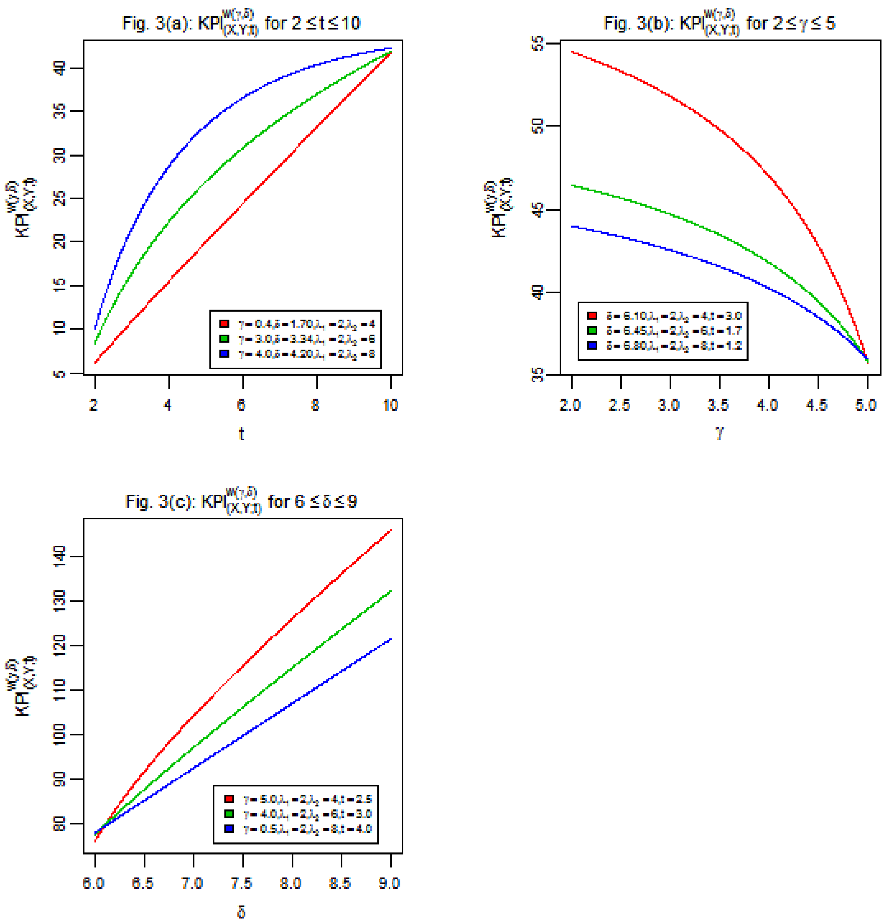
Case II:

When $\lambda_2 = k\lambda_1$, that is $\overline{G}(x) = \left(\overline{F}(x)\right)^k$. This corresponds to k-components series system each component having identically and independently distributed lifetime $X_i,\ i = 1, 2, ..., k$, with distribution function $F(x)$, and here $G(x)$ is the distribution of the lifetime $Y = \min(X_1, X_2, ..., X_k)$ of the system. The weighted generalized past inaccuracy given in (31) in this case reduces to

$$KPI_{(X,Y;t)}^{w(\gamma,\delta)} = \frac{\delta}{\delta-\gamma}\log\left[\frac{k^{\gamma-\delta}\left(\lambda_1\right)^{\gamma-\delta+1}\left(1-e^{-Ct}\left(1+Ct\right)\right)}{C^2\left(1-e^{-\lambda_1 t}\right)\left(1-e^{-k\lambda_1 t}\right)^{\gamma-\delta}}\right], \tag{33}$$

where, $C=\lambda_1\left(1+k\left(\gamma-\delta\right)\right)$.

The plots of the expression given in (33) for different values of $\gamma,\delta,\lambda_1,\lambda_2,t\,and\,k$ is exhibited as



Thus, from the above plots of the expressions of (32) and (33), we observe that both $SPE_{(X,Y;t)}^{w(\gamma,\delta)}$ and $KPI_{(X,Y;t)}^{w(\gamma,\delta)}$ are respectively, increasing with respect to $t\,\&\,\delta$ and decreasing with respect to γ.

5. CONCLUSION

In this paper, we proposed a new length-biased parametric generalized inaccuracy measure by using two parameters γ and type δ between the two past lifetime distributions. By making use of the concept of proportional reversed hazard model (PRHM), some significant characterization results of the proposed inaccuracy measure have been studied. It has been observed that the proposed generalized measure uniquely determines the distribution function. Further, some bounds to the length-biased generalized inaccuracy measure have also been derived. Finally, based on two independent exponential distributions, we studied the monotonic behavior of the proposed length-biased generalized past inaccuracy measure.

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