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TRANSMUTED GUMBEL TYPE-II DISTRIBUTION WITH APPLICATIONS IN DIVERSE FIELDS OF SCIENCE

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ABSTRACT

The family of distributions which are derived from baseline distributions drag the attention of researchers from past recent years. In this paper we introduce an extension of Gumbel type-II by using transmutation approach suggested by Shaw and Buckley (2007). Several distributional properties of the explored distribution, such as moments, moment generating function, quantile function, reliability analysis, order statistics, and Renyi entropy, have been discussed. Various diagrams depict the behaviour of the probability density function (pdf) and cumulative distribution function (cdf). The parameters of the explored distribution are calculated by using the well-known maximum likelihood procedure. Eventually, the utility of the explored distribution is shown using real life data sets.

KEY WORDS

Transmutation technique, Gumbel Type-II distribution, Renyi entropy, reliability analysis, maximum likelihood estimation.

Mathematics subject classification: 60-XX, 62-XX, 11-KXX.

1. INTRODUCTION

In the probability theory, various univariate standard distributions have been widely used for analysing data which were collected from many areas of science such as engineering, bio-medicine, actuarial science, finance and economics. During various investigations it was observed that data obtained from different applied subjects such as finance, environmental science and other related areas, does not follow these distributions. Clearly there is a need to enhance the utility of these distributions by making their extensions and generalisations so that they become suitable for such data. In this direction Statisticians have made a lot of work and introduced various methods and techniques to construct new models from baseline distributions. The modified models proved more flexible than the standard distributions. In this article, we attempt to add another generalisation of the Gumbel type-II distribution by employing the quadratic rank transmutation map to the baseline distribution advocated by Shaw and Buckley (2007). Suppose $G(y)$ denotes the cdf of the baseline distribution, the quadratic transmutation can be determined as

$$F(y)=\int_0^{G(y)} p(t)\,dt$$

where $p(t)=(1+\lambda)-2\lambda t$

$$F(y)=(1+\lambda)G(y)-\lambda G^2(y)\,;\ |\lambda|\le 1 \tag{1.1}$$

The corresponding pdf of (1.1) is given by

$$f(y)=g(y)\left[1+\lambda-2\lambda G(y)\right];\ y>0\,,\ |\lambda|\le 1 \tag{1.2}$$

We noticed that at $\lambda=0$, we obtain baseline distribution. There are plenty of studies that have been focussed on the transmuted generalizations of probability models. Among them are Aryal and Tsokos (2011) studied various structural properties of transmuted Weibull distribution. Faton Mervoci (2013) introduced transmuted Rayleigh distribution. Ashour et al. (2013) presented the transmuted exponential lomax distribution. Ahmad et al. (2015) studied transmuted Kumaraswamy and shows its performance through real data sets. Aijaz et al. (2020) formulated the transmuted inverse Lindley distribution and through examples they performed the versatility of the model. In present study we explored Transmuted Gumbel Type-II distribution and discussed its various structural properties.

2. TRANSMUTED GUMBEL TYPE-II DISTRIBUTION

Definition 2.1

The cumulative distribution function (cdf) of the Gumbel type-II distribution is defined as

$$G(y,a,b)=e^{-by^{-a}}\ ;y>0,a,b>0 \tag{2.1}$$

The associated probability density function (pdf) is given by

$$g(y,a,b)=aby^{-a-1}e^{-by^{-a}}\ ;y>0,a,b>0 \tag{2.2}$$

Now substituting the value of equation (2.1) in equation (1.1) we obtain the cumulative distribution function of the transmuted Gumbel type-II distribution

$$F(y,a,b,\lambda)=e^{-by^{-a}}\left(1+\lambda-\lambda e^{-by^{-a}}\right)\ ;y>0,a,b>0\ \ ;|\lambda|\le 1 \tag{2.3}$$

The probability density function (pdf) of the transmuted Gumbel type-II distribution is given by

$$f(y,a,b,\lambda)=aby^{-a-1}e^{-by^{-a}}\left(1+\lambda-2\lambda e^{-by^{-a}}\right)\ ;y>0,a,b>0\ ;|\lambda|\le 1 \tag{2.4}$$

Figure (1.1) and (1.2) expounds few possible layouts of pdf of transmuted Gumbel type-II distribution for distinct values of parameters

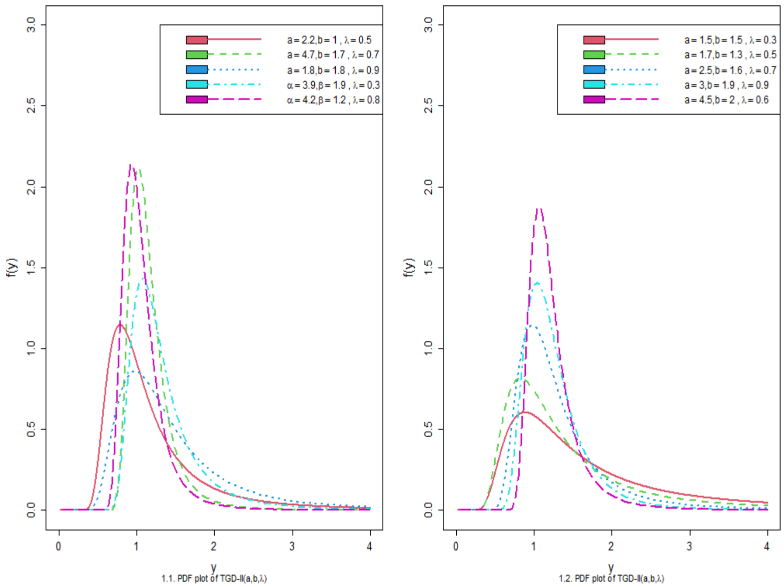
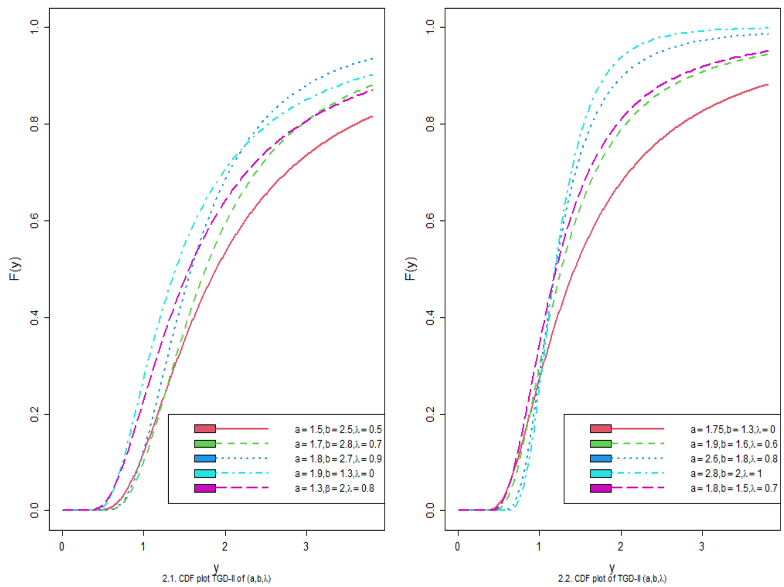


Figure (2.1) and (2.2) expounds few possible layouts of pdf of transmuted Gumbel type-II distribution for distinct values of parameters

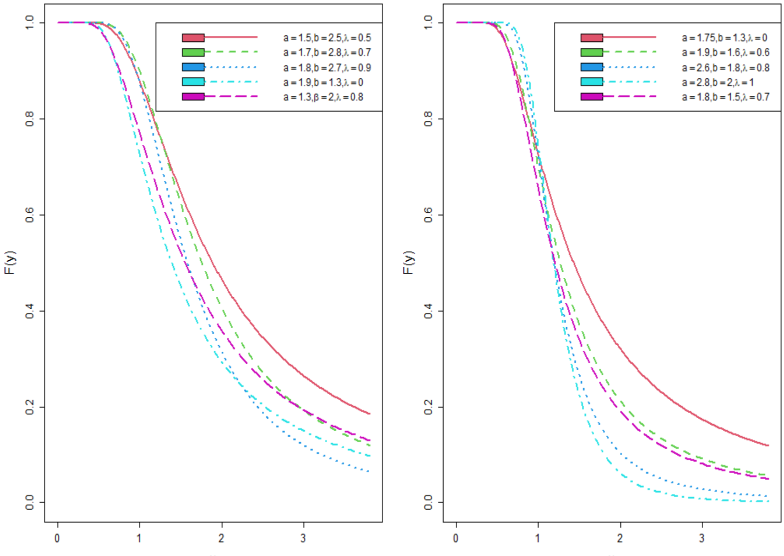


3. RELIABILITY ANALYSIS

The reliability function can be defined as the probability of equipment which could not fizzle before certain specified period of time

$$R(y,a,b,\lambda)=1-F(y,a,b,\lambda)$$
$$=1-e^{-by^{-a}}\left(1+\lambda-\lambda e^{-by^{-a}}\right) \tag{3.1}$$

Figure (3.1) and (3.2) expounds few possible layouts of pdf of transmuted Gumbel type-II distribution for distinct values of parameters

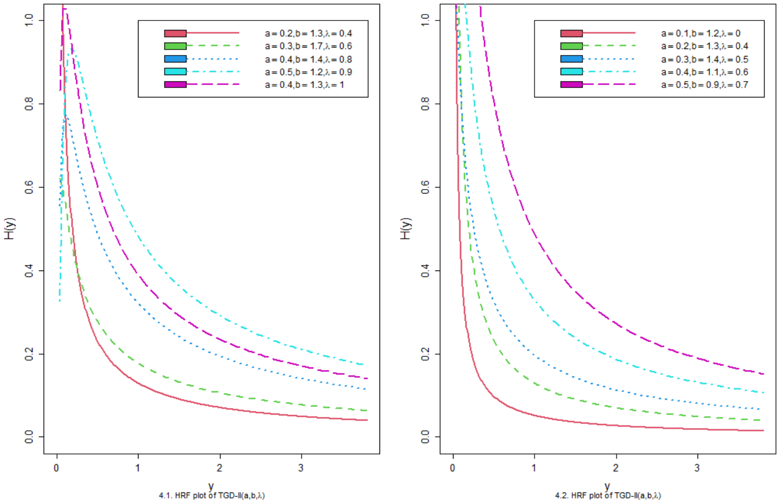


3.1. SF plot TGD-II of (a,b,λ)   3.2. SF plot of TGD-II (a,b,λ)

The hazard rate function of transmuted Gumbel type-II is defined by

$$H(y,a,b,\lambda)=\frac{f(y,a,b,\lambda)}{R(y,a,b,\lambda)}$$
$$=\frac{aby^{-a-1}e^{-by^{-a}}\left(1+\lambda-2\lambda e^{-by^{-a}}\right)}{1-e^{-by^{-a}}\left(1+\lambda-\lambda e^{-by^{-a}}\right)} \tag{3.2}$$

Figure (4.1) and (4.2) expounds few possible layouts of pdf of transmuted Gumbel type-II distribution for distinct values of parameters

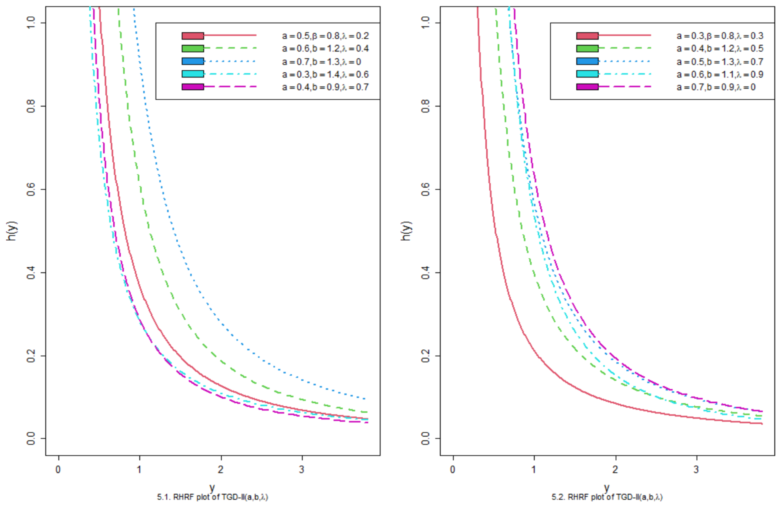


Reverse hazard rate function of transmuted Gumbel type-II is defined by

$$h_r\left(y,a,b,\lambda\right)=\frac{f\left(y,a,b.\lambda\right)}{F\left(y,a,b,\lambda\right)}$$

$$=\frac{aby^{-a-1}\left(1+\lambda-2\lambda e^{-by^{-a}}\right)}{\left(1+\lambda-\lambda e^{-by^{-a}}\right)} \tag{3.3}$$

Figure (5.1) and (5.2) expounds few possible layouts of pdf of transmuted Gumbel type-II distribution for distinct values of parameters



4. STRUCTURAL PROPERTIES OF TRANSMUTED GUMBEL TYPE-II DISTRIBUTION

Theorem 4.1

Suppose y denotes random variable follows TGDII with p.d.f $f\left(y,a,b,\lambda\right)$. Then the r^{th} moment of the transmuted Gumbel type-II distribution is given by

$$\mu_r{}' = \left(1+\lambda-2^{\frac{r}{a}}\lambda\right)b^{\frac{r}{a}}\Gamma\left(1-\frac{r}{a}\right)$$

Proof:

Let Y denotes random variable follows TGDII. Then r^{th} moment denoted by μ_r is given as

$$\mu_r{}' = E\left(Y^r\right) = \int_0^\infty y^r f\left(y,a,b,\lambda\right)dy$$

$$\mu_r{}' = ab\int_0^\infty y^{r-a-1}e^{-by^{-a}}\left(1+\lambda-2\lambda e^{-by^{-a}}\right)dy$$

$$= ab\left(1+\lambda\right)\int_0^\infty y^{r-a-1}e^{-by^{-a}}dy - 2ab\lambda\int_0^\infty y^{r-a-1}e^{-2by^{-a}}dy$$

Making substitution $by^{-a} = z$ so that, $-aby^{-a-1}dy = dz$

$$\mu_r{}' = \left(1+\lambda\right)b^{\frac{r}{a}}\int_0^\infty z^{-\frac{r}{a}}e^{-z}dz - 2\lambda\int_0^\infty z^{-\frac{r}{a}}e^{-2z}dz$$

After solving the integral, we get

$$\mu_r{}' = \left(1+\lambda-2^{\frac{r}{a}}\lambda\right)b^{\frac{r}{a}}\Gamma\left(1-\frac{r}{a}\right)$$

The mean and variance of the TGD-II is given by

$$\mu = E\left(Y\right) = \left(1+\lambda-2^{\frac{1}{a}}\lambda\right)b^{\frac{1}{a}}\Gamma\left(1-\frac{1}{a}\right)$$

$$\sigma^2 = \left(1+\lambda-2^{\frac{2}{a}}\lambda\right)b^{\frac{2}{a}}\Gamma\left(1-\frac{2}{a}\right) - \left[\left(1+\lambda-2^{\frac{1}{a}}\lambda\right)b^{\frac{1}{a}}\Gamma\left(1-\frac{1}{a}\right)\right]^2$$

Theorem 4.2

Suppose y denotes random variable follows TGDII with pdf $f\left(y,a,b,\lambda\right)$. Then the moment generating function of the transmuted Gumbel type-II distribution denoted as $M_Y\left(t\right)$ is given by

$$M_Y\left(t\right)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\left(1+\lambda-2^{\frac{r}{a}}\lambda\right)b^{\frac{r}{a}}\Gamma\left(1-\frac{r}{a}\right)$$

Proof:

Let Y denotes random variable follows TGDII. Then moment generating function denoted by $M_Y\left(t\right)$ is given as

$$M_Y\left(t\right)=E\left(e^{ty}\right)=\int_0^{\infty}e^{ty}f\left(y,a,b,\lambda\right)dy$$

$$=\int_0^{\infty}\left(1+ty+\frac{(ty)^2}{2!}+\frac{(ty)^3}{3!}+...\right)f\left(y,a,b,\lambda\right)dy$$

$$=\int_0^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}y^r f\left(y,a,b,\lambda\right)dy$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty}y^r f\left(y,a,b,\lambda\right)dy$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}E\left(Y^r\right)$$

$$M_Y\left(t\right)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\left(1+\lambda-2^{\frac{r}{a}}\lambda\right)b^{\frac{r}{a}}\Gamma\left(1-\frac{r}{a}\right)$$

The characteristics function of the TGD-II distribution denoted as $\varphi_Y\left(t\right)$ can be obtained by replacing $t=it$ is given by

$$\varphi_Y\left(t\right)=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\left(1+\lambda-2^{\frac{r}{a}}\lambda\right)b^{\frac{r}{a}}\Gamma\left(1-\frac{r}{a}\right)$$

4.3 Quantile function of Transmuted Gumbel Type-II Distribution

The quantile function of random variable Y, where $Y\sim TGD-II\left(a,b,\lambda\right)$, can be obtained by inverting equation $\left(2.3\right)$, we have

$$y_q=\left[-\frac{1}{b}\log\left(\frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4\lambda q}}{2\lambda}\right)\right]^{-\frac{1}{a}}$$

In particular, the median of the distribution is

$$y_{0.5} = \left[-\frac{1}{b} \log \left(\frac{(1+\lambda) - \sqrt{1+\lambda^2}}{2\lambda} \right) \right]^{-\frac{1}{a}}$$

4.4 Random number generation of Transmuted Gumbel Type-II Distribution

Suppose y denotes a random variable with pdf given in equation (2.4). The random number of transmuted Gumbel distribution type-II can be generated as

$$F(y) = u \Rightarrow y = F^{-1}(u)$$

So that

$$y = \left[-\frac{1}{b} \log \left(\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda u}}{2\lambda} \right) \right]^{-\frac{1}{a}}$$

Where u is the uniform random variable defined in an open interval $(0,1)$.

5. RENYI ENTROPY OF TRANSMUTED GUMBEL TYPE-II DISTRIBUTION

If Y is a continuous random variable with probability density function $f(y,a,b,\lambda)$. Then

Renyi entropy is then defined as

$$T_R(\gamma) = \frac{1}{1-\gamma} \log \left\{ \int_0^\infty f^\gamma(y)\, dy \right\}, \text{ where } \gamma > 0 \text{ and } \gamma \neq 1$$

Thus, the Renyi entropy for transmuted Gumbel type-II distribution (2.4), is given as

$$T_R(\gamma) = \frac{1}{1-\gamma} \log \left\{ \int_0^\infty \left[aby^{-a-1} e^{-by^{-a}} \left(1+\lambda-2\lambda e^{-by^{-a}}\right) \right]^\gamma dy \right\}$$

$$= \frac{1}{1-\gamma} \log \left\{ (ab)^\gamma (1+\lambda)^\gamma \int_0^\infty y^{-\gamma(a+1)} e^{-b\gamma y^{-a}} \left(1 - \frac{2\lambda}{1+\lambda} e^{-by^{-a}}\right)^\gamma dy \right\}$$

Using binomial theorem

$$(1-x)^n = \sum_{r=0}^{\infty}\binom{n}{r}(-1)^r x^r$$

$$= \frac{1}{1-\gamma}\log\left\{\left(ab(1+\lambda)\right)^{\gamma}\int_0^{\infty} y^{-\gamma(a+1)}e^{-b\gamma y^{-a}}\sum_{r=0}^{\infty}\binom{\gamma}{r}(-1)^r\left(\frac{2\lambda}{1+\lambda}\right)^r e^{-bry^{-a}}dy\right\}$$

$$= \frac{1}{1-\gamma}\log\left\{(ab)^{\gamma}\sum_{r=0}^{\infty}\binom{\gamma}{r}(-1)^r\frac{(2\lambda)^r}{(1+\lambda)^{r-\gamma}}\int_0^{\infty} y^{-\gamma(a+1)}e^{-b(r+\gamma)y^{-a}}dy\right\}$$

Making substitution $by^{-a} = z$ so that, $-aby^{-a-1}dy = dz$

$$= \frac{1}{1-\gamma}\log\left\{\sum_{r=0}^{\infty}\binom{\gamma}{r}(-1)^r\frac{a^{\gamma-1}}{(b)^{\frac{\gamma-1}{a}}}\frac{(2\lambda)^r}{(1+\lambda)^{r-\gamma}}\int_0^{\infty} z^{\frac{\gamma(a+1)-1}{a}-1}e^{-(r+\gamma)z}dz\right\}$$

After solving the integral, we get

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\left\{\sum_{\mathrm{r}=0}^{\infty}\binom{\gamma}{r}(-1)^r\frac{a^{\gamma-1}}{(b)^{\frac{\gamma-1}{a}}}\frac{(2\lambda)^r}{(1+\lambda)^{r-\gamma}}\frac{\Gamma\left(\frac{\gamma(a+1)}{a}\right)}{(r+\gamma)^{\frac{\gamma(a+1)}{a}}}\right\}.$$

6. TSALLIS ENTROPY OF TRANSMUTED GUMBEL TYPE-II DISTRIBUTION

Tsallis entropy of order γ for Transmuted Gumbel Type-II distribution (2.4), is given as

$$S_{\gamma} = \frac{1}{\gamma-1}\left\{1-\int_0^{\infty} f^{\gamma}(y)dy\right\}, \text{ where } \gamma > 0 \text{ and } \gamma \neq 1$$

$$S_{\gamma} = \frac{1}{\gamma-1}\left\{1-\int_0^{\infty}\left[aby^{-a-1}e^{-by^{-a}}\left(1+\lambda-2\lambda e^{-by^{-a}}\right)\right]^{\gamma}dy\right\}$$

After solving the integral, we get

$$S_{\gamma} = \frac{1}{\gamma-1}\left\{1-\sum_{\mathrm{r}=0}^{\infty}\binom{\gamma}{r}(-1)^r\frac{a^{\gamma-1}}{(b)^{\frac{\gamma-1}{a}}}\frac{(2\lambda)^r}{(1+\lambda)^{r-\gamma}}\frac{\Gamma\left(\frac{\gamma(a+1)-1}{a}\right)}{(r+\gamma)^{\frac{\gamma(a+1)-1}{a}}}\right\}$$

7. ORDER STATISTICS OF TRANSMUTED GUMBEL TYPE-II DISTRIBUTION

Let us suppose $Y_1, Y_2, ..., Y_n$ be random samples of size n from transmuted Gumbel type-II distribution with p.d.f $f(y)$ and $F(y)$. The probability density function of order statistics is then calculated as follows:

$$f_{Y_{(k)}}(y) = \frac{n!}{(k-1)!(n-k)!} f(y)\left[F(y)\right]^{k-1}\left[1-F(y)\right]^{n-k} \tag{7.1}$$

Now using the equation (2.3) and (2.4) in (7.1). The probability of k^{th} order statistics of transmuted Gumbel type-II distribution is given as

$$f_{Y_{(k)}}(y) = \frac{n!}{(k-1)!(n-k)!} aby^{-a-1}e^{-by^{-a}}\left(1+\lambda-2\lambda e^{-by^{-a}}\right)\left[e^{-by^{-a}}\left(1+\lambda-\lambda e^{-by^{-a}}\right)\right]^{k-1}$$
$$\left[1-e^{-by^{-a}}\left(1+\lambda-\lambda e^{-by^{-a}}\right)\right]^{n-k}$$

Then, the p.d.f of first order Y_1 transmuted Gumbel type-II distribution is given as

$$f_{Y_{(1)}}(y) = naby^{-a-1}e^{-by^{-a}}\left(1+\lambda-2\lambda e^{-by^{-a}}\right)\left[1-e^{-by^{-a}}\left(1+\lambda-\lambda e^{-by^{-a}}\right)\right]^{n-k}$$

Then, the p.d.f of first order Y_n transmuted Gumbel type-II distribution is given as

$$f_{Y_{(n)}}(y) = naby^{-a-1}e^{-by^{-a}}\left(1+\lambda-2\lambda e^{-by^{-a}}\right)\left[e^{-by^{-a}}\left(1+\lambda-\lambda e^{-by^{-a}}\right)\right]^{n-1}$$

8. MAXIMUM LIKELIHOOD ESTIMATION OF GUMBEL TYPE-II DISTRIBUTION

Let $Y_1, Y_2, ..., Y_n$ denotes random sample of size n from transmuted Gumbel type-II distribution then its likelihood function is given by

$$l = \prod_{i=1}^{n} f(y, a, b, \lambda)$$
$$= \prod_{i=1}^{n} aby_i^{-a-1}e^{-by_i^{-a}}\left(1+\lambda-2\lambda e^{-by_i^{-a}}\right)$$

The log likelihood function is

$$\log l = n\log a + n\log b - (a+1)\sum_{i=1}^{n}\log y_i + \sum_{i=1}^{n}\log\left(1+\lambda-2\lambda e^{-by_i^{-a}}\right) - b\sum_{i=1}^{n} y_i^{-a}$$

Differentiate w.r.t parameters a, b and λ, we have

$$\frac{\partial \log l}{\partial a}=\frac{n}{a}-\sum_{i=1}^{n}\log y_i-2b\lambda\sum_{i=1}^{n}\frac{y_i^{-a}e^{-by_i^{-a}}\log y_i}{1+\lambda-2\lambda e^{-by_i^{-a}}}+b\sum_{i=1}^{n}y_i^{-a}\log y_i$$

$$\frac{\partial \log l}{\partial b}=\frac{n}{b}+2\lambda\sum_{i=1}^{n}\frac{y_i^{-a}e^{-by_i^{-a}}}{1+\lambda-2\lambda e^{-by_i^{-a}}}-\sum_{i=1}^{n}y_i^{-a}$$

$$\frac{\partial \log l}{\partial \lambda}=\sum_{i=1}^{n}\frac{\left(1-2e^{-by_i^{-a}}\right)}{1+\lambda-2\lambda e^{-by_i^{-a}}}.$$

The above equations are non-linear equations which cannot be expressed in compact form and it is difficult to solve these equations explicitly for a,b and λ. Applying the iterative methods such as Newton–Raphson method, secant method, Regula-falsi method etc. The MLE of the parameters denoted as $\hat{\omega}\left(\hat{a},\hat{b},\hat{\lambda}\right)$ of $\omega\left(a,b,\lambda\right)$ can be obtained by using the above methods.

Since the MLE of ω follows asymptotically normal distribution which is given as

$$\sqrt{n}\left(\hat{\omega}-\omega\right)\rightarrow N\left(0,I^{-1}\left(\omega\right)\right)$$

where $I^{-1}\left(\omega\right)$ is the limiting variance-covariance matrix of $\hat{\omega}$ and $I^{-1}\left(\omega\right)$ is a 3×3 Fisher information matrix i.e.

$$I^{-1}\left(\omega\right)=-\frac{1}{n}\begin{bmatrix} E\left(\frac{\partial^2 \log l}{\partial a^2}\right) & E\left(\frac{\partial^2 \log l}{\partial a\partial b}\right) & E\left(\frac{\partial^2 \log l}{\partial a\partial \lambda}\right) \\ E\left(\frac{\partial^2 \log l}{\partial b\partial a}\right) & E\left(\frac{\partial^2 \log l}{\partial b^2}\right) & E\left(\frac{\partial^2 \log l}{\partial b\partial \lambda}\right) \\ E\left(\frac{\partial^2 \log l}{\partial \lambda\partial a}\right) & E\left(\frac{\partial^2 \log l}{\partial \lambda\partial b}\right) & E\left(\frac{\partial^2 \log l}{\partial \lambda^2}\right) \end{bmatrix}$$

where

$$\frac{\partial^2 \log l}{\partial a^2}=\frac{-n}{a^2}-2b\lambda\sum_{i=1}^{n}\frac{\left\{\begin{array}{l}\left(1+\lambda-2\lambda e^{-by_i^{-a}}\right)\left(by_i^{-a}-1\right)y_i^{-a}e^{-by_i^{-a}}\left(\log y_i\right)^2 \\ \qquad -2b\lambda y_i^{-a}\left(\log y_i\right)^2 e^{-2by_i^{-a}}\end{array}\right\}}{\left(1+\lambda-2\lambda e^{-by_i^{-a}}\right)^2}$$
$$-b\sum_{i=1}^{n}y_i^{-a}\left(\log y_i\right)^2$$

$$\frac{\partial^2 \log l}{\partial b^2} = \frac{-n}{b^2} - 2\lambda \sum_{i=1}^{n} \frac{\left(1+\lambda-\lambda e^{-by_i^{-a}}\right) y_i^{-2a} e^{-by_i^{-a}} \log y_i}{\left(1+\lambda-2\lambda e^{-by_i^{-a}}\right)^2}$$

$$\frac{\partial^2 \log l}{\partial \lambda^2} = \sum_{i=1}^{n} \frac{2e^{-by_i^{-a}}\left(1-2e^{-by_i^{-a}}\right)}{\left(1+\lambda-2\lambda e^{-by_i^{-a}}\right)^2}$$

$$\frac{\partial^2 \log l}{\partial a \partial b} = \frac{\partial^2 \log l}{\partial b \partial a} = 2\lambda \sum_{i=1}^{n} \frac{\left\{ \begin{array}{r} \left(1+\lambda-2\lambda e^{-by_i^{-a}}\right)\left(by_i^{-2a} e^{-by^{-a}} - y_i^{-a}\right) \\ -2b\lambda y_i^{-2a} e^{-2by_i^{-a}} \end{array} \right\} \log y_i}{\left(1+\lambda-2\lambda e^{-by_i^{-a}}\right)^2}$$

$$\frac{\partial^2 \log l}{\partial a \partial \lambda} = \frac{\partial^2 \log l}{\partial \lambda \partial a} = -2b \sum_{i=1}^{n} \frac{y_i^{-a} e^{-by_i^{-a}} \log y_i}{\left(1+\lambda-2\lambda e^{-by_i^{-a}}\right)^2}$$

$$\frac{\partial^2 \log l}{\partial b \partial \lambda} = \frac{\partial^2 \log l}{\partial \lambda \partial b} = \sum_{i=1}^{n} \frac{2y_i^{-a} e^{-by_i^{-a}}}{\left(1+\lambda-2\lambda e^{-by_i^{-a}}\right)^2}$$

Hence the approximate $100(1-\xi)\%$ confidence interval for a,b and λ are respectively given by

$$\hat{a} \pm Z_{\frac{\xi}{2}} \sqrt{I_{aa}^{-1}(\hat{\omega})}\ , \hat{b} \pm Z_{\frac{\xi}{2}} \sqrt{I_{bb}^{-1}(\hat{\omega})} \ \text{ and } \ \hat{\lambda} \pm Z_{\frac{\xi}{2}} \sqrt{I_{\lambda\lambda}^{-1}(\hat{\omega})}$$

Where $Z_{\frac{\xi}{2}}$ is the ξ^{th} percentile of the standard normal distribution.

9. DATA ANALYSIS

This section is dedicated to authenticate the effectiveness of the established distribution by taking into account real data sets taken from various scientific disciplines. The established distribution is compared to its base model.

Data Set I: The data collection contains the tensile strength (in GPa) of 69 carbon fibers checked under tension at gauge of 20 millimeters in length, as stated by Bader and Priest (1982). Shanker (2016) previously examined this content, which is presented below.

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958	1.966	1.997
2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179	2.224	2.240	2.253	2.270
2.272	2.274	2.301	2.301	2.359	2.382,	2.382	2.426	2.434	2.435	2.478	2.490

2.511 2.514 2.535 2.554 2.566 2.57 2.586 2.629 2.633 2.642 2.648 2.684 2.697 2.726 2.770 2.773 2.800 2.809 2.818 2.821 2.848 2.88 2.954 3.012 3.067 3.084 3.090 3.096 3.128 3.233 3.433 3.585 3.585.

Data Set II: The dataset reflects the breaking stress of carbon fibers of 50mm length (GPa) and was previously used by Al-Aqtash et al. (2014) to show the suitability of the Gumbell-Weibull distribution. The following is the data set:

0.39 0.85 1.08 1.25 1.47 1.57 1.61 1.61 1.69 1.80 1.84 1.87 1.89 2.03 2.03 2.05 2.12 2.35 2.41 2.43 2.48 2.50 2.53 2.55 2.55 2.56 2.59 2.67 2.73 2.74 2.79 2.81 2.82 2.85 2.87 2.88 2.93 2.95 2.96 2.97 3.09 3.11 3.11 3.15 3.15 3.19 3.22 3.22 3.27 3.28 3.31 3.31 3.33 3.39 3.39 3.56 3.60 3.65 3.68 3.70 3.75 4.20 4.38 4.42 4.70 4.90.

Data Set III: Bjerkedel observed and recorded the survival periods (in days) of 72 guinea pigs infected with virulent turbercle bacilli (1960). The information is as follows:

0.10 0.33 0.44 0.56 0.59 0.59 0.72 0.74 0.92 0.93 0.96 1. 10 1.02 1.05 1.07 1.07 1.08 1.08 1.08 1.09 1.12 1.13 1.15 1.16 1.20 1.21 1.22 1.22 1.24 1.30 1.34 1.36 1.39 1.44 1.46 1.53 1.59 1.6 1.63 1.68 1.71 1.72 1.76 1.83 1.95 1.96 1.97 2.02 2.13 2.15 2.16 2.22 2.30 2.31 2.4 2.45 2.51 2.53 2.54 2.78 2.93 3.27 3.42 3.47 3.61 4.02 4.32 4.58 5.55 2.54 0.77.

To compare the versatility of the explored distribution, we consider the criteria like AIC (Akaike information criterion), CAIC (Consistent Akaike information criterion), BIC (Bayesian information criterion) and HQIC (Hanan-Quinn information criterion). Distribution having lesser AIC, CAIC, BIC and HQIC values is considered better.

$$AIC = 2k - 2\ln l\,;\, CAIC = \frac{2kn}{n-k-1} - 2\ln l\,;\, BIC = k\ln n - 2\ln l$$

and $HQIC = 2k\ln\left(\ln\left(n\right)\right) - 2\ln l$.

Table 1
Descriptive Statistics of Data Set I, II, III

	Min	**Q_1**	**Median**	**Mean**	**Q_3**	**Skew**	**Kurt.**	**Max**
Data Set I	1.312	2.098	2.478	2.451	2.773	-0.028	2.94	3.585
Data Set II	0.39	2.17	2.83	2.76	3.27	-0.131	3.22	4.90
Data Set III	0.1	1.07	1.45	1.75	2.24	1.328	4.91	5.55

Table 2
MLES and their Standard Errors, Criteria for Good of Fit of Data Sets I, II, III

	Distribution	Estimates	Standard Error	$-2\log l$	AIC	CAIC	BIC	HQIC
Data Set I	TGD-II	$\hat{a}=4.588$ $\hat{b}=20.889$ $\hat{\lambda}=$ -0.751	$\hat{a}=0.373$ $\hat{b}=6.178$ $\hat{\lambda}=0.160$	121.21	127.21	127.79	133.92	129.27
	GD-II	$\hat{a}=4.126$ $\hat{b}=23.266$	$\hat{a}=$ 0.338 $\hat{b}=5.711$	127.24	131.24	131.52	135.71	132.61
Data Set II	TGD-II	$\hat{a}=1.857$ $\hat{b}=2.123$ $\hat{\lambda}=$ -0.896	$\hat{a}=0.134$ $\hat{b}=0.317$ $\hat{\lambda}=0.098$	231.78	237.78	238.35	244.35	239.84
	GD-II	$\hat{a}=1.645$ $\hat{b}=3.226$	$\hat{a}=0.122$ $\hat{b}=0.419$	242.38	246.38	246.66	250.76	247.75
Data Set III	TGD-II	$\hat{a}=1.316$ $\hat{b}=0.591$ $\hat{\lambda}=$ -0.910	$\hat{a}=0.093$ $\hat{b}=0.079$ $\hat{\lambda}=0.088$	223.92	229.92	230.50	236.75	231.98
	GD-II	$\hat{a}=1.175$ $\hat{b}=1.047$	$\hat{a}=0.084$ $\hat{b}=0.130$	234.65	238.65	238.93	243.20	240.02

Since it has been observed from Table 2 that the transmuted Gumbel type-II distribution has smaller values for the AIC, CAIC, BIC and HQIC as compared with Gumbel type-II distribution. Accordingly we achieved the conclusion that transmuted Gumbel type-II distribution provides an adequate fit than Gumbel type-II distribution.

The following figures represents estimated densities, distribution plots, PP and QQ plots for transmuted Gumbel type II and Gumbel type II for the data sets I, II and III.

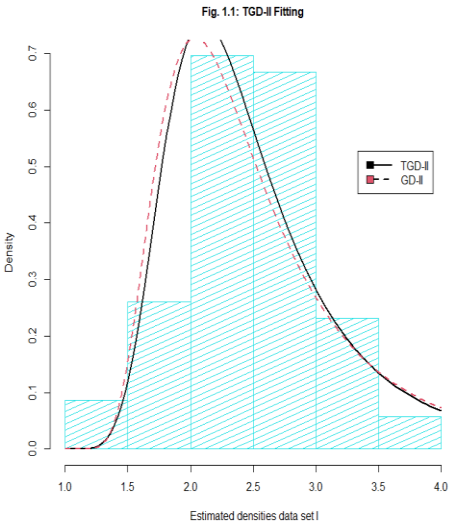


Fig. 1.1: TGD-II Fitting
Density
TGD-II
GD-II
Estimated densities data set I

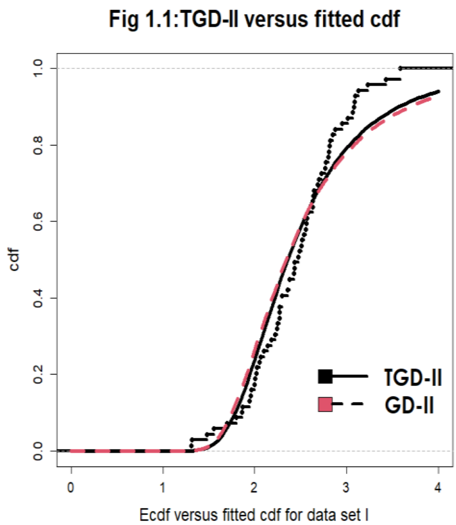
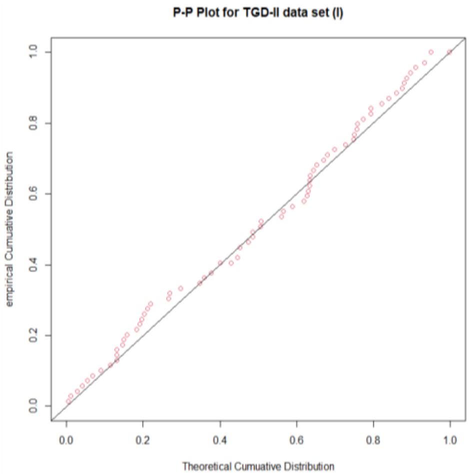
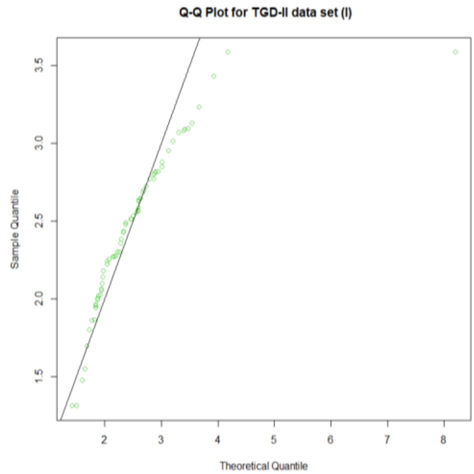


Fig 1.1:TGD-II versus fitted cdf
cdf
TGD-II
GD-II
Ecdf versus fitted cdf for data set I


P-P Plot for TGD-II data set (I)
empirical Cumulative Distribution
Theoretical Cumulative Distribution


Q-Q Plot for TGD-II data set (I)
Sample Quantile
Theoretical Quantile

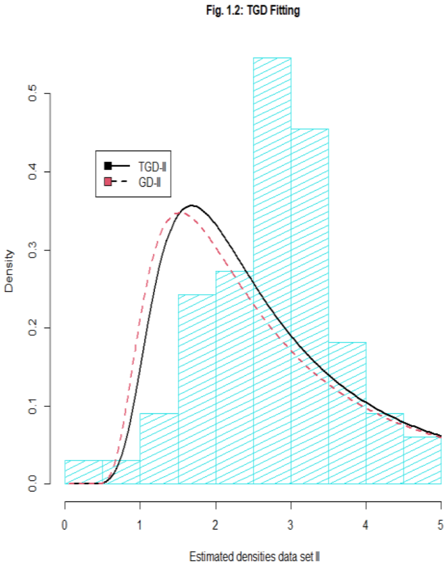


Fig. 1.2: TGD Fitting
Density
TGD-II
GD-II
Estimated densities data set II

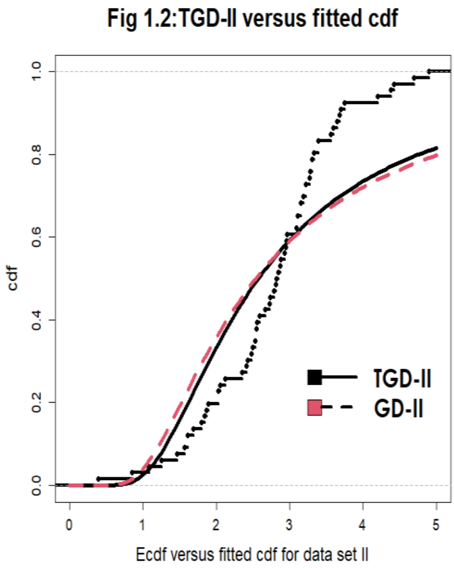
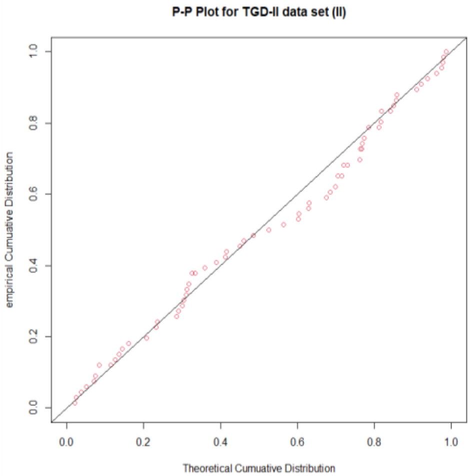
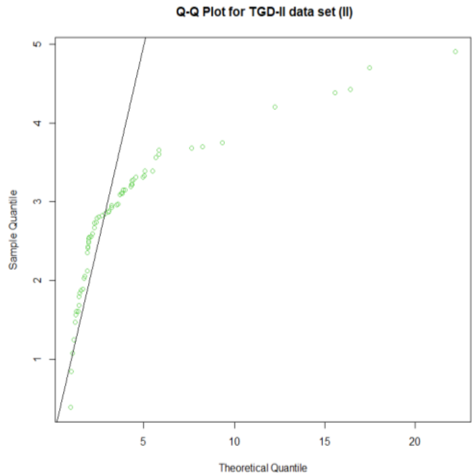


Fig 1.2:TGD-II versus fitted cdf
cdf
TGD-II
GD-II
Ecdf versus fitted cdf for data set II


P-P Plot for TGD-II data set (II)
empirical Cumuative Distribution
Theoretical Cumuative Distribution


Q-Q Plot for TGD-II data set (II)
Sample Quantile
Theoretical Quantile

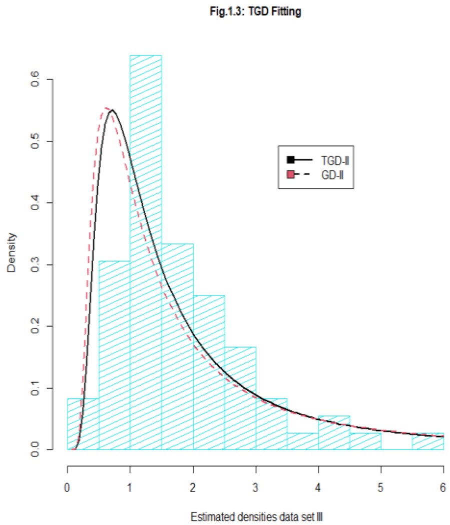


Fig.1.3: TGD Fitting
Density
TGD-II
GD-II
Estimated densities data set III

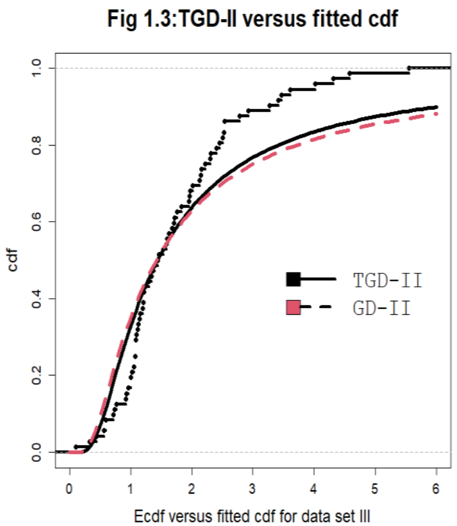
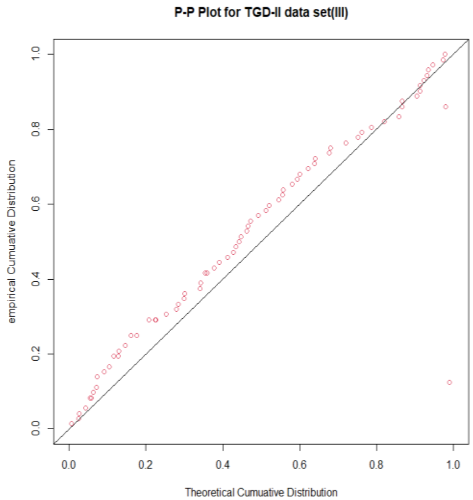
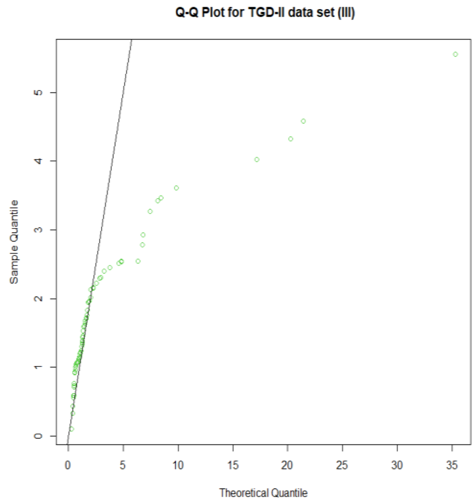


Fig 1.3:TGD-II versus fitted cdf
cdf
TGD-II
GD-II
Ecdf versus fitted cdf for data set III


P-P Plot for TGD-II data set(III)
empirical Cumuative Distribution
Theoretical Cumuative Distribution


Q-Q Plot for TGD-II data set (III)
Sample Quantile
Theoretical Quantile

CONCLUDING REMARKS

In this paper, we introduce a transmuted Gumbel type-II distribution (TGD-II) as an extension of the Gumbel type-II distribution. We provide various statistical properties of the explored distribution. The parameters of the distribution are calculated using the well-known maximum likelihood procedure. The effectiveness of the investigated distribution is evaluated using three real life data sets, and it is shown that the suggested distribution offers a better fitting than the Gumbel type II distribution.

REFERENCES

1. Aryal, G.R. and Tsokos, C.P. (2011). Transmuted Weibull distribution: A generalization of the Weibull probability distribution. *European Journal of Pure and Applied Mathematics*, 4(2), 89-102.
2. Ashour, S.K. and Eltehiwy, M.A. (2013). Transmuted exponentiated Lomax distribution. *Australian Journal of Basic and Applied Sciences*, 7(7), 658-667.
3. Aijaz, A., Afaq, A. and Rajnee, T. (2020). Transmuted inverse Lindley distribution statistical properties and applications. *Science Technology and Development*, IX(IV), 1-9.
4. Afaq, A., Sheikh, P.A. and Ahmad, A. (2015). Transmuted Kumaraswamy distribution. *Mathematical Theory and Modelling*, 5(9), 168-174.
5. Aqtash, A., Lee, C. and Famoye, F. (2014). Gumbel-weibull distribution: Properties and applications. *Journal of Applied Statistical Methods*, 13, 201-225.
6. Bekker, A., Roux, J. and Mostert, P. (2002). A generalization of the compound Rayleigh distribution using a Bayesian methods on cancer survival times. *Communication in statistics-theory and Methods*, 29, 1419-178.
7. Badar, M.G. and Priest, A.M. (1982). Statistical aspects of fiber and bundle strength in hybrid composites. In T. Hayashi, K. Kawata and S. Umekawa (Eds), *Progress in Science and Engineering Composites*. ICCM-IV, Tokoyo, 1129-1136.
8. Merovci, F. (2013). Transmuted rayleigh distribution. *Austrian Journal of Statistics*, 42(1), 21-31.
9. Shanker, R. (2016). Sujatha Distribution and its Applications. *Statistics in Transition-New series*, 17(3), 391-410.
10. Shaw, W.T. and Buckley, I.R. (2009). *The alchemy of probability distributions: Beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map*. arXiv preprint arXiv:0901.0434.