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LAPLACE AND INVERSE LAPLACED TRANSFORM AND GENERALIZED INCOMPLETE HYPERGEOMETRIC FUNCTIONS

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ABSTRACT

The behavior of Laplace Transformation and Inverse Laplace Transform involving generalized incomplete hypergeometric function. By Appling Laplace operator and Inverse Laplace operator containing continuous functions to incomplete hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$ we get the composition of the result.

1. INTRODUCTION AND PRELIMINARIES

Fraction Calculus is rising in the world of mathematics that shows relations of derivatives and integration of fractions order. Fractional calculus is an effectual subject to study the complex problems which have an immense effect in real-world systems. In past, fractional calculus produced so many results that represent properties, extend fractional derivative and integration, and application of special functions. Special functions have improper integration, some of the famous special functions are hypergeometric function and Gamma function. Incomplete hypergeometric function is a part of the systematic interpretation of convergent hypergeometric function.

Euler and Gauss had found several important properties about the hypergeometric series. Riemann created hypergeometric functions from a different point of view a half-century ago, which allowed the simple formulas with a minimum of computations available. Recently Bansal et al. [4] have studied the Fredholm-type integral equation involving the incomplete H-function (IHF) and incomplete $\bar{H}$ function in the kernel. They solved an integral equation associated with the IHF with the aid of the theory of fractional calculus and Mellin transform. Nadir et al. [12] considered the integral transformation like Mellin transforms, Hankel transform, Laplace transforms, and K transforms of incomplete hypergeometric function. Nadir and Khan [11] have applied MSM Differential operators Generalizing Incomplete hypergeometric Functions. Marichev-Saigo-Maeda fractional operator there is an Appell function F_3 as a kernel. Marichev [10] had developed the MSM generalized fractional operator and studied in recent researches, which include the researched by Saigo and Maeda. Perhaps the significance of the MSM fractional integral operators, several scientists used the MSM fractions integral operator to generate many interesting differential-integral formulas, including special functions. Under the specified behavior of MSM fractional integral operators, there is a range of new and well-known solutions involving Erdélyi-Kober, Saigo, Riemann-Liouville and fractional integral operators follow as special cases of their main formulas.

Bansal et al. [1] have established some interesting integrals associated with the product of M-series and incomplete H-functions, which are expressed in terms of incomplete H-functions. They also proved some special cases by specializing the parameters of M-series and incomplete H-functions (for example, Fox's H-Function, Incomplete Fox Wright functions, Fox Wright functions and Incomplete generalized hypergeometric functions).

Bansal et al. [2] have derived the Lambert's law for incomplete H-functions (IHF's) the results of their paper are very useful in derivation of several new and known results having applications in science and engineering.

Kumar et al. [9] have established an equation of internal blood pressure involving incomplete $\bar{H}$ functions. They also give some special cases by specializing the parameters of incomplete $\bar{H}$ functions.

Hilfer [8] give some of the well-known application which has a major role in developing the different scientific knowledge. Choi and Agarwal [7] used different transforms like Beta transform, Laplace transforms to solve the transformation of incomplete hypergeometric function. Srivastava and Agarwal [15] introduced the incomplete Pochhammer symbols that are decomposed and generalize a group of hypergeometric functions that are intensively helpful in semi-infinite integration of various special functions.

Bansal et al. [3] have studied certain interesting and useful properties of incomplete $\aleph$-functions. They found several useful classical integral transforms of these functions. Further, they examined the fractional calculus with the incomplete $\aleph$-functions. They also gave the applications of incomplete $\aleph$-functions in detecting glucose supply in human blood.

Bansal et al. [5] have derived the image formula of the incomplete H-functions by using the Srivastava-Luo-Raina M-transform.

Incomplete gamma functions $\gamma(p;u)$ and $\Gamma[p;u]$ are define,

$$\gamma(p;u)=\int_0^x u^{p-1}e^{-u}du \qquad \left(\Re(p)>0, u\geq 0\right) \tag{1}$$

and

$$\Gamma[p;u]=\int_x^\infty u^{p-1}e^{-u}du \qquad \left(\Re(p)>0, u\geq 0\right) \tag{2}$$

The decomposition formula for gamma is

$$\gamma(p;x)\ +\ \Gamma[p;x]=\ \Gamma(p) \qquad \left(\Re(p)>0, X\geq 0\right) \tag{3}$$

$$\Gamma(p)=\int_0^\infty u^{p-1}e^{-u}du \qquad \left(\Re(p)>0\right) \tag{4}$$

$$(\alpha : X)_n = \frac{\gamma(\alpha + nX)}{\gamma(X)} \tag{5}$$

and

$$[\alpha : X]_n = \frac{\Gamma(\alpha + nX)}{\Gamma(X)} \tag{6}$$

The decomposed formula for Pochhammer symbol is

$$(\alpha : X)_n + [\alpha : X]_n = (\alpha)_n \tag{7}$$

$$(\alpha)_n = \prod_{t=1}^{n} (\alpha + t - 1) \tag{8}$$

$$(\alpha)_n = (\alpha)(\alpha+1)(\alpha+2)...(\alpha+n-1) \tag{9}$$

$$(\alpha)_0 = 1 \qquad\qquad \alpha \neq 0 \tag{10}$$

See Rainville [13], for detail.

Srivastava et al. [14] introduced the following equations

$${}_p\gamma_q\left[\begin{matrix}(\alpha_1; X),(\alpha_2),(\alpha_3),...,(\alpha_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad z\right] = \left\{\sum_{n=0}^{x} \frac{(\alpha_1; X),(\alpha_2),...,(\alpha_p)}{(b_1),...,(b_q)} \frac{z^n}{n!}\right\} \tag{11}$$

and

$${}_p\Gamma_q\left[\begin{matrix}[a_1; X],(a_2),(a_3),...,(a_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad z\right] = \left\{\sum_{n=x}^{\infty} \frac{[a_1; X],(a_2),...,(a_p)}{(b_1),...,(b_q)} \frac{z^n}{n!}\right\} \tag{12}$$

where the parameters $(a_1; X)_n$ and $[a_1; X]_n$ are shown in the interval $[0, X]$ and $[X, \infty)$ respectively. So generalized hypergeometric function ${}_pF_q; (p; q \in N_0)$, is

$$\begin{aligned}&{}_p\gamma_q\left[\begin{matrix}(\alpha_1; X),(\alpha_2),(\alpha_3),...,(\alpha_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad z\right] + {}_p\Gamma_q\left[\begin{matrix}[\alpha_1; X],(\alpha_2),(\alpha_3),...,(\alpha_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad z\right] \\ &= {}_pF_q\left[\begin{matrix}(\alpha_1),(\alpha_2),(\alpha_3),...,(\alpha_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad z\right]\end{aligned} \tag{13}$$

The Gaussian hypergeometric function by Rainville [13] represented by hypergeometric series.

$${}_2F_1(a, b; c; x) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \cdot \frac{x^m}{m!} \tag{14}$$

The Laplace transform of the function $f(x)$ on an interval $[0, \infty)$ by Chauhan et al. [6] is defined as

$$\mathcal{L}\left[f(x);s\right]=\int_0^\infty e^{-sx}f(x)\,dx \tag{15}$$

$$=F(s) \tag{16}$$

where $s\in C$ and $x\geq 0$.

2. THE LAPLACE TRANSFORM WITH INCOMPLETE HYPERGEOMETRIC FUNCTION AT t^c AS LAPLACE FUNCTION

Theorem 1

Let $a,b\,\epsilon\,C$ and a,b independent of z for $\Re(c)\in\left[Z\cup\frac{n}{2}\right]\geq 0$ where $n\in O$ and $\Re(s)>0$ suppose $\left[(a_1),(a_2),...,(a_p)\right]>1$ and $\Re(t)\geq 1$ is satisfied then the incomplete hypergeometric function

$$L\left\{t^c\times{}_p\gamma_q\left[\begin{matrix}(a_1;X),(a_2),(a_3),...,(a_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad zt\right]\right\}$$

$$=\frac{\Gamma(1+c)}{s^{1+c}}\;{}_{p+1}\gamma_q\left[\begin{matrix}(1+c),(a_1;X),(a_2),...,(a_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad \frac{z}{s}\right] \tag{17}$$

and

$$L\left\{t^c\times{}_p\Gamma_q\left[\begin{matrix}(a_1;X),(a_2),(a_3),...,(a_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad zt\right]\right\}$$

$$=\frac{\Gamma(1+c)}{s^{1+c}}\;{}_{p+1}\Gamma_q\left[\begin{matrix}(1+c),(a_1;X),(a_2),...,(a_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad \frac{z}{s}\right] \tag{18}$$

Proof:

In this theorem first use the incomplete interval of hypergeometric function ${}_p\gamma_q$ (11) and applying the Laplace operator and making some amendment with the summation than we have,

$$L\left\{t^c\times{}_p\Gamma_q\left[\begin{matrix}(a_1;X),(a_2),(a_3),...,(a_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad zt\right]\right\}$$

$$=L\left\{t^c\sum_{n=0}^{x}\frac{(a_1;X)_n,(a_2)_n,...,(a_p)_n}{(b_1)_n,...,(b_q)_n}\frac{z^nt^n}{n!}\right\} \tag{19}$$

$$=\left\{\sum_{n=0}^{x}\frac{(a_1;X)_n,(a_2)_n,...,(a_p)_n}{(b_1)_n,...,(b_q)_n}\frac{z^n}{n!}\right\}\qquad L\left(t^{c+n}\right) \tag{20}$$

$$=\left\{\sum_{n=0}^{x}\frac{(a_1;X)_n,(a_2)_n,\ldots,(a_p)_n}{(b_1)_n,\ldots,(b_q)_n}\frac{z^n}{n!}\right\}\quad\frac{(n+c)!}{s^{n+1+c}}$$

$$=\left\{\sum_{n=0}^{x}\frac{(1+c)_n,(a_1;X)_n,(a_2)_n,\ldots,(a_p)_n}{(b_1)_n,\ldots,(b_q)_n}\frac{z^n}{n!s^n}\right\}\frac{(c)\Gamma(c)}{s^{1+c}} \tag{21}$$

$$=\left\{\sum_{n=0}^{x}\frac{(1+c)_n,(a_1;X),(a_2),\ldots,(a_p)}{(b_1),\ldots,(b_q)}\frac{z^n}{n!s^n}\right\}\frac{\Gamma(1+c)}{s^{1+c}} \tag{22}$$

Hence, it proves the RHS of equation (17). For the equation (18) of this theorem using an incomplete interval of hypergeometric function (12) and proceed the same.

3. SPECIAL CASES

In the special cases we considered some discrete value of 'c' in theorem 1.

Case 1: Consider $\Re\ (T) \geq 1$ and when $C = 0$

$$L\left\{1\times{}_p\gamma_q\left[\begin{matrix}(a_1;X),(a_2),(a_3),\ldots,(a_p);\\(b_1),(b_2),\ldots,(b_q);\end{matrix}\quad zt\right]\right\}$$

$$=\frac{1}{s}\ {}_{p+1}\gamma_q\left[\begin{matrix}(1),(a_1;X),(a_2),\ldots,(a_p);\\(b_1),(b_2),\ldots,(b_q);\end{matrix}\quad\frac{z}{s}\right] \tag{23}$$

and

$$L\left\{1\times{}_p\Gamma_q\left[\begin{matrix}(a_1;X),(a_2),(a_3),\ldots,(a_p);\\(b_1),(b_2),\ldots,(b_q);\end{matrix}\quad zt\right]\right\}$$

$$=\frac{1}{s}\ {}_{p+1}\Gamma_q\left[\begin{matrix}(1),(a_1;X),(a_2),\ldots,(a_p);\\(b_1),(b_2),\ldots,(b_q);\end{matrix}\quad\frac{z}{s}\right] \tag{24}$$

Case 2: consider $R(t) \geq 1$ and when $c = 1$

$$L\left\{t^1\times{}_p\gamma_q\left[\begin{matrix}(a_1;X),(a_2),(a_3),\ldots,(a_p);\\(b_1),(b_2),\ldots,(b_q);\end{matrix}\quad zt\right]\right\}$$

$$=\frac{\Gamma(2)}{s^2}\ {}_{p+1}\gamma_q\left[\begin{matrix}(2),(a_1;X),(a_2),\ldots,(a_p);\\(b_1),(b_2),\ldots,(b_q);\end{matrix}\quad\frac{z}{s}\right] \tag{25}$$

and

$$L\left\{t^{1}\times {}_{p}\Gamma_{q}\left[\begin{matrix}(a_1;X),(a_2),(a_3),...,(a_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad zt\right]\right\}$$

$$=\frac{\Gamma(2)}{s^{2}}\,{}_{p+1}\Gamma_{q}\left[\begin{matrix}(2),(a_1;X),(a_2),...,(a_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad \frac{z}{s}\right] \tag{26}$$

CASE 3: Consider $\Re\,(t)\geq 1$ and when $c=\frac{1}{2}$

$$L\left\{t^{\frac{1}{2}}\times {}_{p}\gamma_{q}\left[\begin{matrix}(a_1;X),(a_2),(a_3),...,(a_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad zt\right]\right\}$$

$$=\frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}}\times {}_{p+1}\gamma_{q}\left[\begin{matrix}\left(\frac{3}{2}\right),(a_1;X),(a_2),...,(a_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad \frac{z}{s}\right] \tag{27}$$

and

$$L\left\{t^{\frac{1}{2}}\times {}_{p}\Gamma_{q}\left[\begin{matrix}(a_1;X),(a_2),(a_3),...,(a_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad zt\right]\right\}$$

$$=\frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}}\times {}_{p+1}\Gamma_{q}\left[\begin{matrix}\left(\frac{3}{2}\right),(a_1;X),(a_2),...,(a_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad \frac{z}{s}\right] \tag{28}$$

4. THE LAPLACE TRANSFORM WITH INCOMPLETE HYPERGEOMETRIC FUNCTION AT e^{ct} AS LAPLACE FUNCTION

Theorem 2:

Let $a,b\,\epsilon\,C$ and a,b independent of z for $R(c)\geq 1$ and $R(s)>c$ suppose $\left[(a_1),(a_2),...,(a_p)\right]>1$ and $R(t)\geq 1$ is satisfied then the incomplete hypergeometric function

$$L\left\{e^{ct}\times {}_{p}\gamma_{q}\left[\begin{matrix}(a_1;X),(a_2),(a_3),...,(a_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad zt\right]\right\}$$

$$=\frac{1}{s-c}\times {}_{p+1}\gamma_{q}\left[\begin{matrix}(1),(a_1;X),(a_2),...,(a_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad \frac{z}{s-c}\right] \tag{29}$$

and

$$L\left\{e^{ct}\times {}_{p}\Gamma_{q}\left[\begin{matrix}(a_1;X),(a_2),(a_3),...,(a_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad zt\right]\right\}$$

$$=\frac{1}{s-c}\times {}_{p+1}\Gamma_{q}\left[\begin{matrix}(1),(a_1;X),(a_2),...,(a_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad \frac{z}{s-c}\right] \tag{30}$$

Proof:

By using incomplete hypergeometric function ${}_p\gamma_q$ (11) and applying the Laplace operator then use the translation theorem, we have

$$L\left\{e^{ct}\times {}_p\gamma_q\left[\begin{matrix}(a_1;X),(a_2),(a_3),...,(a_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad zt\right]\right\}=$$

$$L\left\{e^{ct}\sum_{n=0}^{x}\frac{(a_1;X)_n,(a_2)_n,...,(a_p)_n}{(b_1),(b_2),...,(b_q)}\frac{z^n t^n}{n!}\right\} \tag{31}$$

$$=\left\{\sum_{n=0}^{x}\frac{(a_1;X)_n,(a_2)_n,...,(a_p)_n}{(b_1),(b_2),...,(b_q)}\frac{z^n}{n!}\right\}L\left(e^{ct}t^n\right) \tag{32}$$

$$=\left\{\sum_{n=0}^{x}\frac{(a_1;X)_n,(a_2)_n,...,(a_p)_n}{(b_1),(b_2),...,(b_q)}\frac{z^n}{n!}\right\}L\left\{t^n\right\}_{s\to s-c} \tag{33}$$

$$=\left\{\sum_{n=0}^{x}\frac{(1)_n(a_1;X)_n,(a_2)_n,...,(a_p)_n}{(b_1),(b_2),...,(b_q)}\frac{z^n}{n!(s-c)^n}\right\}\frac{1}{s-c}$$

Hence, it proves the RHS of equation (29). For the equation (30) of this theorem using an incomplete interval of hypergeometric function (12) and proceed the same.

5. SPECIAL CASES

Case 1: Consider $\Re(t)\geq 1$ and when $c=1$

$$L\left\{e^{t}\times {}_p\gamma_q\left[\begin{matrix}(a_1;X),(a_2),(a_3),...,(a_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad zt\right]\right\}$$

$$=\frac{1}{s-1}\times {}_{p+1}\gamma_q\left[\begin{matrix}(1),(a_1;X),(a_2),...,(a_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad \frac{z}{s-1}\right] \tag{34}$$

and

$$L\left\{e^{t}\times {}_p\Gamma_q\left[\begin{matrix}(a_1;X),(a_2),(a_3),...,(a_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad zt\right]\right\}$$

$$=\frac{1}{s-1}\times {}_{p+1}\Gamma_q\left[\begin{matrix}(1),(a_1;X),(a_2),...,(a_p); \\ (b_1),(b_2),...,(b_q);\end{matrix}\quad \frac{z}{s-1}\right] \tag{35}$$

Case 2: Consider $\Re\,(t) \geq 1$ and when $c = 2$

$$L\left\{e^{2t} \times {}_p\gamma_q\left[\begin{matrix}(a_1;X),(a_2),(a_3),\ldots,(a_p); \\ (b_1),(b_2),\ldots,(b_q);\end{matrix}\; zt\right]\right\}$$

$$= \frac{1}{s-2} \times {}_{p+1}\gamma_q\left[\begin{matrix}(1),(a_1;X),(a_2),\ldots,(a_p); \\ (b_1),(b_2),\ldots,(b_q);\end{matrix}\; \frac{z}{s-2}\right] \tag{36}$$

and

$$L\left\{e^{2t} \times {}_p\Gamma_q\left[\begin{matrix}(a_1;X),(a_2),(a_3),\ldots,(a_p); \\ (b_1),(b_2),\ldots,(b_q);\end{matrix}\; zt\right]\right\}$$

$$= \frac{1}{s-2} \times {}_{p+1}\Gamma_q\left[\begin{matrix}(1),(a_1;X),(a_2),\ldots,(a_p); \\ (b_1),(b_2),\ldots,(b_q);\end{matrix}\; \frac{z}{s-2}\right] \tag{37}$$

6. THE LAPLACE TRANSFORM WITH INCOMPLETE HYPERGEOMETRIC FUNCTION AT e^{-ct} AS LAPLACE FUNCTION

Theorem 3:

Let $a, b \epsilon C$ and a, b independent of z for $\Re\,(c) \geq 0$ and suppose $\left[(a_1),(a_2),\ldots,(a_p)\right] > 1$ and $\Re\ (t) \geq 1$ is satisfied then the incomplete hypergeometric function is given by

$$L\left\{e^{-ct} \times {}_p\gamma_q\left[\begin{matrix}(a_1;X),(a_2),(a_3),\ldots,(a_p); \\ (b_1),(b_2),\ldots,(b_q);\end{matrix}\; zt\right]\right\}$$

$$= \frac{1}{s+c} \times {}_{p+1}\gamma_q\left[\begin{matrix}(1),(a_1;X),(a_2),\ldots,(a_p); \\ (b_1),(b_2),\ldots,(b_q);\end{matrix}\; \frac{z}{s+c}\right] \tag{38}$$

and

$$L\left\{e^{-ct} \times {}_p\Gamma_q\left[\begin{matrix}(a_1;X),(a_2),(a_3),\ldots,(a_p); \\ (b_1),(b_2),\ldots,(b_q);\end{matrix}\; zt\right]\right\}$$

$$= \frac{1}{s+c} \times {}_{p+1}\Gamma_q\left[\begin{matrix}(1),(a_1;X),(a_2),\ldots,(a_p); \\ (b_1),(b_2),\ldots,(b_q);\end{matrix}\; \frac{z}{s+c}\right] \tag{39}$$

Proof:

By using incomplete hypergeometric function ${}_p\gamma_q$ (11) and applying the Laplace operator then use the translation theorem, we have

$$L\left\{e^{-ct}\times{}_p\gamma_q\left[\begin{matrix}(a_1;X),(a_2),(a_3),\dots,(a_p); \\ (b_1),(b_2),\dots,(b_q);\end{matrix}\quad zt\right]\right\}$$

$$=L\left\{e^{-ct}\sum_{n=0}^{x}\frac{(a_1;X)_n,(a_2)_n,\dots,(a_p)_n}{(b_1),(b_2),\dots,(b_q)}\frac{z^nt^n}{n!}\right\} \tag{40}$$

$$=\left\{\sum_{n=0}^{x}\frac{(a_1;X)_n,(a_2)_n,\dots,(a_p)_n}{(b_1),(b_2),\dots,(b_q)}\frac{z^nt^n}{n!}\right\}L\left(e^{-ct}t^n\right) \tag{41}$$

$$=\left\{\sum_{n=0}^{x}\frac{(a_1;X)_n,(a_2)_n,\dots,(a_p)_n}{(b_1),(b_2),\dots,(b_q)}\frac{z^nt^n}{n!}\right\}L\left\{t^n\right\}_{s\to s-(-c)} \tag{42}$$

$$=\left\{\sum_{n=0}^{x}\frac{(1)_n(a_1;X)_n,(a_2)_n,\dots,(a_p)_n}{(b_1),(b_2),\dots,(b_q)}\frac{z^n}{n!(s+c)^n}\right\}\frac{1}{s+c}.$$

Hence, it proves the RHS of equation (38). For the equation (39) of this theorem using an incomplete interval of hypergeometric function (12) and proceed the same.

7. SPECIAL CASE

Case 1: Consider $\boldsymbol{R(t)\geq 1}$ and When $\boldsymbol{c=1}$

$$L\left\{e^{-t}\times{}_p\gamma_q\left[\begin{matrix}(a_1;X),(a_2),(a_3),\dots,(a_p); \\ (b_1),(b_2),\dots,(b_q);\end{matrix}\quad zt\right]\right\}$$

$$=\frac{1}{s+1}\times{}_{p+1}\gamma_q\left[\begin{matrix}(1),(a_1;X),(a_2),\dots,(a_p); \\ (b_1),(b_2),\dots,(b_q);\end{matrix}\quad \frac{z}{s+1}\right] \tag{43}$$

and

$$L\left\{e^{-t}\times{}_p\Gamma_q\left[\begin{matrix}(a_1;X),(a_2),(a_3),\dots,(a_p); \\ (b_1),(b_2),\dots,(b_q);\end{matrix}\quad zt\right]\right\}$$

$$=\frac{1}{s+1}\times{}_{p+1}\Gamma_q\left[\begin{matrix}(1),(a_1;X),(a_2),\dots,(a_p); \\ (b_1),(b_2),\dots,(b_q);\end{matrix}\quad \frac{z}{s+1}\right] \tag{44}$$

8. THE INVERSE LAPLACE TRANSFORM WITH INCOMPLETE HYPERGEOMETRIC FUNCTION

Theorem 4:

Let $a,b\in C$ and a,b independent of z for $\Re(c)\geq 0$ suppose $\left[(a_1),(a_2),\dots,(a_p)\right]>1$ and $(t\in\Re)$ is satisfied then the incomplete hypergeometric function is given by

$$L^{-1}\left\{\frac{1}{s}\times {}_p\gamma_{q+1}\left[\begin{matrix}(a_1;X),(a_2),(a_3),...,(a_p); \\ (1+s),(b_1),(b_2),...,(b_q);\end{matrix}\; z\right]\right\}$$

$$= {}_p\gamma_{q+1}\left[\begin{matrix}(a_1;X),(a_2),...,(a_p); \\ (1)(b_1),(b_2),...,(b_q);\end{matrix}\; z(1-e^{-t})\right] \tag{45}$$

and

$$L^{-1}\left\{\frac{1}{s}\times {}_p\Gamma_{q+1}\left[\begin{matrix}(a_1;X),(a_2),(a_3),...,(a_p); \\ (1+s),(b_1),(b_2),...,(b_q);\end{matrix}\; z\right]\right\}$$

$$= {}_p\Gamma_{q+1}\left[\begin{matrix}(a_1;X),(a_2),...,(a_p); \\ (1)(b_1),(b_2),...,(b_q);\end{matrix}\; z(1-e^{-t})\right] \tag{46}$$

Proof:

By using the incomplete hypergeometric function (11) and Laplace inverse and by applying partial fraction we required the main result of the theorem.

$$L^{-1}\left\{\frac{1}{s}\times {}_p\gamma_{q+1}\left[\begin{matrix}(a_1;X),(a_2),(a_3),...,(a_p); \\ (1+s),(b_1),(b_2),...,(b_q);\end{matrix}\; z\right]\right\}$$

$$= L^{-1}\left\{\frac{1}{s}\sum_{n=0}^{x}\frac{(a_1;X)_n,(a_2)_n,...,(a_p)_n}{(1+s)_n,(b_1)_n,...,(b_q)_n}\frac{z^n}{n!}\right\} \tag{47}$$

$$= \left\{\sum_{n=0}^{x}\frac{(a_1;X)_n,(a_2)_n,...,(a_p)_n}{(b_1)_n,...,(b_q)_n}\frac{z^n}{n!}\right\}L^{-1}\left\{\frac{1}{s(s+1)_n}\right\} \tag{48}$$

Take

$$L^{-1}\left\{\frac{1}{s(s+1)_n}\right\} \tag{49}$$

We can write after processing with partial fraction

$$\left\{\frac{1}{s(s+1)_n}\right\} = \frac{1}{(1)_n}\frac{1}{s} - \frac{n}{(1)_n}\frac{1}{s+1} + \frac{n(n-1)}{(1)_n(2!)}\frac{1}{s+2}$$

$$-\frac{n(n-1)(n-2)}{(1)_n(3!)}\frac{1}{s+3} + ... + \frac{(-1)^{-n}}{(1)_n}\frac{1}{s+n} \tag{50}$$

$$= \frac{1}{(1)_n}L^{-1}\left[\frac{1}{s} - n\frac{1}{s+1} + \frac{n(n-1)}{(2!)}\frac{1}{s+2} - \frac{n(n-1)(n-2)}{(3!)}\frac{1}{s+3} + ... + (-1)^{-1}\frac{1}{s+n}\right] \tag{51}$$

$$= \frac{1}{(1)_n}\left\{(1-e^{-t})^n\right\} \tag{52}$$

The equation (1.14) becomes

$$= \left\{\sum_{n=0}^{x} \frac{(a_1;X)_n,(a_2)_n,...,(a_p)_n}{(b_1)_n,...,(b_q)_n}\frac{z^n}{n!}\right\} L^{-1}\left\{\frac{1}{s(s+1)_n}\right\} \tag{53}$$

$$= \left\{\sum_{n=0}^{x} \frac{(a_1;X)_n,(a_2)_n,...,(a_p)_n}{(b_1)_n,...,(b_q)_n}\frac{z^n}{n!}\frac{1}{(1)_n}\left\{(1-e^{-t})^n\right\}\right\} \tag{54}$$

Hence, it proves the RHS of equation (45). For the equation (46) of this theorem using an incomplete interval of hypergeometric function (12) and proceed the same.

Remarks:

Thus we can see the composition of the incomplete hypergeometric function due to the Laplace transforms in the above theorems and cases.

9. CONCLUSION

We compromised the generalized incomplete hypergeometric functions with the help of Laplace transformation and inverse Laplace transformation. These transformation helps incomplete hypergeometric function to become more generalized and composite. With help of changing some variables we conclude some cases. These result have effective in physics and biological science.

REFERENCES

1. Bansal, M.K., Kumar, D., Khan, I., Singh, J. and Nisar, K.S. (2019). Certain unified integrals associated with product of M-series and incomplete H-functions. *Mathematics*, 7(12), 1191.
2. Bansal, M.K., Kumar, D., Nisar, K.S. and Singh, J. (2019). Application of incomplete H–functions in determination of Lambert's law. *Journal of Interdisciplinary Mathematics*, 22(7), 1205-1212.
3. Bansal, M.K., Kumar, D., Nisar, K.S. and Singh, J. (2020). Certain fractional calculus and integral transform results of incomplete ℵ-functions with applications. *Mathematical Methods in the Applied Sciences*, 43(8), 5602-5614.
4. Bansal, M.K., Kumar, D., Singh, J. and Nisar, K.S. (2020). On the solutions of a class of integral equations pertaining to incomplete H-function and incomplete $\bar{H}$-function. *Mathematics*, 8(5), 819.
5. Bansal, M.K., Kumar, D., Singh, J., Tassaddiq, A. and Nisar, K.S. (2020). Some new results for the Srivastava–Luo–Raina M-transform pertaining to the incomplete H-functions. *AIMS Math*., 5(1), 717-722.
6. Chauhan, R., Kumar, N. and Aggarwal, S. (2019). Dualities between Laplace and Some Useful Integration. *International Journal of Innovative Technology and Exploring Engineering (IJITEE)*, 8(12), 2278-3075.

7. Choi, J. and Agarwal, P. (2015). Certain Integral transforms for the incomplete functions. *Applied Mathematics and Information Sciences*, 9, 2161-2167.
8. Hifler, R. (2000). Application of Fractional Calculus in Physics. *World Scientific*. Singapore.
9. Kumar, D., Bansal, M.K., Nisar, K.S. and Singh, J. (2019). Mathematical modelling of internal blood pressure involving incomplete H-functions. *Journal of Interdisciplinary Mathematics*, 22(7), 1213-1221.
10. Marichev, O.I. (1974). Volterra equation of Mellin convolution type with a Horn function in the kernel. *Izv. Akad. Nauk BSSR, Ser. Fiz.-Mat. Nauk*, 1, 128-129.
11. Nadir, A. and Khan, A. (2018). Marichev-Saigo-Maeda differential operator and Generalized Incomplete Hypergeometric Functions. *Punjab University Journal of Mathematics*, 50(3), 123-129.
12. Nadir, A., Khan, A. and Kalim, M. (2014). Integral transform of the generalized Mittag-Leffler function. *Applied Mathematical Sciences*, 8, 5145-5154.
13. Rainville, E.D. (1971). Special Functions. *The Macmillan Company*, New York.
14. Srivastava, H.M., Choudhry, M.A. and Agarwal, R.P. (2012). The incomplete pochhammer symbol and their applications to hypergeometric and related functions. Integral transforms. *Spec. Funct.*, 23, 659-683.
15. Srivastava, H.M. and Agarwal, P. (2013). Certain fractional Integration operator and the generalized incomplete hypergeometric functions. *Application and Applied Mathematics*, 8, 333-345.