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CHI-SQUARED GOODNESS-OF-FIT TEST FOR BOUNDED EXPONENTIATED WEIBULL DISTRIBUTION

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ABSTRACT

In this paper, we attempt to supplement the distribution theory literature by incorporating a new bounded distribution, called the bounded exponentiated Weibull (BEW) distribution in the (0,1) intervals by transformation method. The proposed distribution exhibits decreasing, increasing, bathtub and right-skewed unimodal density while the hazard rate can have decreasing and bathtub shaped. Although our main focus is on the construction of chi-squared goodness-of-fit tests for the BEW distribution for right censored data based on Nikulin-Rao-Robson (NRR) statistic and its modification, in addition, we derive some basic statistical properties of the proposed BEW distribution. The test statistic used is the modified chi-squared statistic Y^2, developed by Bagdonavicius and Nikulin (2011) for some parametric models when data are censored. The performances of the proposed test is investigated through an extensive simulation study. An application to a real data set is also provided. The main purpose of this work is the construction of chi-squared goodness of-fit tests for the transmuted generalized linear exponential distribution with unknown parameters and right censoring. The criterion test used is the modified chi-squared statistic Y^2, developed by Bagdonavicius and Nikulin (2011) for some parametric models when data are censored. The performances of the proposed test is investigated through an extensive simulation study. An application to a real data set is also provided.

KEYWORDS

Right censored data, chi-squared test, maximum likelihood estimation.

1. INTRODUCTION

One of the important aspects of statistical studies is to obtain information about the form of the population from which the sample is drawn. Towards this end, goodness of fit (GOF) tests are employed to determine how well the observed sample data "fits" some proposed model. And for the purpose of validating the chosen model, graphical tests, chi-squared tests, Kolmogorov-Smirnov test, Anderson-Darling test may be employed. The principle behind these tests is to measure the distance between the observed values and the expected theoretical values. When this distance is found to be greater than the critical value, we may conclude that the chosen model should be rejected at the specified significance level. Further, when the parameters are unknown, the standard tables for these tests are not valid. In case of censored samples, the complete sample procedures of goodness of fit tests are inappropriate Badr (2019).

The literature is rich and varied on GOF when the model is well specified. In this regard, detailed study has been carried out by the following authors: Stephens (1970, 1974), Durbin (1975) and Green and Hegazy (1976) studied on GOF testing for normal and exponential distribution with unknown parameters. Chandra et al. (1981) studied the Kolmogorov statistics for tests of fit for the extreme-value and Weibull distributions. Murthy et al. (2004) and Abdelfattah (2008) discussed GOF tests for the Weibull distribution. Yen and Moore (1988) discussed GOF for Laplace distribution. The exponential distribution was discussed by Balakrishnan and Basu (1995). Hassan (2005) studied GOF for the generalized exponential distribution. Abd-Elfattah (2011) discussed GOF for generalized Rayleigh distribution. Wang (2008) discussed the GOF test for the exponential distribution based on progressively Type II censored sample. Al-Omari and Zamanzade (2016) discussed different GOF tests for Rayleigh distribution based on ranked set sampling. Aidi and Seddik-Ameur (2016) studied Chi-square tests for generalized exponential AFT distributions with censored data. Mahdizadeh and Zamanzade (2017, 2019) introduced goodness of fit tests for Cauchy distribution and apply these test on financial data. Zamanzade and Mahdizadeh (2017) studied the GOF test for Rayleigh distribution based on Phi-divergence. However, if data are censored and the parameters are unknown, which often happens in reliability and medical studies, the problem remains open. The adequacy of many newly introduced distributions have not yet been investigated.

Over the last five decades or so several studies were carried out to define new families of Weibull distribution, such as exponentiated Weibull[Mudholkar and Srivastava (1993), Mudholkar et al. (1995)], extended Weibull [Marshall and Olkin (1997), Zhang and Xie (2007)], modified Weibull [Jiang et al. (2008), Lai et al. (2003)], odd Weibull [Cooray (2006)], Weibull-X family (Alzaatreh et al. (2013)), Weibull-G family (Bourguignon et al. (2014)), extended Weibull-G family (Korkmaz (2019)) and so on. The aforesaid distributions are actually extension of Weibull distribution and are generally derived by adding some additional parameters to the original probability distribution. Besides, one common aspect of these distributions is that they are based on the support over positive part of the real line. However, probability distributions with support on finite range are also of importance in many studies. But there is a scarcity of distributions with finite support. Moreover, many life test experiments quite often lead to data which may lie in some finite range, like data on fractions, percentages, per capita income growth, fuel efficiency of vehicles, height and weight of individuals, survival times from a deadly disease etc. are likely to lie in some bounded positive intervals (see Kumaraswamy (1980), Gomez-Deniz et al. (2013), Mazucheli et al. (2018a, 2018b, 2018c), Mazucheli et al. (2019)).

In this paper, first we derive a new bounded distribution from the exponentiated Weibull distribution by transformation of the type $x = T/(1+T)$, where T has the exponentiated Weibull distribution. We obtain a new distribution with support on (0,1), which we call it bounded exponentiated Weibull (BEW) distribution. This distribution is capable of modelling decreasing and bathtub shaped hazard rate. Second, we obtain maximum likelihood estimators for unknown parameters of the model based on right-censored data. Next, we construct chi-squared tests for the BEW model when data are right censored. We use modified chi-squared statistic developed by Bagdonavicius and Nikulin (2011) for some parametric accelerated failure times models. This technique has been used to validate some models like, generalized Birnbaum Saunders distribution (Nikulin and Tran, 2013) and competing risk model (Chouia and Seddik-Ameur, 2017).

In Section 2, model description and some basic properties are provided. In Section 3, characterization of BEW is presented. In Section 4, maximum likelihood estimates based on complete and right censored data are discussed. In Section 5, test statistic for right censored data is proposed for the model. In Section 6, construction of a modified chi-square goodness-of-fit test for BEW distribution when data are right censored is proposed. In Section 7, in order to confirm the practibility of the proposed goodness-of-fit test, and the usefulness of this model, simulation study is carried out and one real data set is analyzed. At the end of this paper, conclusions are given in Section 8.

2. MODEL DESCRIPTION

Let T be a random variable follows exponentiated Weibull distribution, then $x = T/(1+T)$ follows BEW distribution.

The cumulative distribution function of exponentiated Weibull distribution is given by

$$F(t) = \left(1 - e^{-t^{\beta}}\right)^{\alpha}, t > 0, \alpha, \beta > 0$$

Thus the BEW distribution with two parameters α and β has the density function

$$f(x,\alpha,\beta) = \frac{\alpha\beta}{(1-x)^2}\left(\frac{x}{1-x}\right)^{\beta-1} e^{-\left(\frac{x}{1-x}\right)^{\beta}}\left[1 - e^{-\left(\frac{x}{1-x}\right)^{\beta}}\right]^{\alpha-1}, \quad 0 < x < 1, \alpha, \beta > 0 \tag{2.1}$$

Special cases of BEW distribution are: for $\beta = 1$ it reduces to Bounded exponentiated exponential distribution, for $\beta = 2$ it reduces to Bounded exponentiated Rayleigh distribution, for $\alpha = 1$ it reduces to Bounded Weibull distribution (Mazucheli et al., 2019), for $\alpha = 1$ and $\beta = 1$ it reduces to Bounded exponential distribution and for $\alpha = 1$ and $\beta = 2$ it reduces to Bounded Rayleigh distribution.

The cumulative distribution function, reliability and hazard rate functions are, respectively given by

$$F(x,\alpha,\beta) = \left[1 - e^{-\left(\frac{x}{1-x}\right)^{\beta}}\right]^{\alpha} \tag{2.2}$$

$$S(x,\alpha,\beta) = 1 - \left[1 - e^{-\left(\frac{x}{1-x}\right)^{\beta}}\right]^{\alpha} \tag{2.3}$$

$$h(x,\alpha,\beta) = \frac{\alpha\beta e^{-\left(\frac{x}{1-x}\right)^{\beta}}\left[1 - e^{-\left(\frac{x}{1-x}\right)^{\beta}}\right]^{\alpha-1}}{(1-x)^2\left(1 - \left[1 - e^{-\left(\frac{x}{1-x}\right)^{\beta}}\right]^{\alpha}\right)}\left(\frac{x}{1-x}\right)^{\beta-1} \tag{2.4}$$

and the cumulative hazard rate function is

$$H(x,\alpha,\beta) = -lnS(t;\alpha,\beta) = -ln\left(1 - \left[1 - e^{-\left(\frac{x}{1-x}\right)^{\beta}}\right]^{\alpha}\right) \tag{2.5}$$

2.1 Reversed Hazard Function

The reversed hazard function is the ratio of the probability density function and the distribution function. It uniquely defines the distribution function. It plays a vital role in analyzing left censored data and can be obtained using the following relationship:

$$r(x,\alpha,\beta) = \frac{f(x,\alpha,\beta)}{F(x,\alpha,\beta)} \tag{2.6}$$

So, the reversed hazard function of the BEW distribution becomes:

$$r(x,\alpha,\beta) = \frac{\alpha\beta\left(\frac{x}{1-x}\right)^{\beta-1} e^{-\left(\frac{x}{1-x}\right)^{\beta}}}{(1-x)^2\left(1 - e^{-\left(\frac{x}{1-x}\right)^{\beta}}\right)}. \tag{2.7}$$

2.2 The Odd Function

The odd function is the ratio of distribution function and survival function and it can be obtained using the following relationship:

$$O(t,\alpha,\beta) = \frac{F(x,\alpha,\beta)}{S(x,\alpha,\beta)} \tag{2.8}$$

So, the odd function for the BEW distribution becomes:

$$O(x,\alpha,\beta) = \frac{\left[1 - e^{-\left(\frac{x}{1-x}\right)^{\beta}}\right]^{\alpha}}{1 - \left[1 - e^{-\left(\frac{x}{1-x}\right)^{\beta}}\right]^{\alpha}}. \tag{2.9}$$

2.3 Quantile Function

The quantile function of a scalar random variable X is the inverse of its distribution function. The quantile function provides a complete description of the statistical properties of the random variable and can be defined as

$$Q(x,\theta) = z = F^{-1}(x,\theta) \tag{2.10}$$

Thus the quantile function for the BEW distribution is obtained as follows:

$$x = \frac{\left[-ln\left(1 - z^{1/\alpha}\right)\right]^{\frac{1}{\beta}}}{1 + \left[-ln\left(1 - z^{1/\alpha}\right)\right]^{\frac{1}{\beta}}} \tag{2.11}$$

where z is uniformly distributed [0,1].

The probability density function (pdf) and hazard rate function (hrf) of BEW distribution have been plotted in Figures 1 and 2, respectively, for different values of α and β.

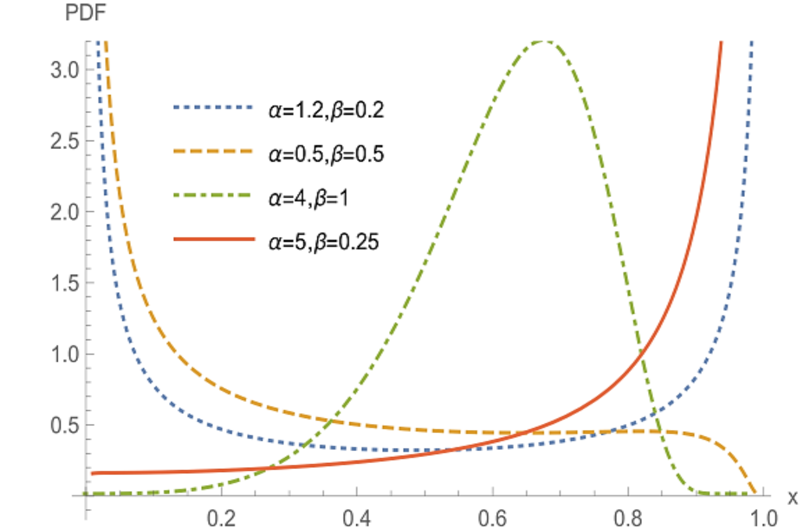

Figure 1: Probability Density Plot

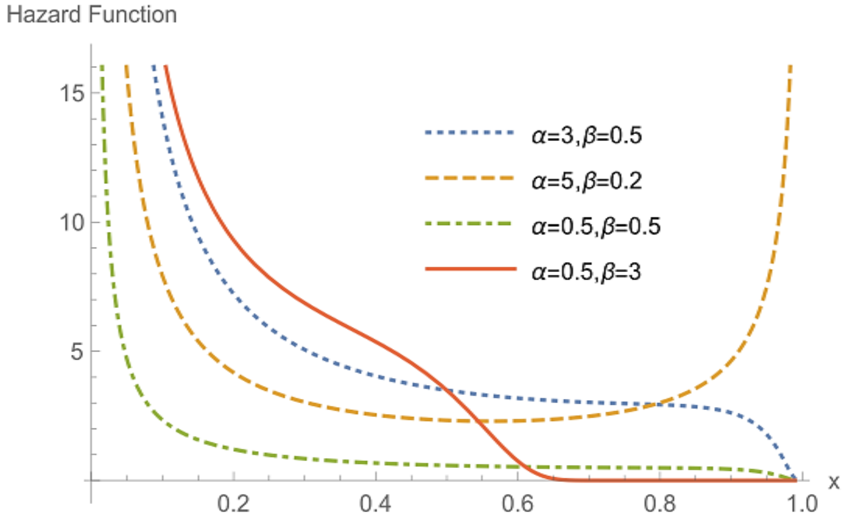

Figure 2: Hazard Rate Plot

2.4 Moments

To understand the probability distribution, one should take into account that most of its characteristics are based on moments. So it is worthwhile to derive rth moments of the distribution and which can be used to obtain first order moment, second order moment etc. by replacing r values. The rth raw moment is defined as

$$E\left(x^{\{r\}}\right) = \int_0^1 x^r\, f(x)dx$$

$$E(x^r) = \alpha\beta \int_0^1 \frac{x^r}{(1-x)^2}\left(\frac{x}{1-x}\right)^{\beta-1} e^{-\left(\frac{x}{1-x}\right)^{\beta}} \left[1 - e^{-\left(\frac{x}{1-x}\right)^{\beta}}\right]^{\alpha-1} dx$$

on simplification, we get

$$E(x^r) = \frac{\alpha}{\alpha - i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j \binom{-r}{i}\binom{\alpha-1}{j} G\left[\frac{1}{\beta}(r+i)+1\right] \tag{2.12}$$

where $G[.]$ is the complete Gamma function.

2.5 Central Moments

Central moments gives us direct access to many characteristics of the distribution such as variance, skewness and kurtosis. The rth central moments is defined as

$$\mu_r = \sum_{k=0}^{r}\binom{r}{k}(-1)^k\left(E(x^r)\right)^k E(x^{r-k})$$

On simplification, we get

$$\mu_r = \sum_{k=0}^{r}\left\{\begin{array}{c}\binom{r}{k}(-1)^k\left(\frac{\alpha}{\alpha-i}\right)^{k+1}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-r}{i}\binom{\alpha-1}{j}G\left[\frac{1}{\beta}(r+i)+1\right]\right)^k \\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-r-k}{i}\binom{\alpha-1}{j}G\left[\frac{1}{\beta}(r-k+i)+1\right]\end{array}\right. \tag{2.13}$$

2.6 Skewness

Skewness is the degree of distribution from the symmetrical bell curve or the normal distribution. It measures the lack of symmetry in data distribution. Thus, Skewness is defined as

$$\beta_1 = \frac{(\mu_3)^2}{(\mu_2)^{3/2}}$$

Thus β_1 is given by

$$\beta_1 = \frac{\left(\sum_{k=0}^{3}\left\{\begin{array}{c}\binom{3}{k}(-1)^k\left(\frac{\alpha}{\alpha-i}\right)^{k+1}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-3}{i}\binom{\alpha-1}{j}G\left[\frac{1}{\beta}(3+i)+1\right]\right)^k\\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-3-k}{i}\binom{\alpha-1}{j}G\left[\frac{1}{\beta}(3-k+i)+1\right]\end{array}\right.\right)^2}{\left(\sum_{k=0}^{2}\left\{\begin{array}{c}\binom{2}{k}(-1)^k\left(\frac{\alpha}{\alpha-i}\right)^{k+1}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-2}{i}\binom{\alpha-1}{j}G\left[\frac{1}{\beta}(2+i)+1\right]\right)^k\\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-2-k}{i}\binom{\alpha-1}{j}G\left[\frac{1}{\beta}(2-k+i)+1\right]\end{array}\right.\right)^{3/2}} \tag{2.14}$$

2.7 Kurtosis

Kurtosis is a measure of whether the data are heavy-tailed or light-tailed relative to a normal distribution. Thus, Kurtosis is defined as

$$\beta_2 = \frac{\mu_4}{(\mu_2)^2}$$

Thus β_2 is given by

$$\beta_2 = \frac{\sum_{k=0}^{4}\left\{\begin{array}{c}\binom{4}{k}(-1)^k\left(\frac{\alpha}{\alpha-i}\right)^{k+1}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-4}{i}\binom{\alpha-1}{j}G\left[\frac{1}{\beta}(4+i)+1\right]\right)^k\\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-4-k}{i}\binom{\alpha-1}{j}G\left[\frac{1}{\beta}(4-k+i)+1\right]\end{array}\right.}{\left(\sum_{k=0}^{4}\left\{\begin{array}{c}\binom{2}{k}(-1)^k\left(\frac{\alpha}{\alpha-i}\right)^{k+1}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-2}{i}\binom{\alpha-1}{j}G\left[\frac{1}{\beta}(2+i)+1\right]\right)^k\\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-2-k}{i}\binom{\alpha-1}{j}G\left[\frac{1}{\beta}(2-k+i)+1\right]\end{array}\right.\right)^2} \tag{2.15}$$

In Table 1, we have presented the mean, variances, skewness and kurtosis of the BEW distribution for various choices of α =2, 3, 4, 5 and β =1, 2, 3, respectively. One can see from Table 1 that the means are increasing with respect to α while decreasing with respect to β. Variance are decreasing with respect to α and β. (α, β): (3,1), (4,1) and (5,1) shows distribution approximately symmetric.

Table 1
Mean, Variances, Skewness and Kurtosis of the BEW Distribution for Different Values of the Parameters

(α, β)	**Mean**	**Variance**	**Skewness**	**Kurtosis**
(2,1)	0.5299	0.0316	-0.4443	2.5578
(2,2)	0.5147	0.0097	-0.5843	3.2886
(2,3)	0.5097	0.0045	-0.6448	3.6166
(3,1)	0.5926	0.0222	-0.5900	2.9823
(3,2)	0.5493	0.0065	-0.5290	3.3343
(3,3)	0.5332	0.0030	-0.5192	3.4547
(4,1)	0.6309	0.0168	-0.6404	3.2157
(4,2)	0.5699	0.0050	-0.4638	3.2861
(4,3)	0.5472	0.0023	-0.4233	3.3226
(5,1)	0.6571	0.0134	-0.6518	3.1336
(5,2)	0.5841	0.0041	-0.4065	3.2292
(5,3)	0.5568	0.0019	-0.3507	3.2316

3. CHARACTERIZATION

In this section, we characterize BEW distribution. Glanzel (1987) derived theorem of ratio of two truncated moments to characterize some distribution families. i.e. Let (Ω, F, P) be given probability space and let $H = [a_1, a_2]$ an interval with $a_1 < a_2 (a_1 = -\infty, a_2 = \infty)$. Let $X: \Omega \to [a_1, a_2]$ be a continuous random variable (RV) with distribution function F and let $g(x)$ be a real function defined on $H = [a_1, a_2]$ such that $E[g(X)|X \geq x] = h(x),\ x \in H$ is defined with some real function $h(x)$ should be in simple form. Assume that $g(x) \varepsilon C([a_1, a_2]), h(x) \varepsilon C^2([a_1, a_2])$ and F is twofold continuously differentiable and strictly monotone function on the set $[a_1, a_2]$: To conclude, assume that the equation $g(x) = h(x)$ has no real solution in the inside of $[a_1, a_2]$. Then F is obtained from the functions g(x) and $h(x)$ as $F(x) = \int_a^x k \left| \frac{h'(t)}{h(t) - g(t)} \right| exp(-s(t))dt$, where $s(t)$ is the solution of equation $s'(t) = \frac{h'(t)}{h(t) - g(t)}$ and k is a constant, chosen to make $\int_{a_1}^{a_2} dF = 1$.

Theorem-1

Let $X: \Omega \to (0,1)$ be a continuous RV and let $g(x) = \frac{1}{\alpha}\left[1 - e^{-\left(\frac{x}{1-x}\right)^{\beta}}\right]^{-\alpha+1}$, and $h(x) = \frac{2}{\alpha} e^{-\left(\frac{x}{1-x}\right)^{\beta}} \left[1 - e^{-\left(\frac{x}{1-x}\right)^{\beta}}\right]^{-\alpha+1}$. The RV $X \sim BEW(\alpha, \beta)$, iff the function $\tau(x)$ has the form $\tau(x) = e^{-\left(\frac{x}{1-x}\right)^{\beta}}, x > 0$.

Proof:

If $X \sim BEW(\alpha,\beta)$, then

$$\big(1 - F(x)\big)E(g(X) \mid X \geq x) = e^{-\left(\frac{x}{1-x}\right)^{\beta}},$$

and

$$\big(1 - F(x)\big)E(h(X) \mid \geq x) = e^{-2\left(\frac{x}{1-x}\right)^{\beta}}$$

$$\frac{E[g(x) \mid X \geq x]}{E[g(x) \mid X \geq x]} = \tau(x) = e^{\left(\frac{x}{1-x}\right)^{\beta}}, \tag{3.16}$$

also $\tau(x)h(x) - g(x) \neq 0$ for $x > 0$. The differential equation $s'(x) = \frac{\tau'(x)h(x)}{\tau(x)h(x)-g(x)} = \frac{2\beta}{(1-x)^2}\left(\frac{x}{1-x}\right)^{\beta-1}$ has solution $s(x) = 2\left(\frac{x}{1-x}\right)^{\beta}$. Therefore, in the light of Glanzel (1987) $X \sim BEW(\alpha,\beta)$.

Corollary 1

Let $X: \Omega \to (0,1)$ be a continuous RV and let $h(x) = \left(\frac{2}{\alpha}\right) e^{-\left(\frac{x}{1-x}\right)^{\beta}}\left[1 - e^{-\left(\frac{x}{1-x}\right)^{\beta}}\right]^{-\alpha+1}$. Then $X \sim BEW(\alpha,\beta)$, iff there exist functions $\tau(x)$ and $g(x)$, using (Glanzel, 1987) justifying differential equation

$$\frac{\tau'(x)}{\tau(x)h(x) - g(x)} = \frac{\alpha\beta}{(1-x)^2}\left(\frac{x}{1-x}\right)^{\beta-1} e^{-\left(\frac{x}{1-x}\right)^{\beta}}\left[1 - e^{-\left(\frac{x}{1-x}\right)^{\beta}}\right]^{\alpha-1}.$$

General solution of above equation is

$$\tau(x) = e^{2\left(\frac{x}{1-x}\right)^{\beta}} \int \left[-\frac{\alpha\beta}{(1-x)^2}\left(\frac{x}{1-x}\right)^{\beta-1} e^{-\left(\frac{x}{1-x}\right)^{\beta}}\left[1 - e^{-\left(\frac{x}{1-x}\right)^{\beta}}\right]^{\alpha-1} g(x)dx\right] + D \tag{3.17}$$

where D is arbitrary constant.

Definition 1

Let $X: \Omega \to (0,1)$ be a continuous RV with CDF $F(x)$ and PDF $f(x)$ provided the reverse hazard function $\tau_F(x)$ is twice differentiable function justifying differential equation

$$\frac{d}{dx}[ln\, f(x)] = \frac{r_F'(x)}{r_F(x)} + r_F(x). \tag{3.18}$$

Theorem 2

Let $X: \Omega \to (0,1)$ be a continuous RV and $X \sim BEW(\alpha,\beta)$, iff its reverse hazard function r_F justifies the first order differential equation

$$x r_F'(x) + r_F(x) = \frac{\alpha\beta x^{\beta}(1-x)^{-\beta-1}}{\left[exp\ \left(\frac{x}{1-x}\right)^{\beta} - 1\right]}\left\{\frac{\beta}{x} - \frac{(\beta+1)}{(1-x)} + \frac{x^{\beta-1}(1-x)^{-\beta-1}}{\left\{1 - exp\ \left(-\left(\frac{x}{1-x}\right)^{\beta}\right)\right\}}\right\}$$

(3.19)

Proof:

Consider

$$\frac{d}{dx}\{xr_F(x)\} = \frac{d}{dx}\left\{x^{\beta}(1-x)^{-\beta-1}\left[exp\left(\frac{x}{1-x}\right)^{\beta} - 1\right]^{-1}\right\}$$

or

$$r_F(x) = \frac{\alpha\beta x^{\beta}}{x(1-x)^{\beta+1}}\left[\exp\left(\frac{x}{1-x}\right)^{\beta} - 1\right]^{-1}$$

which is the reverse hazard rate of BEW distribution.

4. PARAMETER ESTIMATION

4.1 Maximum Likelihood Estimation with Complete Data

Here, the parameters of the BEW distribution are estimated using the method of maximum likelihood. Let $x_1, x_2, \ldots x_n$ be random samples distributed according to the BEW distribution, the likelihood function is obtained by the relationship;

$$L_n(\theta) = \prod_{i=1}^{n} f(x_i, \alpha, \beta)$$

By taking the natural logarithm, the log-likelihood function is obtained as;

$$logL_n(\theta) = nln(\alpha\beta) - \sum_{i=1}^{n} \ln(1-x_i^2) + (\beta-1)\sum_{i=1}^{n} \ln(u_i) - u_i^{\beta} + (\alpha-1)\sum_{i=1}^{n} ln(1-e^{-u_i^{\beta}}) \tag{4.20}$$

we suppose

$$u_i = \left(\frac{x_i}{1-x_i}\right).$$

The components of the score function are

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} ln(1-e^{u_i^{\beta}}) = 0 \tag{4.21}$$

$$\frac{\partial L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} ln(u_i) - \sum_{i=1}^{n} u_i^{\beta}\, ln\,(u_i) + (\alpha-1)\sum_{i=1}^{n} \frac{u_i^{\beta}\, ln\,(u_i)\, e^{-u_i^{\beta}}}{1-e^{-u_i^{\beta}}} = 0 \tag{4.22}$$

4.2 Maximum Likelihood Estimation with Right Censorship

Let us consider $X = (X_1, X_2, \ldots, X_n)^T$ a sample from the new distribution BEW with the parameter vector $\theta = (\alpha, \beta)^T$ which can contain right censored data with fixed censoring time τ. Each x_i can be written as $x_i = (X_i, \delta_i)$ where

$$\delta_i = \begin{cases} 0 & \text{if } x_i \text{ is a censoring time} \\ 1 & \text{if } x_i \text{ is a failure time} \end{cases}$$

The right censoring is assumed to be non informative, so the log-likelihood function can be written as:

$$L_n(\theta) = \sum_{i=1}^{n} \delta_i ln\, h(x_i, \theta) + \sum_{i=1}^{n} lnS(x_i, \theta)$$

$$L_n(\theta) = \sum_{i=1}^{n} \delta_i \Big[ln\,(\alpha\beta) \,-2\, ln\,\,(1-x_i) \,+(\beta-1)\ln\,(u_i) - u_i^{\beta} \,+(\alpha-1)$$
$$ln\left(1-e^{-u_i^{\beta}}\right) - ln\left(1-\left(1-e^{-u_i^{\beta}}\right)^{\alpha}\right)\Big]$$
$$+\sum_{i=1}^{n} ln\left(1-\left(1-e^{-u_i^{\beta}}\right)^{\alpha}\right)$$

The maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$ of the unknown parameters α and β are derived from the nonlinear following score equations:

$$\frac{\partial L}{\partial \alpha} = \sum_{i=1}^{n} \delta_i \left[\frac{1}{\alpha} + ln\left(1-e^{-u_i^{\beta}}\right) + \frac{\left(1-e^{-u_i^{\beta}}\right)^{\alpha} ln\left(1-e^{-u_i^{\beta}}\right)}{1-\left(1-e^{-u_i^{\beta}}\right)^{\alpha}} \right]$$
$$-\sum_{i=1}^{n} \frac{\left(1-e^{-u_i^{\beta}}\right)^{\alpha} ln\left(1-e^{-u_i^{\beta}}\right)}{1-\left(1-e^{-u_i^{\beta}}\right)^{\alpha}} \tag{4.23}$$

$$\frac{\partial L}{\partial \beta} = \sum_{i=1}^{n} \delta_i \left[\frac{1}{\beta} + ln(u_i)(1-u_i^{\beta}) + \frac{u_i^{\beta}\, ln\,(u_i)\, e^{-u_i^{\beta}} \left(\left(1-e^{-u_i^{\beta}}\right)^{\alpha} + \alpha - 1\right)}{\left(1-e^{-u_i^{\beta}}\right)\left(1-\left(1-e^{-u_i^{\beta}}\right)^{\alpha}\right)} \right]$$
$$-\sum_{i=1}^{n} \frac{u_i^{\beta}\, ln\,(u_i)\, e^{-u_i^{\beta}} \left(1-e^{-u_i^{\beta}}\right)^{\alpha-1}}{1-\left(1-e^{-u_i^{\beta}}\right)^{\alpha}} = 0 \tag{4.24}$$

The explicit form of $\hat{\alpha}$ and $\hat{\beta}$ cannot be obtained, so we use numerical methods.

4.3 Estimated Fisher Information Matrix $\hat{I}$

The components of the estimated information matrix $I = (\iota_{ij})_{(2\times 2)}$ are obtained by

$$\hat{\iota}_{11} = \frac{1}{n} \sum_{i=1}^{n} \delta_i \left[\frac{1}{\alpha} + ln\left(1-e^{-u_i^{\beta}}\right) + \frac{\left(1-e^{-u_i^{\beta}}\right)^{\alpha} ln\left(1-e^{-u_i^{\beta}}\right)}{1-\left(1-e^{-u_i^{\beta}}\right)^{\alpha}} \right]^2 \tag{4.25}$$

$$\hat{\iota}_{22} = \frac{1}{n}\sum_{i=1}^{n}\delta_i\left[\frac{1}{\beta} + ln(u_i)(1 - u_i^{\beta}) + \frac{u_i^{\beta}\ ln\,(u_i)\ e^{-u_i^{\beta}}\left(\left(1 - e^{-u_i^{\beta}}\right)^{\alpha} + \alpha - 1\right)}{\left(1 - e^{-u_i^{\beta}}\right)\left(1 - \left(1 - e^{-u_i^{\beta}}\right)^{\alpha}\right)}\right]^2 \tag{4.26}$$

$$\hat{\iota}_{12} = \frac{1}{n}\sum_{i=1}^{n}\delta_i\left(\frac{1}{\alpha} + ln\left(1 - e^{-u_i^{\beta}}\right) + \frac{\left(1 - e^{-u_i^{\beta}}\right)^{\alpha} ln\left(1 - e^{-u_i^{\beta}}\right)}{1 - \left(1 - e^{-u_i^{\beta}}\right)^{\alpha}}\right)$$
$$\left(\frac{1}{\beta} + \ln(u_i)\left(1 - u_i^{\beta}\right) + \frac{u_i^{\beta}\ ln\,(u_i)\ e^{-u_i^{\beta}}\left(\left(1 - e^{-u_i^{\beta}}\right)^{\alpha} + \alpha - 1\right)}{\left(1 - e^{-u_i^{\beta}}\right)\left(1 - \left(1 - e^{-u_i^{\beta}}\right)^{\alpha}\right)}\right) \tag{4.27}$$

where α and β are replaced by their MLEs $\hat{\alpha}$ and $\hat{\beta}$.

5. TEST STATISTIC FOR RIGHT CENSORED DATA

Let $X_1, \dots, X_n$ be n i.i.d. random variables grouped into k classes I_j. To assess the adequacy of a parametric model F_0, we consider

$$H_0: P(X_i \leq x \mid H_0) = F_0(x;\theta), x \geq 0, \theta = (\theta_1, \dots, \theta_s)^T \in \Theta \subset R^s$$

when data are right censored and the parameter vector θ is unknown, Bagdonavicius and Nikulin (2011) proposed a statistic test Y^2 based on the vector

$$Z_j = \frac{1}{\sqrt{n}}\left(U_j - e_j\right), j = 1,2, \dots, k, \ \text{ with } \ k > s.$$

This represents the differences between observed and expected numbers of failures (U_j and e_j) to fall into these grouping intervals $I_j = (a_{j-1}, a_j]$ with $a_0 = 0, a_r = \tau$, where τ is a finite time. The authors considered a_j as random data functions such as the k intervals chosen have equal expected numbers of failures e_j.

The statistic test Y^2 is defined by

$$Y^2 = Z^T \hat{\Sigma}^- Z \ = \sum_{j=1}^{k}\frac{\left(U_j - e_j\right)^2}{U_j} + Q \tag{5.28}$$

where $Z = (Z^1, \dots, Z_k)^T$ and $\hat{\Sigma}^-$ is a generalized inverse of the covariance matrix $\hat{\Sigma}$ and

$$Q = W^T \hat{G}^- W, \qquad \hat{A}_J = \frac{U_j}{n}, \qquad U_j = \sum_{i: X_i \in I_j}\delta_i$$

$$W = (W_1, \dots, W_s)^T, \quad \hat{G} = \left[\hat{g}_{ll'}\right]_{s\times s}, \quad \hat{g}_{ll'} = \hat{\iota}_{ll'} - \sum_{j=1}^{k}\hat{C}_{lJ}\hat{G}_{l'J}\hat{A}_J^{-1}$$

$$\hat{C}_{lj} = \frac{1}{n}\sum_{i:X_i\in I_j} \delta_i \frac{\partial lnh(x_i,\hat{\theta})}{\partial\theta}, \qquad \hat{\imath}_{ll'} = \frac{1}{n}\sum_{i=1}^{n} \delta_i \frac{\partial lnh(x_i,\hat{\theta})}{\partial\theta_l}\frac{\partial lnh(x_i,\hat{\theta})}{\partial\theta_{l'}}$$

$$\widehat{W}_l = \sum_{j=1}^{k} \hat{C}_{lj}\hat{A}_j^{-1}Z_j, \qquad l,l' = 1,\dots,s$$

$\hat{\theta}$ is the maximum likelihood estimator of θ on initial non-grouped data.

Under the null hypothesis H_0, the limit distribution of the statistic Y^2 is a chi-square with $k = rank(\Sigma)$ degrees of freedom. The description and applications of modified chi-square tests are discussed in Voinov et al. (2013).

The interval limits a_j for grouping data into j classes I_j are considered as data functions and defined by

$$\hat{a}_j = H^{-1}\left(\frac{E_j - \sum_{l=1}^{i-1} H(x_l,\hat{\theta})}{n-i+1}, \hat{\theta}\right), \qquad \hat{a}_k = \max(X_{(n)},\tau)$$

such as the expected failure times e_j to fall into these intervals are $e_j = \frac{E_k}{k}$ for any j, with $E_k = \sum_{i=1}^{n} H(x_i,\theta)$. The distribution of this statistic test Y_n^2 is chi-square (see Voinov et al., 2013).

6. CRITERIA TEST FOR BEW

For testing the null hypothesis H_0 that data belong to the BEW model, we construct a modified chi-squared type goodness-of-fit test based on the statistic Y^2. Suppose that τ is a finite time, and observed data are grouped into $k > s$ sub-intervals $I_j = (a_{j-1}, a_j]$ of $[0,\tau]$. The limit intervals a_j are considered as random variables such that the expected numbers of failures in each interval I_j are the same, so the expected numbers of failures e_j are obtained as

$$E_j = -\frac{j}{k-1}\sum_{i=1}^{n} ln\left\{1 - \left(1 - e^{-u_i^{\beta}}\right)^{\alpha}\right\} \; j = 1,..,k-1$$

and

$$\hat{a}_j = \frac{\left(-\,ln\left(1-\left(1-exp\left\{\frac{\sum_{l=1}^{i-1} H(x_l,\hat{\theta})-E_j}{n-i+1}\right\}\right)^{1/\alpha}\right)\right)^{1/\beta}}{1+\left(-\,ln\left(1-\left(1-exp\left\{\frac{\sum_{l=1}^{i-1} H(x_l,\hat{\theta})-E_j}{n-i+1}\right\}\right)^{1/\alpha}\right)\right)^{1/\beta}}, j = 1,..,k-1 \tag{6.29}$$

6.1 Estimated Matrix $\widehat{W}$ et $\hat{C}$

The components of the estimated matrix $\widehat{W}$ are derived from the estimated matrix $\hat{C}$ which is given by:

$$\hat{C}_{1j} = \frac{1}{n}\sum_{i:x_i\in I_j}^{n} \delta_i \left[\frac{1}{\alpha} + ln\left(1-e^{-u_i^{\beta}}\right) + \frac{\left(1-e^{-u_i^{\beta}}\right)^{\alpha} ln\left(1-e^{-u_i^{\beta}}\right)}{1-\left(1-e^{-u_i^{\beta}}\right)^{\alpha}}\right] \quad (6.30)$$

$$\hat{C}_{2j} = \frac{1}{n}\sum_{i:x_i\in I_j}^{n} \delta_i \left[\frac{1}{\beta} + ln(u_i)(1-u_i^{\beta}) + \frac{u_i^{\beta}\ ln\ (u_i)\, e^{-u_i^{\beta}}\left(\left(1-e^{-u_i^{\beta}}\right)^{\alpha} + \alpha - 1\right)}{\left(1-e^{-u_i^{\beta}}\right)\left(1-\left(1-e^{-u_i^{\beta}}\right)^{\alpha}\right)}\right] \quad (6.31)$$

and

$$\widehat{W}_l = \sum_{j=1}^{k} \hat{C}_{lj}\hat{A}_j^{-1} Z_j \quad l = 1,..,m \qquad j = 1,\dots,k \quad (6.32)$$

Therefore the quadratic form of the test statistic can be obtained easily:

$$Y_n^2(\hat{\theta}) = \sum_{j=1}^{k} \frac{\left(U_j - e_j\right)^2}{U_j} + \widehat{W}^T \left[\hat{\imath}_{ll'} - \sum_{j=1}^{k} \hat{C}_{lj}\hat{G}_{l'j}\hat{A}_J^{-1}\right]^{-1} \widehat{W}. \quad (6.33)$$

7. SIMULATIONS

7.1 Maximum Likelihood Estimation with Right Censorship

We generated N =10,000 right censored samples with different sizes (n =15, 25, 50, 130, 350, 500) from the BEW model with parameters α =2 and β =1.5. Using R statistical software and the Barzilai-Borwein (BB) algorithm (Varadhan and Gilbert, 2009), we calculate the maximum likelihood estimators of the unknown parameters and their mean squared errors (MSE).The code used to generate Table 2 is given in Appendix-1 and the results are presented below.

Table 2
Mean Simulated Values of MLEs $\hat{\alpha}$ and $\hat{\beta}$ and their Corresponding Mean Square Errors

N=10,000	n_1 = 15	n_2 = 25	n_3 = 50	n_4 = 130	n_5 = 350	n_6 = 500
α	1.8246	1.8563	1.8756	1.9254	1.9563	1.9999
MSE	0.0045	0.0032	0.0028	0.0019	0.0014	0.0009
β	1.6625	1.6250	1.5994	1.5835	1.5562	1.5012
MSE	0.0074	0.0065	0.0038	0.0022	0.0015	0.0010

The maximum likelihood estimates of the parameters, presented in Table 2, agree closely with the true parameter values.

7.2 Test Statistic Y^2

Using $N =$ 10,000 right censored simulated samples with different percentage (15% and 30%) of right censoring and different sample sizes (n =25, 50, 130, 350, 500), we calculate the test statistic Y^2 for each sample with respect to the BEW model and we compare the obtained values with the theoretical levels of significance (ε =0.01, 0.05, 0.1). The results are summarized in Tables 3 and 4.

Table 3
Simulated Levels of Significance for Y^2 against their Theoretical Values (15% of Censorship)

N=10,000	n_1=25	n_2=50	n_3=130	n_4=350	n_5=500
ε=1%	0.0087	0.0089	0.0092	0.0098	0.0103
ε=5%	0.0482	0.0488	0.0492	0.0498	0.0502
ε=10%	0.0975	0.0981	0.0987	0.0993	0.1001

Table 4
Simulated Levels of Significance for Y^2 against their Theoretical Values (30% of Censorship)

N=10,000	n_1=25	n_2=50	n_3=130	n_4=350	n_5=500
ε=1%	0.0062	0.0069	0.0078	0.0083	0.0099
ε=5%	0.0468	0.0472	0.0479	0.0486	0.0498
ε=10%	0.0970	0.0973	0.0982	0.0989	0.0999

We can see that empirical proportions of rejection of the null hypothesis H_0 for ε =1%, 5% and 10% levels of significance for all sample sizes and for different percentage of censorship (Table 3 and Table 4) are very close to the theoretical ones. Therefore, the test statistic Y^2, proposed in this work, can be applied to fit data to BEW.

7.3 Data Analysis

A study was conducted on the effects of ploidy on the prognosis of patients with cancers of the mouth. Patients were selected who had a paraffin-embedded sample of the cancerous tissue taken at the time of surgery. Follow-up survival data was obtained on each patient. The tissue samples were examined using a flow cytometer to determine if the tumor had an aneuploidy (abnormal) or diploid (normal) DNA profile using a technique discussed in Sickle-Santanello et al. (1988). The data in Table 4 represents times (in weeks) taken from patients with cancer of the tongue. The data are:

Aneuploid Tumors:

Death Times: 1, 3, 3, 4, 10, 13, 13, 16, 16, 24, 26, 27, 28, 30, 30, 32, 41, 51, 65, 67, 70, 72, 73, 77, 91, 93, 96, 100, 104, 157, 167.

Censored Observations: 61, 74, 79, 80, 81, 87, 87, 88, 89, 93, 97, 101, 104, 108, 109, 120, 131, 150, 231, 240, 400.

We first transform the variables by using $(x/410)$.

We use the statistic test provided above to verify whether these data can be modeled by BEW distribution, and for this, we first calculate the maximum likelihood estimates of the unknown parameters

$$\theta = (\alpha, \beta)^T = (0.9526, 1.2351)^T$$

Data are grouped into $k = 5$ intervals I_j. We give the necessary calculations in the following Table 5.

Table 5
Values of $\hat{a}_j, e_j, U_j, \hat{C}_{1j}, \hat{C}_{2j}$ for Lifetime Data

$\hat{a}_j$	0.0623	0.170	0.225	0.273	0.979
U_j	10	11	13	10	8
$\hat{C}_{1j}$	0.0239	0.1948	0.2351	0.4856	0.9485
$\hat{C}_{2j}$	1.2516	0.9748	1.1574	0.9241	0.8364
e_j	4.2745	4.2745	4.2745	4.2745	4.2745

Then we obtain the value of the statistic test Y_n^2:

$$Y_n^2 = X^2 + Q = 3.145 + 2.351 = 5.496$$

For significance level ε = 0.05, the critical value $\chi_5^2 = 11.0705$ is superior than the value of Y_n^2 = 5.496, so we can say that the proposed model BEW fit these data. The test statistics Y_n^2 to fit these data to the sub models are also calculated and given in Table 6.

Table 6
Values of the Test Statistics Y_n^2 for BEW

Modeling distribution	Y_n^2
BEW	5.496
EW	8.956

8. CONCLUSION

In this work, we first introduced a new univariate version of the exponentiated Weibull distribution called the BEW model. The new PDF can be decreasing, increasing, bathtub and right-skewed unimodal density while the hazard rate can have decreasing and bathtub shaped. Some basic statistical properties of the BEW model are derived. We give the formulas of the criteria statistic of modified chi-squared goodness-of-fit test for BEW model when data are right censored and the parameters are unknown. The statistic Y^2 can be used to check the validity of the BEW model. The main advantage of the chi-square goodness-of-fit tests for censored data is that the limiting distribution of these statistics is the well-known χ^2 distribution. We hope that the results obtained through this study will be useful for practitioners in several fields. The performances of the results and the effectiveness of the proposed test are shown by simulation study and real data analysis.

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APPENDIX

```
    rm(list=ls())
y < runif(50,min=0, max=1)
alpha< 2
beta< 1.5
t < (( log(1-y^(1/alpha)))^(1/beta))/(1+(( log(1 y^(1/alpha)))^(1/beta)))
library(BB)
# Q represents parameters
# dd represents score fonctions
g<  function(Q){n=50;
        dd <  rep(NA, length(Q))
        dd[1]<  (n/Q[1])+sum(log(1-exp(-(t/(1-t))^(Q[2]))))
        dd[2]<  (n/Q[2])+sum(log(t/(1-t)))-(t/(1-t))^(Q[2])*
        log(t/(1-t))+(Q[1]-1)*sum(((t/(1- t))^Q[2]*log(t/(1-t))*
        exp(-(t/(1-t))^Q[2]))/(1-exp(-(t/(1-t))^Q[2])))
        dd
}
Q0 <  rep(1.5,2) # we can chage it #
g(Q0)
BBsolve(par = Q0, fn = g)
BBsolve(par = Q0, fn = g) $par
```