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LINDLEY-QUASI XGAMMA DISTRIBUTION: PROPERTIES AND APPLICATIONS

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ABSTRACT

The probability model Lindley-Quasi Xgamma distribution which we have formulated in this particular article has been obtained by mixture technique as a genuine mixture of two parameter continuous Lindley distribution and Quasi Xgamma distribution with their respective proportions. The vital most statistical properties like moments, reliability measure, order statistics and moment generating function are obtained. Then for estimating the unknown parameters we used method of maximum likelihood estimation. At last we examined the significance and comparability of our proposed model with other related models by fitting our proposed model and its related models to two real life data sets.

KEYWORDS

Two parameter Lindley Distribution, Quasi Xgamma Distribution, Lindley-Quasi Xgamma Distribution. Mixture technique, Structural Properties, Life time data, Maximum Likelihood Estimation.

1. INTRODUCTION

There are many occasions where mixture of probability models fit well to the real life data. Mixture of two models (both continuous or both discrete or one continuous and one discrete) are obtained through linear combination of two probability models with their respective proportions. There are many practical situations which demand for mixture distributions. By combining two models and obtaining a new model a lot of flexibility is obtained in fitting model to real life data. Need of mixture models arise when the population or distribution from which the data is obtained is a genuine mixture of k distinct populations or distributions and our aim is to estimate the proportions $(p_1, p_2, ... p_k)$ in which these k distinct populations in which these occur. In the research work of combination of two probability models Rao (1948) used Fisher's method of scoring for a mixture of two univariate distributions with equal variances and introduced utilization of multiple measurements in problems of biological classification. Stacy (1962) obtained generalization of gamma distribution using power transformation of gamma distribution.

Shanker, Fesshaye & Sharma (2016) proposed two parameter Lindley distribution with applications to lifetime data. Sen, Maiti & Chandra (2016) formulated the Quasi Xgamma distribution and obtained its statistical properties and applications. Shanker (2016) obtained a Quasi Akash distribution with applications. Shanker and Shukla (2017) obtained Ishita distribution by using mixture technique. Shanker & Mishra (2013) obtained Quasi Lindley distribution and studied its properties. We have obtained Lindley-Quasi Xgamma Distribution (LQXD) as a linear combination of two parameter continuous Lindley distribution and Quasi Xgamma distribution. Quasi Xgamma distribution and Lindley distribution are mixture distributions widely used in daily life especially for modeling of survival time data. But these two distributions doesn't cover much variation from data and sometimes data times data may possess characteristics of both these distributions, in those situations we need mixture of these two models known as Lindley-Quasi Xgamma distribution. The vital most statistical properties and some other properties of this distribution are obtained.

A continuous random variable X is said to have a mixture distribution with p.d.f $f(x)$ its p.d.f is obtained as a mixture of k distinct populations having density functions $f_1(x), f_2(x),... f_k(x)$ and with mixing proportions $p_1, p_2,... p_k$ respectively. Mathematically

$$f(x) = p_1 f_1(x) + p_2 f_2(x) + ... + p_k f_k(x)$$

where $0 \le p_i \le 1$ and

$$\sum_{i=1}^{k} p_i = 1 .$$

We have used mixture technique to obtain Lindley-Quasi Xgamma distribution in this paper

2. LINDLEY-QUASI XGAMMA DISTRIBUTION

Lindley Quasi Xgamma Distribution with two parameters α and θ is defined by its probability density function as

$$f(x,\alpha,\theta) = \frac{\theta e^{-\theta x}}{(\alpha+\theta)^2} \left\{ (\alpha+\theta)(\alpha + \frac{x^2\theta^2}{2}) + \theta(\theta-1)(1+\alpha x) \right\} \quad x>0, \theta>0, \alpha>0 \qquad (2.1)$$

where θ is a scale parameter and α is a shape parameter.

The p.d.f of Lindley-Quasi Xgamma distribution is a mixture of Quasi Xgamma distribution with parameters (α,θ) and p.d.f given by (2.2) and two parameter Lindley (α,θ) with p.d.f given by (2.3).

$$f_1(x) = \frac{\theta}{(1+\alpha)} \left(\alpha + \frac{\theta^2}{2} x^2 \right) e^{-\theta x} \qquad x>0, \theta>0 , \alpha>0 \qquad (2.2)$$

$$f_2(x) = \frac{\theta^2}{(\theta+\alpha)}(1+\alpha x)e^{-\theta x} \quad x>0, \theta>0,\ \alpha>0 \tag{2.3}$$

The p.d.f of Lindley Quasi Xgamma distribution $f(x,\alpha,\theta)$ is given by

$$f(x,\alpha,\theta) = pf_1 x + (1-p) f_2(x)$$

where

$$p = \frac{(1+\alpha)}{(a+\theta)}$$

where p is a mixing proportion.

So p.d.f (2.1) is obtained as a linear combination of two parameter continuous Lindley distribution and Quasi Xgamma distribution with mixing proportion p.

$$p = \frac{(1+\alpha)}{(\alpha+\theta)}.$$

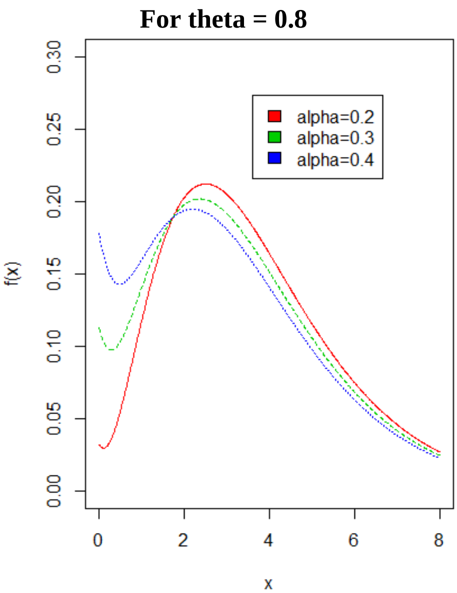

Figure 1(a): Graph of Density Function

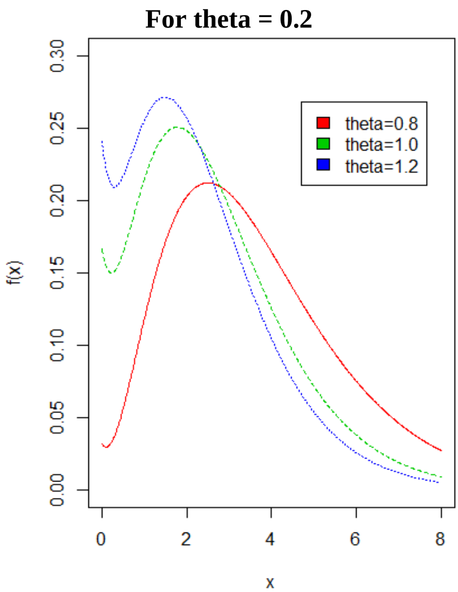

Figure 1(b): Graph of Density Function

The above graphs represent p.d.f of Lindley-Quasi Xgamma distribution for different values of parameters showing our proposed model is positively skewed.

The corresponding cumulative distribution function of Lindley-Quasi Xgamma Distribution (LQXD) is obtained as

$$F_X(x,\alpha,\theta)=\frac{\theta}{(\theta+\alpha)^2}\int_0^x\left(\begin{array}{l}(\alpha+\theta)(\alpha+\dfrac{\theta^2x^2}{2})\\+\theta(\theta-1)(1+\alpha x)\end{array}\right)e^{-\theta x}dx$$

$$F_x(X)=\frac{1}{2(\theta+\alpha)^2}\left[\begin{array}{l}(\alpha+\theta)\left\{2\alpha+2-\left(2\alpha+x^2\theta^2+2\theta x+2\right)e^{-\theta x}\right\}\\+2(\theta-1)\left\{\theta+\alpha-\left(\theta+\alpha\theta x+\alpha\right)e^{-\theta x}\right\}\end{array}\right]$$

$$x>0,\theta>,\alpha>0 \qquad (2.4)$$

where θ,α and are positive parameters.

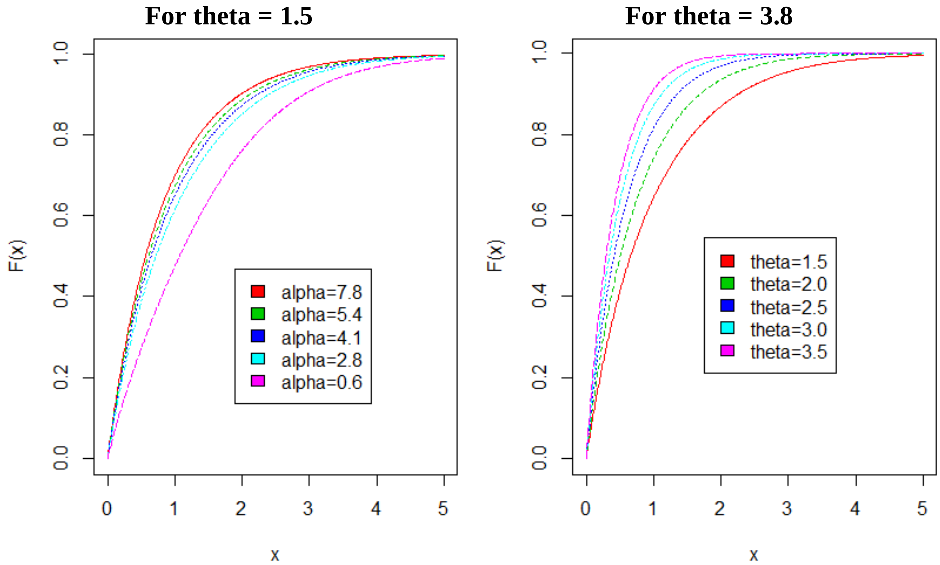

Figure 2(a): Graph of Distribution Function

Figure 2(b): Graph of Distribution Function

The above graphs represent the c.d.f of proposed model for different parameter values.

3. RELIABILITY ANALYSIS

In this segment we have explored various important properties (reliability, hazard rate, reverse hazard rate) of the newly formulated Lindley-Quasi Xgamma distribution.

3.1 Reliability Function R(x)

The reliability of any system is important aspect for decision making. For a system of given age (X), the numerical measure of uncertainty of the survival and functioning of that system beyond given age (X) is known as reliability of that system. Reliability function $R(X)$ can be obtained from cumulative distribution function $F(X)$ of a system by using the relation

$$R(X)=1-F(X)$$

The reliability function of our formulated Lindley-Quasi Xgamma distribution is as:

$$R\left(x,\alpha,\theta\right)=1-\frac{1}{2(\theta+\alpha)^2}\left[\begin{array}{l}(\alpha+\theta)\left\{2\alpha+2-\left(2\alpha+x^2\theta^2+2\theta x+2\right)e^{-\theta x}\right\}\\ +2(\theta-1)\left\{\theta+\alpha-\left(\theta+\alpha\theta x+\alpha\right)e^{-\theta x}\right\}\end{array}\right].$$

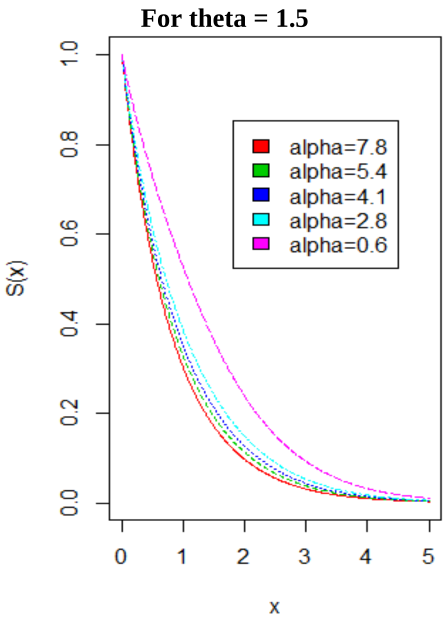

Figure 3(a): Graph of Survival Function

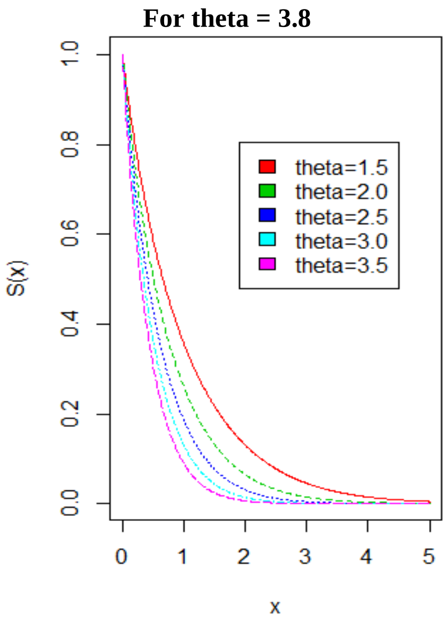

Figure 3(b): Graph of Survival Function

3.2 Hazard Function

The hazard function or hazard rate for a system having attained the age (X) is defined as rate at which this system will fail in within a narrow period of time. It is computed as:

$$H.R=h\left(x;\alpha,\theta\right)=\frac{f\left(x,\theta,\alpha\right)}{R\left(x,\theta,\alpha\right)}.$$

$$=\frac{2\theta e^{-\theta x}\left\{(\alpha+\theta)(\alpha+\frac{x^2\theta^2}{2})+\theta(\theta-1)(1+\alpha x)\right\}}{2(\theta+\alpha)^2-\left[\begin{array}{l}(\alpha+\theta)\left\{2\alpha+2-\left(2\alpha+x^2\theta^2+2\theta x+2\right)e^{-\theta x}\right\}\\ +2(\theta-1)\left\{\theta+\alpha-\left(\theta+\alpha\theta x+\alpha\right)e^{-\theta x}\right\}\end{array}\right]}$$

3.3 Reverse Hazard Rate

The reverse hazard rate for our proposed model (Lindley- Quasi Xgamma distribution) is computed as:

$$R.H.R = h\left(x,\alpha,\theta\right) = \frac{f\left(x,\alpha,\theta\right)}{F\left(x,\alpha,\theta\right)} = \frac{2\theta e^{-\theta x}\left\{(\alpha+\theta)(\alpha+\frac{x^2\theta^2}{2})+\theta(\theta-1)(1+\alpha x)\right\}}{\begin{bmatrix}(\alpha+\theta)\left\{2\alpha+2-\left(2\alpha+x^2\theta^2+2\theta x+2\right)e^{-\theta x}\right\}\\ +2(\theta-1)\left\{\theta+\alpha-\left(\theta+\alpha\theta x+\alpha\right)e^{-\theta x}\right\}\end{bmatrix}}.$$

4. STATISTICAL PROPERTIES

This segment represents the vital structural properties of our newly introduced Lindley-Quasi Xgamma model. These include moments, dispersion measures, moment generating function and characteristic function

4.1 Moments about Origin

Suppose X is a random variable following Lindley-Quasi Xgamma distribution with parameters α & θ, and then the r^{th} moment for a given probability distribution is given by

$$\mu_r' = E\left(X_w{}^r\right) = \int_0^\infty x^r f\left(x,\alpha,\theta\right)dx$$

$$= \frac{\theta}{(\theta+\alpha)^2}\int_0^\infty x^r e^{-\theta x}[(\alpha+\theta)(\alpha+\frac{x^2\theta^2}{2})+\theta(\theta-1)(1+\alpha x)]dx$$

$$\mu_r' = \frac{r!}{(\theta+\alpha)^2\theta^r}\begin{Bmatrix}(\theta+\alpha)(\alpha+\frac{(r+1)(r+2)}{2})\\ +(\theta-1)(\theta+\alpha(r+1))\end{Bmatrix} \qquad (4.1.1)$$

Put r=1 in equation (4.1.1) we get

$$\mu_1' = \frac{\{(\theta+\alpha)(\alpha+3)+(\theta-1)(\theta+2\alpha)\}}{\theta(\theta+\alpha)^2}$$

Which is mean of the Lindley-Quasi Xgamma distribution

Put r=2 in equation (4.1.1) we get

$$\mu_2' = \frac{2\left\{(\theta+\alpha)\left(\alpha+6\right)+(\theta-1)(\theta+3\alpha)\right)\}}{\theta^2\{\theta+\alpha)^2}$$

Put r=3 in (4.1.1)

$$\mu_3' = \frac{6\left\{(\theta+\alpha)\left(\alpha+10\right)+(\theta-1)(\theta+4\alpha)\right)\}}{\theta^3\{\theta+\alpha)^2}$$

Put r=4 in (4.1.1)

$$\mu_4' = \frac{24\left\{(\theta+\alpha)\left(\alpha+15\right)+(\theta-1)(\theta+5\alpha)\right)\}}{\theta^4\{\theta+\alpha)^2}$$

4.2 Central Moments

$$\mu_2 = \frac{\begin{Bmatrix} 2(\theta+\alpha)^2((\theta+\alpha)\left(\alpha+6\right)+(\theta-1)(\theta+3\alpha)) \\ -((\theta+\alpha)(\alpha+3)+(\theta-1)(\theta+2\alpha)^2) \end{Bmatrix}}{\theta^2\{\theta+\alpha)^4}$$

Which is the variance of Lindley-Quasi Xgamma distribution

$$\mu_3 = \left\{ \frac{2\begin{Bmatrix} 3(\theta+\alpha)^4((\theta+\alpha)(\alpha+10)+(\theta-1)(\theta+4\alpha))-3(\theta+\alpha)^2 \\ \{((\theta+\alpha)(\alpha+3)+(\theta-1)(\theta+2\alpha))((\theta+\alpha)(\alpha+6)+(\theta-1)(\theta+3\alpha))\} \\ +\{(\theta+\alpha)(\alpha+3)+(\theta-1)(\theta+2\alpha)\} \end{Bmatrix}}{\theta^3\{\theta+\alpha)^6} \right\}$$

$$\mu_4 = \left\{ \frac{3\begin{Bmatrix} 8(\theta+\alpha)^6((3(\theta+\alpha)^4((\theta+\alpha)(\alpha+15)+(\theta-1)(\theta+5\alpha))-6(\theta+\alpha)^4 \\ \{((\theta+\alpha)(\alpha+3)+(\theta-1)(\theta+2\alpha))((\theta+\alpha)(\alpha+10)+(\theta-1)(\theta+4\alpha))\} \\ +4(\theta+\alpha)^2\{((\theta+\alpha)(\alpha+3)+(\theta-1)(\theta+2\alpha))^2 \\ ((\theta+\alpha)(\alpha+6)+(\theta-1)(\theta+3\alpha))\}-\{(\theta+\alpha)(\alpha+3)+(\theta-1)(\theta+2\alpha)\}^4 \end{Bmatrix}}{\theta^4\{\theta+\alpha)^8} \right\}$$

4.3 Coefficient of Variation, Skewness, Kurtosis and Index of Dispersion

The coefficient of variation (C.V), coefficient of skewness ($\sqrt{\beta_1}$), coefficient of kurtosis (β_2) and index of dispersion (γ) of the LQXD are thus obtained as

$$C.V = \frac{(\mu_2)^{\frac{1}{2}}}{\mu_1'} = \left\{ \frac{\left\{ \begin{array}{l} 2(\theta+\alpha)^2((\theta+\alpha)\left(\alpha+6\right)+(\theta-1)(\theta+3\alpha)\right) \\ -((\theta+\alpha)(\alpha+3)+(\theta-1)(\theta+2\alpha)^2) \end{array} \right\}^{\frac{1}{2}}}{\left\{(\theta+\alpha)(\alpha+3)+(\theta-1)(\theta+2\alpha)\right\}} \right\}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2{}^{3/2}} = \left\{ \frac{2\left\{ \begin{array}{l} 3(\theta+\alpha)^4((\theta+\alpha)(\alpha+10)+(\theta-1)(\theta+4\alpha)) \\ -3(\theta+\alpha)^2\{((\theta+\alpha)(\alpha+3)+(\theta-1)(\theta+2\alpha)) \\ ((\theta+\alpha)(\alpha+6)+(\theta-1)(\theta+3\alpha))\} \\ +\{(\theta+\alpha)(\alpha+3)+(\theta-1)(\theta+2\alpha)\} \end{array} \right\}}{\left\{ \begin{array}{l} 2(\theta+\alpha)^2((\theta+\alpha)\left(\alpha+6\right)+(\theta-1)(\theta+3\alpha)) \\ -((\theta+\alpha)(\alpha+3)+(\theta-1)(\theta+2\alpha)^2) \end{array} \right\}^{3/2}} \right\}$$

$$\beta_2 = \frac{\mu_4}{\mu_2{}^2} = \left\{ \frac{3\left\{ \begin{array}{l} 8(\theta+\alpha)^6((3(\theta+\alpha)^4((\theta+\alpha)(\alpha+15) \\ +(\theta-1)(\theta+5\alpha))-6(\theta+\alpha)^4\{((\theta+\alpha)(\alpha+3)+ \\ (\theta-1)(\theta+2\alpha))((\theta+\alpha)(\alpha+10) \\ +(\theta-1)(\theta+4\alpha))\}+4(\theta+\alpha)^2\{((\theta+\alpha)(\alpha+3)+ \\ (\theta-1)(\theta+2\alpha))^2((\theta+\alpha)(\alpha+6)+(\theta-1)(\theta+3\alpha))\} \\ -\{(\theta+\alpha)(\alpha+3)+(\theta-1)(\theta+2\alpha)\}^4 \end{array} \right\}}{\left\{ \begin{array}{l} 2(\theta+\alpha)^2((\theta+\alpha)\left(\alpha+6\right)+(\theta-1)(\theta+3\alpha)) \\ -((\theta+\alpha)(\alpha+3)+(\theta-1)(\theta+2\alpha)^2) \end{array} \right\}^{2}} \right\}$$

$$\gamma = \frac{\mu_2}{\mu_1'} = \left\{ \frac{\left\{ \begin{array}{l} 2(\theta+\alpha)^2((\theta+\alpha)\left(\alpha+6\right)+(\theta-1)(\theta+3\alpha)) \\ -((\theta+\alpha)(\alpha+3)+(\theta-1)(\theta+2\alpha)^2) \end{array} \right\}}{\theta(\theta+\alpha)^2\left\{(\theta+\alpha)(\alpha+3)+(\theta-1)(\theta+2\alpha)\right\}} \right\}$$

4.4 Moment Generating Function and Characteristic Function of Lindley-Quasi Xgamma distribution (LQXD)

We will now obtain moment generating function and characteristic function of LQXD in this section.

Theorem 1.1

If X has the LQXD (θ,α), then the moment generating function $M_X(t)$ and characteristic generating function $\varphi_X(t)$ are

$$M_X(t)= \frac{\theta}{(\theta+\alpha)^2(\theta-t)^3}\begin{Bmatrix}(\theta+\alpha)(\alpha(\theta\text{-t})^2+\theta^2) \\ +\theta(\theta-1)((\theta-t)^2+\alpha(\theta-t)\end{Bmatrix}$$

&

$$\varphi_X(t)= \frac{\theta}{(\theta+\alpha)^2(\theta-it)^3}\begin{Bmatrix}(\theta+\alpha)(\alpha(\theta\text{-it})^2+\theta^2) \\ +\theta(\theta-1)((\theta-it)^2+\alpha(\theta-it)\end{Bmatrix} \quad \text{respectively.}$$

Proof:

We begin with the well-known definition of the moment generating function given by

$$M_X(t)=E\left(e^{tx}\right)=\int_0^\infty e^{tx}\, f(x;\alpha,\theta)\,dx$$

$$=\int_0^\infty \frac{e^{tx}\theta e^{-\theta x}}{(\alpha+\theta)^2}\left\{(\alpha+\theta)(\alpha+\frac{x^2\theta^2}{2})+\theta(\theta-1)(1+\alpha x)\right\}dx$$

$$=\frac{\theta}{\{\alpha+\theta\}^2}\int_0^\infty e^{-x(\theta-t)}\left\{(\alpha+\theta)(\alpha+\frac{x^2\theta^2}{2})+\theta(\theta-1)(1+\alpha x)\right\}dx$$

$$M_X(t)= \frac{\theta}{(\theta+\alpha)^2(\theta-t)^3}\begin{Bmatrix}(\theta+\alpha)(\alpha(\theta\text{-t})^2+\theta^2) \\ +\theta(\theta-1)((\theta-t)^2+\alpha(\theta-t)\end{Bmatrix} \quad (4.4.1)$$

Which is the m.g.f of Lindley Quasi Xgamma distribution.

Also we know that $\varphi_X(t)=M_X(it)$

Therefore,

$$\varphi_X(t)= \frac{\theta}{(\theta+\alpha)^2(\theta-it)^3}\begin{Bmatrix}(\theta+\alpha)(\alpha(\theta\text{-it})^2+\theta^2) \\ +\theta(\theta-1)((\theta-it)^2+\alpha(\theta-it)\end{Bmatrix}$$

is the characteristic function of Lindley Quasi Xgamma distribution.

5. ORDER STATISTICS

Suppose $X_{(1)}, X_{(2)}, X_{(3)}...., X_{(n)}$ be the ordered statistics of the random sample $X_1, X_2, X_3,....X_n$ obtained from the Lindley-Quasi Xgamma Distribution with cumulative distribution function $F_X(x)$ and probability density function $f_X(x)$ given in (2.4) & (2.1) respectively, then the probability density function of p^{th} order statistics $X_{(p)}$ is given by:

$$f_{(p)}(x,\alpha,\theta) = \frac{n!}{(p-1)!(n-p)!} f(x)\left[F(x)\right]^{p-1}\left[1-F(x)\right]^{n-p}. \quad p = 1,2,...,n$$

Using the equations (2.1) and (2.4), the probability density function of p^{th} order statistics of Lindley Quasi Xgamma distribution is given by:

$$f_{(p)}(x,\alpha,\theta) = \left\{ \begin{array}{l} \frac{n!}{(p-1)!(n-p)!} \frac{\theta e^{-\theta x}}{(\alpha+\theta)^2} \left\{ (\alpha+\theta)(\alpha+\frac{x^2\theta^2}{2}) + \theta(\theta-1)(1+\alpha x) \right\} \\ \left[\frac{1}{2(\theta+\alpha)^2} \left[\begin{array}{l} (\alpha+\theta)\left\{2\alpha+2-\left(2\alpha+x^2\theta^2+2\theta x+2\right)e^{-\theta x}\right\} \\ +2(\theta-1)\left\{\theta+\alpha-\left(\theta+\alpha\theta x+\alpha\right)e^{-\theta x}\right\} \end{array} \right] \right]^{p-1} \\ \left[1-\frac{1}{2(\theta+\alpha)^2} \left[\begin{array}{l} (\alpha+\theta)\left\{2\alpha+2-\left(2\alpha+x^2\theta^2+2\theta x+2\right)e^{-\theta x}\right\} \\ +2(\theta-1)\left\{\theta+\alpha-\left(\theta+\alpha\theta x+\alpha\right)e^{-\theta x}\right\} \end{array} \right] \right]^{n-p} \end{array} \right\}.$$

Then, the p.d.f of first order $X_{(1)}$ of Lindley-Quasi Xgamma distribution is given by:

$$f_{(1)}(x,\alpha,\theta) = \left\{ \begin{array}{l} n \frac{\theta e^{-\theta x}}{(\alpha+\theta)^2} \left\{ (\alpha+\theta)(\alpha+\frac{x^2\theta^2}{2}) + \theta(\theta-1)(1+\alpha x) \right\} \\ \left[1-\frac{1}{2(\theta+\alpha)^2} \left[\begin{array}{l} (\alpha+\theta)\left\{2\alpha+2-\left(2\alpha+x^2\theta^2+2\theta x+2\right)e^{-\theta x}\right\} \\ +2(\theta-1)\left\{\theta+\alpha-\left(\theta+\alpha\theta x+\alpha\right)e^{-\theta x}\right\} \end{array} \right] \right]^{n-1}. \end{array} \right\}$$

and the p.d.f of nth order $X_{(n)}$ of Lindley-Quasi Xgamma distribution is given as:

$$f_{(n)}(x,\alpha,\theta) = \left\{ \begin{array}{l} n \frac{\theta e^{-\theta x}}{(\alpha+\theta)^2} \left\{ (\alpha+\theta)(\alpha+\frac{x^2\theta^2}{2}) + \theta(\theta-1)(1+\alpha x) \right\} \\ \left[\frac{1}{2(\theta+\alpha)^2} \left[\begin{array}{l} (\alpha+\theta)\left\{2\alpha+2-\left(2\alpha+x^2\theta^2+2\theta x+2\right)e^{-\theta x}\right\} \\ +2(\theta-1)\left\{\theta+\alpha-\left(\theta+\alpha\theta x+\alpha\right)e^{-\theta x}\right\} \end{array} \right] \right]^{n-1}. \end{array} \right\}$$

6. METHOD OF MAXIMUM LIKELIHOOD ESTIMATION

Suppose $X_1, X_2, X_3.....X_n$ to be a random sample of size n drawn from Lindley-Quasi Xgamma distribution, then the likelihood function of distribution is given as:

$$L(x \mid \alpha, \theta) = \prod_{i=1}^{n} \left[\frac{\theta e^{-\theta x}}{(\alpha+\theta)^2} \left\{ (\alpha+\theta)(\alpha + \frac{x^2\theta^2}{2}) + \theta(\theta-1)(1+\alpha x) \right\} \right]$$

The log likelihood function becomes:

$$\log L = \left\{ \begin{array}{l} n\log(\theta) + \sum_{i=1}^{n} \log\left((\alpha+\theta)(\alpha + \frac{\theta^2 x^2}{2}) + \theta(\theta-1)(1+\alpha x) \right) \\ -\theta \sum_{i=1}^{n} x_i - 2n\log(\theta+\alpha) \end{array} \right\} \tag{6.1}$$

differentiating the log-likelihood function with respect to α, θ. This is done by partially differentiating (6.1) with respect to θ, α and equating the result to zero, we obtain the following normal equations,

$$\frac{d\log L}{d\theta} = \left[\begin{array}{l} \frac{n}{\theta} - \sum_{i=1}^{n} x_i + \frac{2n}{(\theta+\alpha)} + \\ \sum_{i=1}^{n} \left(\frac{(\alpha + \frac{x^2\theta^2}{2}) + (\theta+\alpha)\theta x^2 + (1+\alpha x)(2\theta-1)}{(\alpha+\theta)(\alpha + \frac{\theta^2 x^2}{2}) + \theta(\theta-1)(1+\alpha x)} \right) \end{array} \right] = 0. \tag{6.2}$$

$$\frac{d\log L}{d\alpha} = \sum_{i=1}^{n} \left(\frac{(\alpha+\theta) + (\alpha + \frac{\theta^2 x^2}{2}) + \theta(\theta-1)x}{(\alpha+\theta)(\alpha + \frac{\theta^2 x^2}{2}) + \theta(\theta-1)(1+\alpha x)} \right) - \frac{2n}{(\alpha+\theta)} = 0 \tag{6.3}$$

As equations (6.2) and (6.3) are complex equations so MLEs of α, θ cannot be obtained by solving these equations. Hence we use iteration method to obtain MLEs of α, θ through R software

7. APPLICATIONS OF LINDLEY-QUASI XGAMMA DISTRIBUTION

We fitted our model and its competitive models to two real life data sets to examine the superiority of our model and to find the applicability of our model in real life.

Data Set 1:

The data set given in table 1 was analyzed by McGilchrist and Aisbett (199)]. The data consists of 58 recurrence times to infection, at the point of insertion of the catheter, for kidney patients using portable dialysis equipment.

Table 1
Recurrence Times to Infection, at the Point of Insertion of the Catheter

8	16	23	22	28	447	318	30
12	24	245	7	9	511	30	53
196	15	154	7	333	141	96	38
536	17	185	177	292	114	15	152
562	402	13	66	39	12	40	201
132	156	34	30	2	25	130	26
27	58	43	152	30	190	119	8
78	63						

Data Set 2:

The data set given in table 2 represents the exceedances of flood peaks (in m3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consists of 72 exceedances for the year 1958-1984, rounded to one decimal place as shown below. Recently, Merovci and Puka (2014), and Bourguignon et al. (2013) analyzed this data using the Transmuted Pareto (TP) distribution and Kumaraswamy Pareto (Kw-P) distribution respectively, demonstrating the superiority of their distributions over the Weibull and Pareto distributions.

Table 2
72 Exceedances of Flood Peaks (in m3/s) of the Wheaton River

1.7	2.2	14.4	1.1	0.4	20.6	5.3	0.7
13.0	12.0	9.3	1.4	18.7	8.5	25.5	11.6
14.1	22.1	1.1	2.5	14.4	1.7	37.6	0.6
2.2	39.0	0.3	15.0	11.0	7.3	22.9	1.7
0.1	1.1	0.6	9.0	1.7	7.0	20.1	0.4
14.1	9.9	10.4	10.7	30.0	3.6	5.6	30.8
13.3	4.2	25.5	3.4	11.9	21.5	27.6	36.4
2.7	64.0	1.5	2.5	27.4	1.0	27.1	20.2
16.8	5.3	9.7	27.5	2.5	27.0	1.9	2.8

These data sets are used here only for illustrative purposes. The required numerical evaluations are carried out using R software. We have fitted Lindley-Quasi Xgamma distribution, Quasi Xgamma distribution & Lindley distribution to the two real life data sets given in table 1 & 2. The summary statistic of data sets 1 & 2 is given in table 3. The MLEs of the parameters with standard errors in parentheses, model functions are displayed in table 4 for these two data sets. The corresponding log-likelihood values, AIC, AICC, HQIC, BIC & Shannon’s entropy are given in tables 5 & 6 for data sets 1 & 2 respectively.

Table 3
Summary Statistic of Data Sets 1 & 2

Data Set	No. of Observations	Min. Value	First Quartile	Median	Mean	Third Quartile	Max. Value
Data Set 1	58	2	23.25	48.00	118.78	155.50	562.00
Data Set 2	72	0.100	2.125	9.500	12.204	20.125	64.000

Table 4
ML Estimates, Standard Error of Estimates in Parenthesis, Model Function of related Models and Proposed Model for Data sets 1 & 2

Data Set	Distribution	ML Estimates with Standard errors	Model Function
Data Set 1	Lindley-Quasi Xgamma Distribution (LQXD)	$\hat{\alpha}=1.64184$ (0.71806) $\hat{\theta}=0.01316$ (0.00220)	$\frac{\theta e^{-\theta x}}{(\alpha+\theta)^2}\left\{(\alpha+\theta)(\alpha+\frac{x^2\theta^2}{2})+\theta(\theta-1)(1+\alpha x)\right\}$
	Quasi Xgamma Distribution (QXD)	$\hat{\theta}=0.011391$ (0.002135) $\hat{\alpha}=4.70305$ (3.70364)	$\frac{\theta}{(1+\alpha)}\left(\alpha+\frac{\theta^2}{2}x^2\right)e^{-\theta x}$
	Lindley Distribution (LD)	$\hat{\theta}=0.16713$ (0.00154)	$\frac{\theta^2}{(\theta+1)}(1+x)e^{-\theta x}$
Data Set 2	Lindley-Quasi Xgamma Distribution (LQXD)	$\hat{\alpha}=1.08629$ (0.26208) $\hat{\theta}=0.16179$ (0.01824)	$\frac{\theta e^{-\theta x}}{(\alpha+\theta)^2}\left\{(\alpha+\theta)(\alpha+\frac{x^2\theta^2}{2})+\theta(\theta-1)(1+\alpha x)\right\}$
	Quasi Xgamma Distribution (QXD)	$\hat{\alpha}=1.93972$ (1.05814) $\hat{\theta}=0.13768$ (0.02180)	$\frac{\theta}{(1+\alpha)}\left(\alpha+\frac{\theta^2}{2}x^2\right)e^{-\theta x}$
	Lindley Distribution (LD)	$\hat{\theta}=0.153007$ (0.12806)	$\frac{\theta^2}{(\theta+1)}(1+x)e^{-\theta x}$

Table 5
Model Comparison of Proposed Model and its related Models for Data Set 1

Distribution	$-\log L$	AIC	BIC	AICC	HQIC	Shannon Entropy *H(X)*
Lindley-Quasi Xgamma Distribution (LQXD)	331.706	667.4133	671.5342	667.6315	669.018	5.71
Quasi Xgamma Distribution (QXD)	333.928	671.856	675.977	672.0743	673.461	5.75
Lindley Distribution (LD)	355.036	712.073	714.1334	712.1444	712.875	6.12

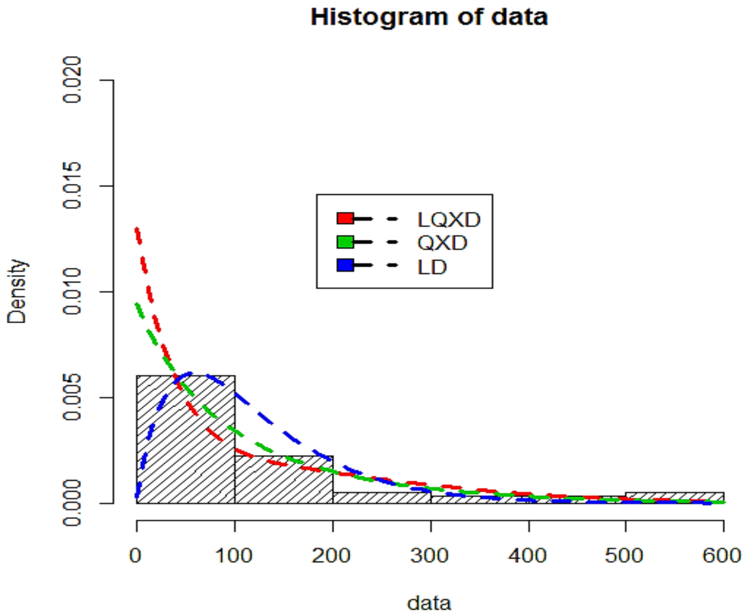

Figure 4: Graph of Data Set 1 by Proposed Model and related Models

Table 6
Model Comparison of Proposed Model and its related Models for Data Set 2

Distribution	$-\log L$	AIC	BIC	AICC	HQIC	Shannon entropy *H(X)*
Lindley-Quasi Xgamma Distribution (LQXD)	248.517	501.035	505.589	501.209	502.848	3.45
Quasi Xgamma Distribution (QXD)	250.865	505.7299	510.2833	505.9038	507.5426	3.48
Lindley Distribution (LD)	264.211	530.4236	532.7002	530.4807	531.3299	3.66

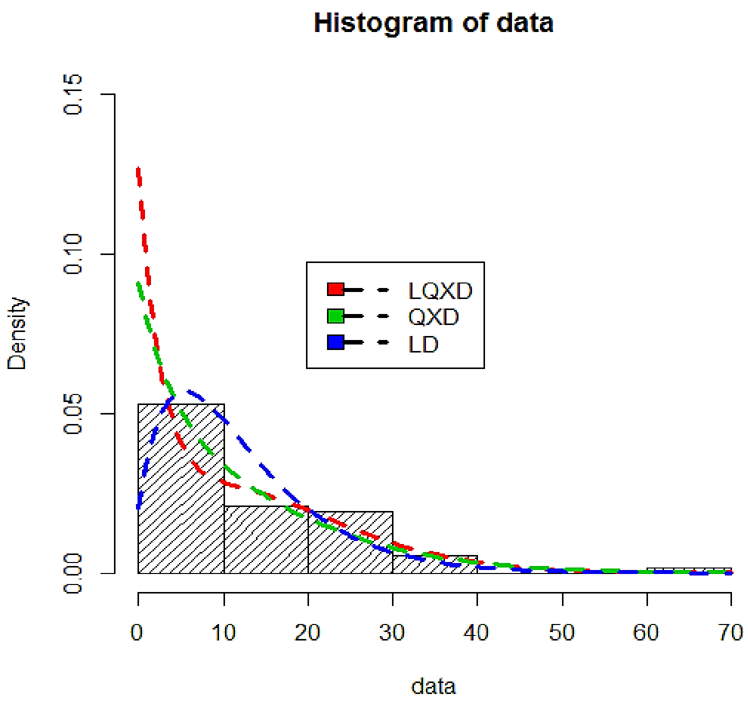

Figure 5: Graph of Data Set 2 Fitted by Proposed Model and related Models

In order to compare the Lindley-Quasi Xgamma distribution with Quasi Xgamma distribution and Lindley distribution, We compute the criteria like AIC (Akaike information criterion), AICC (corrected Akaike information criterion) , BIC (Bayesian information criterion) & HQIC which represent the loss of information resulting from fitting probability models to data. The better distribution corresponds to lesser AIC, AICC, BIC & HQIC values. Also we computed the Shannon's entropy *(H(X))* which represents the average uncertainty. The better model possesses lesser Shannon's entropy value.

$$\text{AIC} = 2\text{k}-2\log L \quad \text{AICC} = \text{AIC}+\frac{2k(k+1)}{n-k-1}$$

$$\text{BIC} = \text{k} \log n-2\log L \quad \text{HQIC}=2\text{k} \log(\log(\text{n}))+2 \log L$$

$$H(X) = -\frac{\log L}{n}$$

where k is the number of parameters in the statistical model, n is the sample size and -2log$L$ is the maximized value of the log-likelihood function under the considered model. From Tables 5 & 6, it has been observed that Lindley-Quasi Xgamma distribution possesses the lesser AIC, AICC BIC, HQIC and *H(X)* values as compared to Quasi Xgamma distribution and Lindley distribution for data sets 1 & 2 respectively. Hence we can conclude that Lindley- Quasi Xgamma distribution leads to a better fit than Quasi Xgamma distribution and Lindley distribution for data sets 1 & 2 respectively.

8. CONCLUSION

In this article we worked on the formulation of new probability model known as Lindley-Quasi Xgamma distribution as a genuine mixture of two parameter continuous Lindley distribution and Quasi Xgamma distribution. We obtained some of the vital statistical properties like moments, order statistics, reliability measures of the proposed model. Then we obtained estimates of unknown parameters of our proposed model. Finally we fitted our formulated model to two real life data sets and compared our model to other related models and showed that our model fits better to real life data sets than its related models as our model possesses lesser values of AIC, BIC, AICC, HQIC & H(X).

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