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TYPE II HALF LOGISTIC EXPONENTIATED EXPONENTIAL DISTRIBUTION: PROPERTIES AND APPLICATIONS

Muritala Abdulkabir[1] and **Ipinyomi, R.A.**[2]
Department of Statistics, University of Ilorin, Ilorin, Nigeria
Email: [1]kaybeedydx@gmail.com
[2]ipinyomira@gmail.com

ABSTRACT

This paper introduce a new distribution called Type II half logistic exponentiated exponential (TIIHLEE) distribution established from Type II half logistic G family of distribution . Some mathematical properties; moments, probability weighted moments, mean deviation, quantile function, Renyi entropy of TIIHLEE distribution are investigated. The expressions of order statistics are derived. Parameters of the derived distribution are obtained using maximum likelihood method. We compare the fits of the TIIHLEE model with some models which shows that the new model is strong and better fit than other models.

KEYWORD

Type II half logistic-G family, Moments, Order statistic, Estimation, Maximum likelihood, quantile function, Renyi Entropy.

1. INTRODUCTION

Since real world data are usually complex and can take a variety of shapes, existing distributions do not always provide an adequate fit. Hence, generalizing distributions and studying their flexibility are of interest of researchers for last decades, the generated family of continuous distributions is a new improvement for producing and extending the usual classical distributions. These families have been broadly studied in several areas as well as yield more flexibility in many applications. Some of the generators are: Beta-generated (B-G) (Eugene et al. (2002)), Gamma-G (Zografos and Balakrishnan (2009)) and Ristic and Balakrishnan (2011)), Kumaraswamy-G (Cordeiro and de Castro (2011)), exponentiated generalized class (Cordeiro et al. (2013)), Weibull-G (Bourguignon et al. (2014)), Garhy-G (Elgarhy et al. (2016)), Kumaraswamy Weibull-G (Hassan and Elgarhy (2016)), exponentiated Weibull-G (Hassan and Elgarhy (2016)), additive Weibull-G (Hassan and Hemeda (2016)), exponentiated extended-G (Elgarhy et al. (2017), Type II half logistic-G(TIIHLG) (Hassan et al. (2017)), generalized additive Weibull-G (Hassan et al. (2017)), odd Frechet-G (Haq and Elgarhy (2018)), power Lindley-G (Hassan and Nassr (2018) and Muth-G (Almarashi and Elgarhy (2018)) among others are Type II Half Logistic Rayleigh Distribution (Muhammad et al (2018)).

The half logistic distribution is a member of the family of logistic distributions which is introduced by (Balakrishnan, 1985) which has the following cumulative distribution function (cdf)

$$F(t)=\frac{1-e^{-\lambda t}}{1+e^{-\lambda t}} \qquad t>0, \quad \lambda>0\,. \qquad (1)$$

The associated probability density function (pdf) corresponding

$$f(t)=\frac{2e^{-\lambda t}}{(1+e^{-\lambda t})^2} \qquad t>0, \quad \lambda>0 \qquad (2)$$

Hassan, et al. (2017) use the half logistic generator instead of gamma generator to obtain type II half logistic family which is denoted by TIIHL-*G*.

Then the cdf and pdf of TIIHL-G is define as follows

$$F(x;\lambda)=1-\int_0^{-\log G(x)}\frac{2\lambda e^{-\lambda}}{\left(1+e^{-\lambda t}\right)^2}dt=\frac{2\left[G(x)\right]^{\lambda}}{1+\left[G(x)\right]^{\lambda}} \qquad x>0,\lambda>0 \qquad (3)$$

$$f(x;\lambda)=\frac{2\lambda g(x)\left[G(x)\right]^{\lambda-1}}{\left[1+\left[G(x)\right]^{\lambda}\right]^2} \qquad x>0,\lambda>0 \qquad (4)$$

where λ is the shape parameter and g(x;λ) and G(x;λ) are the baseline distribution respectively.

However, the TIIHLEE has tractable properties especially for simulation since its quantile function take a simple form.

$$Q(u)=G^{-1}\left[\frac{u}{2-u}\right]^{\frac{1}{\lambda}} \qquad (5)$$

where, u is a uniform distribution on the interval (0,1)and $G^{-1}(.)$ is the inverse function of $G(.)$

Our aim in this work is to study a modified statistical distribution that will be suitable to fit positively skewed and unimodal data and to check the flexibility of the existing and proposed distribution.

2. TYPE II HALF LOGISTIC EXPONENTIATED EXPONENTIAL DISTRIBUTION

The cumulative density function (cdf) and probability density function (pdf) of exponentiated exponential distribution (EE) are define as follows:

$$G(x;\alpha,\beta)=(1-e^{-\beta x})^{\alpha} \qquad (6)$$

$$g(x;\alpha,\beta)=\alpha\beta e^{-\beta x}(1-e^{-\beta x})^{\alpha-1} \qquad (7)$$

Substituting 6 and 7 in 3 and 4 then we define probability density function (pdf) and cumulative density function (cdf) Type II Half Logistic Exponentiated Exponential Distribution (TIIHL-EE) as follows:

$$F(x;\alpha,\beta,\lambda)=\frac{2\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda}}{1+\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda}} \tag{8}$$

$$f(x;\alpha,\beta,\lambda)=\frac{2\lambda\alpha\beta e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1}\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda-1}}{\left[1+\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda}\right]^{2}} \tag{9}$$

Henceforth, a random variable with probability density function (pdf) is denoted by $X \sim TIIHLEE\left(\alpha,\beta,\lambda\right)$.

Hazard Function

The hazard function (or failure rate) of a random variable X with density $X \sim TIIHLEE(\alpha,\beta,\lambda)$ and a cumulative distribution function is given by

$$h(x;\alpha,\beta,\lambda)=\frac{f(x)}{F(x)}=\frac{2\lambda g(x)\left[G(x)\right]^{\lambda-1}}{1-\left[G(x)\right]^{2\lambda}}=\frac{2\lambda\alpha\beta x^{-\beta-1}e^{-\alpha x^{\beta}}\left[1-e^{\alpha x^{\beta}}\right]^{\lambda-1}}{1-\left[1-e^{-\alpha x^{\beta}}\right]^{2\lambda}} \tag{10}$$

Reversed Hazard Function

$$\tau(x;\alpha,\beta,\lambda)=\frac{f(x)}{F(x)}=\frac{\lambda\alpha\beta e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1}}{\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]\left[1+\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda}\right]} \tag{11}$$

Cumulative Hazard Function

$$H(x;\alpha,\beta,\lambda)=-InF(x)=-In\left[\frac{1-\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda}}{1+\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda}}\right] \tag{12}$$

Survival Function

$$\bar{F}(x;\alpha,\beta,\lambda)=1-F(x)=\frac{1-\left[G(x)\right]^{\lambda}}{1+\left[G(x)\right]^{\lambda}}=\frac{1-\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda}}{1+\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda}} \tag{13}$$

Odd Function

$$O(x;\alpha,\beta,\lambda)=\frac{F(x)}{1-F(x)}=\frac{\left(1-e^{-\beta x}\right)^{\alpha}}{1-\left(1-e^{-\beta x}\right)^{\alpha}} \quad (14)$$

Plots of the density function (9) can be represented through Fig. 1. As it seems from Fig. 1, that the pdf of TIIHLEE can take different shapes according to different values of α and λ. It can be symmetric, right skewed, unimodal

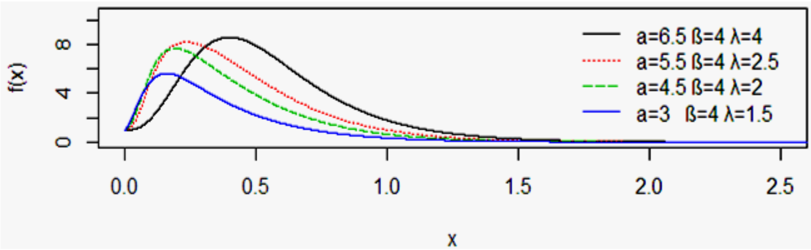

Figure 1: Plot of the Density Function of TIIHLEE

Plots of cumulative distribution function (8) for the TIIHLEE distribution are displayed in Fig. 3. The cdf graph tend to move at a constant value from 0 to 1 on the x-axis before projecting to one on the y-axis

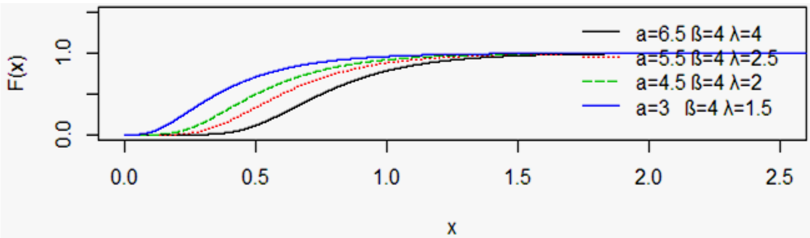

Figure 2: Plot of the Distribution Function (CDF) of TIIHL-EE

Plots of hazard function (10) for the TIIHLEE distribution are displayed in Fig. 3. It was examine from Fig. 3 show that the shape of the h(x) of the TIIHLEE shows the hazard functions can be constant, increasing or decreases, depending on the shape and scale parameters.

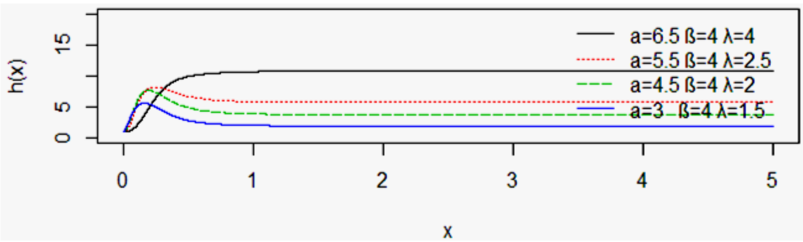

Figure 3: Plot of the Hazard Function of TIIHLEE

Plots of survival function (13) for the TIIHLEE distribution are displayed in Fig. 4. It was deduced from Fig. 4 that the shape parameters has strong effect on the survival plot. A decrease in the shape parameter tend to reduce the movement of the line on the y-axis and no plot is exceeding the bench mark of 1:

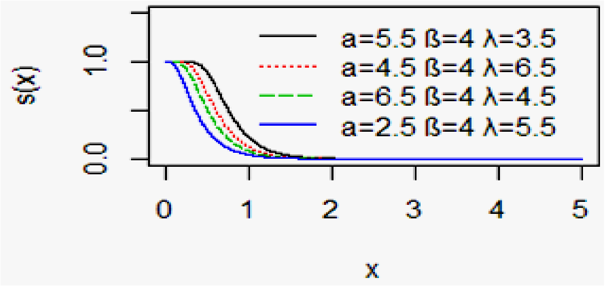

Figure 4: Plot of the Survival Function of TIIHLEE

2.2 Some Special Models of TIIHLEE

An approximation to other probabilistic models shows the flexibility of models for different assumed values. By considering the pdf of TIIHLEE distribution from (9) here we present some special cases of observed model.

- For λ=1, the THIILEE model reduces to Half Logistic Exponentiated Exponential (TIIHLE) model with α and β are shape and scale parameters and the pdf

$$f(x;\alpha,\beta)=\frac{2\alpha\beta e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1}}{\left[1+\left(1-e^{-\beta x}\right)\right]^{2}}$$

- For α=1, the THIILEE model reduces to Type II half Logistic Exponential (HLEE) model with λ and β are shape and scale parameters and the pdf

$$g(x;\beta,\lambda)=\frac{2\lambda\beta e^{-\beta x}\left[1-e^{-\beta x}\right]^{\lambda-1}}{\left[1+\left(1-e^{-\beta x}\right)^{\lambda}\right]^{2}}$$

- For α=λ=1, the TIIHLEE model generates a new model, Half Logistic Exponential (HLE) model with β scale parameters and the pdf

$$g(x;\beta)=\frac{2\beta e^{-\beta x}}{\left[\left(1+e^{-\beta x}\right)\right]^{2}}.$$

3. MATHEMATICAL PROPERTIES

In this section we provide some mathematical properties of the TIIHLEE

Moment of Proposed TIIHLEE Distribution

The r[th] moment for the TIIIHL-*G* family is derived.

$$E(X^r) = \mu^r = \int_0^\infty x^r f(x)dx = \int_0^\infty x^r \alpha\beta e^{-\beta x}(i+1)\left(1-e^{-\beta x}\right)^{\alpha(i+1)-1} dx \quad (15)$$

$$\mu^r = \alpha\beta(i+1)\int x^r e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha(i+1)-1} dx \quad (16)$$

However the equation (16) can written as

$$\mu^r = \frac{\Gamma\left(\frac{r}{2}+1\right)}{\alpha\beta^{r+2}(k+1)^{\frac{r}{2}+1}} w_{i.j,k} \quad (17)$$

The mean of the proposed TIIHLEE distribution is gotten by making μ^r moment equal to one (|r=1)

$$\mu_j^1 = \frac{\Gamma\left(\frac{3}{2}\right)}{\alpha\beta^3(k+1)^{\frac{3}{2}}} w_{i.j,k} \quad (18)$$

When r=2

$$\mu_j^2 = \frac{\Gamma(2)}{\alpha\beta^4(k+1)^2} w_{i.j,k} \quad (19)$$

When r=3

$$\mu_j^3 = \frac{\Gamma\left(\frac{5}{2}\right)}{\alpha\beta^5(k+1)^{\frac{5}{2}}} w_{i.j,k} \quad (20)$$

$$\mu_j^4 = \frac{\Gamma(3)}{\alpha\beta^6(k+1)^3} w_{i.j,k} \quad (21)$$

The variance of TIIHLEE can be obtain as

$$\text{var}(x) = \sigma^2 = E(x^2) - [E(x)]^2 = \mu_j^2 - \left(\mu_j^1\right)^2 \quad (22)$$

Substituting equation 19 and 18 into 22 to obtain the variance of TIIHLEE

$$\sigma^2 = \frac{\Gamma(2)}{\alpha\beta^4 (k+1)^2} w_{i.j,k} - \left(\frac{\Gamma\left(\frac{3}{2}\right)}{\alpha\beta^3 (k+1)^{\frac{3}{2}}} w_{i.j,k} \right)^2$$

$$\sigma^2 = \frac{\Gamma(2)}{\alpha\beta^4 (k+1)^2} w_{i.j,k} - \frac{\left(\Gamma\left(\frac{3}{2}\right)\right)^2}{\alpha\beta^6 (k+1)^{\frac{9}{4}}} w_{i.j,k} \tag{23}$$

Standard Deviation of TIIHLEE

$$\sigma = Std(x) = \sqrt{Var(x)}$$

$$\sigma = \sqrt{\frac{\Gamma(2)}{\alpha\beta^4 (k+1)^2} w_{i.j,k} - \frac{\left(\Gamma\left(\frac{3}{2}\right)\right)^2}{\alpha\beta^6 (k+1)^{\frac{9}{4}}} w_{i.j,k}} \tag{24}$$

Coefficient of Variation of TIIHLEE

$$CV = \frac{\text{Standard deviation } (x)}{E(x)}$$

$$CV = \frac{\sqrt{\frac{\Gamma(2)}{\alpha\beta^4 (k+1)^2} w_{i.j,k} - \frac{\left(\Gamma\left(\frac{3}{2}\right)\right)^2}{\alpha\beta^6 (k+1)^{\frac{9}{4}}} w_{i.j,k}}}{} \times \frac{\Gamma\left(\frac{3}{2}\right)}{\alpha\beta^3 (k+1)^{\frac{3}{2}}} w_{i.j,k}$$

$$CV = \frac{\sqrt{\frac{\Gamma(2)}{\alpha\beta^4 (k+1)^2} w_{i.j,k} - \frac{\left(\Gamma\left(\frac{3}{2}\right)\right)^2}{\alpha\beta^6 (k+1)^{\frac{9}{4}}} w_{i.j,k}} \left(\Gamma\left(\frac{3}{2}\right)\right)}{\alpha\beta^3 (k+1)^{\frac{3}{2}}}. \tag{25}$$

3.5.4 Moment Generating Function of TIIHLEE Distribution

A random variable x with pdf f(x) is defined a

$$M_x(t) = E(e^{tx})$$

$$E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r^1(x)$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r)$$

$$= \int_0^\infty e^{tx} \alpha\beta e^{-\beta x}(i+1)\left(1-e^{-\beta x}\right)^{\alpha(i+1)-1} dx$$

$$= \alpha\beta \int_0^\infty e^{(t-\beta)x}(i+1)\left(1-e^{-\beta x}\right)^{\alpha(i+1)-1} dx$$

$$= \sum_{i,j,k=0}^{\infty} \frac{t^r \Gamma\left(\frac{r}{2}+1\right)}{r!\alpha\beta^{r+2}(k+1)^{\frac{r}{2}+1}} w_{i,j,k} \tag{26}$$

Obtaining the first moment from the moment generating function from 3.51 we differentiate with respect to

$$\mu_j^1 = M_x^1(t)$$

$$M_x^1(t) = \frac{rt^{r-1}}{r!} \sum_{i,j,k=0}^{\infty} \frac{\Gamma\left(\frac{r}{2}+1\right)}{r!\alpha\beta^{r+2}(k+1)^{\frac{r}{2}+1}} w_{i,j,k}$$

$$M_x^1(0) = \sum_{i,j,k=0}^{\infty} \frac{\Gamma\left(\frac{r}{2}+1\right)}{r!\alpha\beta^{r+2}(k+1)^{\frac{r}{2}+1}} w_{i,j,k} \tag{27}$$

3.2 Incomplete Moment

The s^{th} incomplete moment of the TIIHLEE distribution say $E_s(t)$ is obtained by the pdf in equation (9)

$$E_s(t) = \int_0^t x^s f(x)dx = \int x^s \frac{2\lambda\alpha\beta e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1}\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda-1}}{\left[1+\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda}\right]^2} \tag{28}$$

Hence by applying binomial expansion

$$E_s(t) = \sum_{i,j=0}^{\infty} (-1)^{i+j}(i+1)2\lambda\beta \binom{\lambda(i+1)-1}{i} \int_0^t x^{s+1} e^{-\beta(j+1)} dx$$

After some expression, the sth moment of TIIHLEE distribution takes the form

$$E_s(t)=\sum_{i,j=0}^{\infty}\frac{\eta_{i,j}}{\alpha\beta(j+1)^{\frac{s}{2}+1}}\int_0^{\alpha\beta(j+1)t^2} y^{\frac{s}{2}}e^{-y}dy$$

Lower incomplete gamma function, Hence the sth incomplete moment of the TIIHLEE distribution

$$E_s(t)=\sum_{i,j=0}^{\infty}\frac{\eta_{i,j}}{\alpha\beta\left(j+1\right)^{\frac{s}{2}+1}}\gamma\left(\frac{s}{2}+1,\alpha\beta(j+1)t^2\right) \qquad s=1,2,3,.... \tag{29}$$

where $\gamma(.,.)$ is the lower incomplete gamma function

3.3 Probability Weighted Moment (PWMS)

Suppose a random variable X has *TIIHL-G* family of distributions the PWMs, denoted by $\tau_{r,s}$

Then

$$\tau_{r,s}=E\left[X^r F(x)^s\right]=\int_{-\infty}^{\infty}x^r f(x)\left[F(x)\right]^s dx \tag{30}$$

$$\tau_{r,s}=\int_{-\infty}^{\infty}\sum_{i=0}^{\infty}\sum_{z=0}^{\infty}\eta_i x^r g(x)\left[G(x)\right]^{z+\lambda(i+1)-1}dx \tag{31}$$

Substituting equation (6) and (7) in (13) we obtain the TIIHL-EE Probability Weighted Moment (PWMS)

$$\tau_{r,s}=\int_{-\infty}^{\infty}\sum_{i=0}^{\infty}\sum_{z=0}^{\infty}S_z\eta_i x^r\alpha\beta e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1}\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{z+\lambda(i+1)-1}dx \tag{32}$$

$$\tau_{r,s}=\sum_{i=1}^{\infty}\sum_{z=0}^{\infty}S_z\eta_i\tau_{r,z+\lambda(i+1)-1}$$

$$\tau_{r,z+\lambda(i+1)-1}=\int_{-\infty}^{\infty}x^r g(x)\left[G(x)\right]^{z+\lambda(i+1)-1}$$

$$\tau_{r,z+\lambda(i+1)-1}=\int_{-\infty}^{\infty}x^r\alpha\beta e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1}\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{z+\lambda(i+1)-1}$$

Then TIIHL-EE Probability Weighted Moment (PWMS)

$$\tau_{r,s}=\sum_{i,j,l=0}^{\infty}\sum_{k=0}^{s}\frac{\eta_{i,j}\eta_{k,l}\Gamma(r+1)}{\left[\beta(l+j+1)\right]^{r+1}} \tag{33}$$

3.4 Mean Deviations

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. These are known as the mean deviation about the mean and the mean deviation about the median which is defined by

$$\delta_1(X)=\int_0^{\infty}|x-\mu|\,f(x)dx \text{ and } \delta_2(X)=\int_0^{\infty}|x-\mu|\,f(x)dx$$

Respectively, where μ=E(X) and M=Median(X) denote the median. The measure $\delta_1(X)$ and $\delta_2(X)$ can be express as

$$\delta_1(X)=2\mu F(\mu)-2\mu+\int_{\mu}^{\infty}xf(x)dx$$

$$\delta_1(X)=2\mu F(\mu)-2T(\mu) \tag{34}$$

$$\delta_2(X)=2\int_{M}^{\infty}xf(x)dx-\mu$$

$$\delta_2(X)=\mu-2T(m) \tag{35}$$

where T(μ) and T(m) are first incomplete moment which are obtained from (28).

Mean deviation is practically use to explain the behavior of Bonferron and Lovenz curve, this curve are mostly applied theoretically in many field such as economics, reliability, demography, insurance and medicine.

The Lovenz and Boneferroni curve are obtained

Lovenz Curve

$$L_F(X)=\frac{1}{E(X)}\int_0^{x}tf(t)dt=\frac{2\sum\limits_{i,j=0}^{\infty}\frac{\eta_{i,j}}{\left(\beta(j+1)\right)^{\frac{3}{2}}}\gamma\left(\frac{3}{2},\alpha\beta(j+1)\right)}{\sum\limits_{i,j=0}^{\infty}\frac{\eta_{i,j}\sqrt{\pi}}{\left[\alpha\beta(j+1)\right]^{\frac{3}{2}}}} \tag{36}$$

Boneferroni Curve

$$B_F(X)=\frac{L_F(X)}{F(x;\alpha,\beta,\lambda)}=\frac{1+\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda}\sum\limits_{i,j=0}^{\infty}\frac{\eta_{i,j}}{\left[\alpha\beta(i+1)\right]^{\frac{3}{2}}}}{\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda}\sum\limits_{i,j=0}^{\infty}\left[\frac{\eta_{i,j}\sqrt{\pi}}{\left[\alpha\beta(j+1)\right]^{\frac{3}{2}}}\right]} \tag{37}$$

3.5 Asymptotic Behavior

We seek to investigate the behavior of the proposed model (TIIHLEE) as $x \to 0$ and $x \to \infty$. This approach involves considering the $\lim_{x\to 0} f(x)$ and $\lim_{x\to\infty} f(x)$

As $x \to 0$

$$\lim_{x\to 0} \frac{2\lambda\alpha\beta e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1}\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda-1}}{\left[1+\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda}\right]^{2}} = 0$$

$$\lim_{x\to \infty} \frac{2\lambda\alpha\beta e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1}\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda-1}}{\left[1+\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda}\right]^{2}} = 0$$

This implies that as x tends to zero and x tends to infinity the probability density function of TIIHLEE depends only on the two shape parameters λ *and* α, with β scale

3.6 Renyi Entropy

The entropy of a random variable X is a measure of variation of uncertainty and has been used in many field such as physic, engineering and economics

$$I_{\delta}(x) = \frac{1}{1-\delta}\log\int_{-\infty}^{\infty} f(x)^{\delta}dx \qquad \delta > 0 \text{ and } \delta \neq 1 \tag{38}$$

Applying the binomial theory in (16) and (18) to the pdf in equation (36), then pdf can be expressed as

$$f(x)^{\delta} = \sum_{i,j=0}^{\infty} t_{i,j}x^{\delta}e^{\beta(\delta+j)}$$

where

$$t_i = \left(2\lambda\right)^{\delta}(-1)^{i}\binom{2\delta+i+1}{i}$$

$$t_{i,j} = (2\lambda\alpha\beta)^{\delta}(-1)^{i+j}\binom{2\delta+i-1}{i}\binom{\lambda(\delta+i)-\delta}{j}$$

Therefore, the Renyi entropy of TIIHLEE distribution is given by

$$I_{\delta}(X) = \frac{1}{1-\delta}\log\left[\sum_{i,j=0}^{\infty}\frac{t_{i,j}\Gamma\left(\frac{\delta+1}{2}\right)}{\alpha\beta(\delta+j)^{\frac{\delta+1}{2}}}\right] \tag{39}$$

The q-Renyi Entropy of TIIHL-G

$$H_q(x)=\frac{1}{1-q}\log\left(1-\sum_{i=1}^{\infty}t_i\int_{-\infty}^{\infty}g(x)^q\left[G(x)\right]^{\lambda(i+q)-q}dx\right)$$

Then q-Renyi Entropy of TIIHL-EE

$$H_q(x)=\frac{1}{1-q}\log\left(1-\sum_{i=1}^{\infty}t_i\int_{-\infty}^{\infty}\left[\alpha\beta e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1}\right]^q\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda(i+q)-q}dx\right) \quad (40)$$

3.7 Order of Statistics

Order statistics appears in various fields of statistics, such as reliability and life testing data have been widely studied. Let $Y_1, Y_2, \ldots Y_n$ be independently and identically distributed (i .i.d) random variable with their corresponding cumulative function (cdf) $F(x)$. Let $X_{(1)}$ be the smallest (first order statistic) of the $Y_1, Y_2, \ldots Y_n$, $X_{(2)}$ is the second order statistics greater than $X_{(1)}$ and $X_{(n)}$ is the largest order statistics. Hence, $X_{(1)} < X_{(2)} \ldots < X_{(n)}$ is the order statistics corresponding to random variable $Y_1, Y_2, \ldots Y_n$ (see kapur and Sexena (1960)).

The density $f_{n:i}(x)$ of the ith order statistics, for $i = 1, \ldots \ldots n,$ from independent identical distribution random variable $Y_1 \ldots \ldots Y_n$ is given by

$$f_{n:i}(x) = \frac{f(x)}{B(i, n-i+1)}F(x)^{i-1}\left(1-F(x)\right)^{n-i}.$$

3.7.1 Order Statistic of TIIHLEE

$$f(x;\alpha,\beta,\lambda)=\frac{f(x;\alpha,\beta,\lambda)}{B(r,n-r-1)}\sum_{v=0}^{n-r}(-1)^v\binom{n-r}{v}F(x;\alpha,\beta,\lambda)^{v+r-1} \quad (41)$$

$B(.,.)$ is the beta function. The pdf of the rth order statistic for TIIHLEE distribution is derived by substituting (8) and (9) in (41) as follows

$$f(x;\alpha,\beta,\lambda)=\frac{1}{B(r,n-r+1)}\sum_{v=0}^{n-r}(-1)^v\binom{n-r}{v}\frac{\alpha\beta\lambda e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1}\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda(v+r)-1}}{\left[1+\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda+r+1}} \quad (42)$$

Applying the binomial expansion (16) in (40), then we have

$$f(x;\alpha,\beta,\lambda)=\frac{1}{B(r,n-r+1)}\sum_{v=0}^{n-r}(-1)^{v+i}\binom{n-r}{v}\binom{v+r+i}{i}$$

$$\alpha\beta\lambda e^{-\beta x}\left(1-e^{\beta x}\right)^{\alpha-1}\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda(v+r+i)-1}$$

Again, using the binomial expansion (16) in the previous equation, then the pdf of the r^th order statistic for TIIHLEE distribution is obtained as follows

$$f(x;\alpha,\beta,\lambda)=\frac{1}{B(r,n-r+1)}\sum_{v=0}^{n-r}\sum_{I=0}^{v+r-1}\sum_{J=0}^{\infty}\eta^{*}\alpha\beta\lambda e^{-\beta(j+1)x^{2}} \quad (43)$$

$$\eta^{*}=(-1)^{v+i+j}\binom{n-r}{v}\binom{v+r+i}{i}\binom{\lambda(v+r+i)-1}{j}$$

The distribution of the smallest and largest order statistics are obtained by putting *r* =1 and *r* =*n* in (41) respectively as follows

$$f_{(1)}(x;\alpha,\beta,\lambda)=\frac{1}{B(r,n-r+1)}\sum_{v=0}^{n-r}\sum_{I=0}^{v+r-1}\sum_{J=0}^{\infty}\eta^{**}\alpha\beta\lambda e^{-\beta(j+1)x^{2}}$$

$$\eta^{**}=(-1)^{v+i+j}\binom{n-1}{v}\binom{v+1+i}{i}\binom{\lambda(v+1+i)-1}{j}$$

However

$$f_{(n)}(x;\alpha,\beta,\lambda)=n\sum_{v=0}^{n-r}\sum_{j=0}^{v\infty}\eta^{***}\alpha\beta\lambda e^{-\beta(j+1)x^{2}}$$

$$\eta^{***}=(-1)^{v+n+j}\binom{n-1}{v}\binom{v+n+i}{i}\binom{\lambda(v+n+i)-1}{j}$$

3.8 Skewness and Kurtosis of the TIIHLE Distribution

Two approaches are used in obtaining the Skewness and kurtosis of the TIIHLEE distribution. These approaches include the measure of kurtosis (k.u) and skewness (s.k) based on moments and quantiles. In the moments based approach,

$$S.K=\frac{\mu_3^{/}-3\mu\mu_2^{/}+2\mu^2}{\left(\mu_2^{/}-\mu^2\right)^{3/2}} \quad (44)$$

and

$$K.U=\frac{\mu_4^{/}-4\mu\mu_3^{/}+6\mu^2\mu_2^{/}-3\mu^4}{\left(\mu_2^{/}-\mu^2\right)^{2}} \quad (45)$$

The quantile measure based approach of evaluating skewness and kurtosis of a distribution is particularly useful when the quantile function of a distribution exists in closed form or in a simple analytic expression.

4. PARAMETER ESTIMATION

From parameter estimation we obtain the maximum likelihood estimator for TIIHLEE in the case of complete sample and simulation study is performed.

4.1 Maximum Likelihood Estimator (MLE)

Let $x_1; x_2;\dots, x_n$ be a random sample of size *n* from the TIIHL family of distributions (*α; β; λ*). The log-likelihood function for the vector of parameters L = $(\alpha; \beta; \lambda)^T$ can be expressed as

Let

$$f\left(x_1, x_2, x_3, ..., x_n; \alpha, \beta, \lambda\right) = \prod_{i=1}^{n} f(x)$$

$$f(x) = \frac{2\lambda\alpha\beta e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1}\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda-1}}{\left[1+\left[\left(1+e^{-\beta x}\right)^{\alpha}\right]^{\lambda}\right]^{2}}$$

$$\prod_{i=1}^{n}\left[\frac{2\lambda\alpha\beta e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1}\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda-1}}{\left[1+\left[\left(1+e^{-\beta x}\right)^{\alpha}\right]^{\lambda}\right]^{2}}\right]$$

The log-likelihood function is expressed as

$$l = n\log 2 + n\log\lambda + n\log\alpha + n\log\beta + \alpha - 1\sum_{i=1}^{n}\log\left(1-e^{-\beta x}\right) + (\lambda-1)\sum_{i=1}^{n}\log\left(1-e^{-\beta x}\right)^{\alpha}$$

$$-2\sum_{i=1}^{n}\log\left[1+\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda}\right] \tag{46}$$

Taking the first partial derivatives of ℓ(*x*; α, β, λ) of (46) with respect to α, β, and λ and letting them equal zero, we obtain a nonlinear system of equations.

$$\frac{\delta l}{\delta\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(1-e^{-\beta x}\right) - \frac{(\lambda-1)(\alpha-1)\beta\sum_{i=1}^{n} x\left(1-e^{-\beta x}\right)^{\alpha-1}}{\left(1-e^{-\beta x}\right)}$$

$$-2\sum_{i=1}^{n}\frac{\left[\left(1-e^{-\beta x}\right)\right]^{\lambda}\log\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]}{\left[1+\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda}} = 0$$

$$\frac{\delta l}{\delta\beta} = \frac{n}{\beta} + \frac{(\alpha-1)\sum_{i=1}^{n}\beta e^{\beta x}}{\left(1-e^{-\beta x}\right)} - (\lambda-1)\beta\sum_{i=1}^{n} x\left(1-e^{-\beta x}\right) - 2\sum_{i=1}^{n}\frac{\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda-1} x e^{-\beta x}}{\left[1+\left[(1-e^{-\beta x})^{\alpha}\right]^{\lambda}\right]} = 0$$

$$\frac{\delta l}{\delta\lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}\log\left(1-e^{-\beta x}\right)-2\sum_{i=1}^{n}\frac{\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda}\log\left[\left(1-e^{-\beta x}\right)^{\alpha}\right]}{\left[1+\left(1-e^{-\beta x}\right)^{\alpha}\right]^{\lambda}}=0\,.$$

The above derived equations are in the complex form; therefore the exact solution of ML estimator for unknown parameters is not possible. So it is convenient to use nonlinear Newton Raphson algorithm for exact numerically solution to maximize the above likelihood function.

However, the above equation cannot be solved analytically and statistical software can be used to solve them numerically using iterative methods.

Solving the non linear system of equation of $\frac{\delta l}{\delta\alpha}=0$, $\frac{\delta l}{\delta\beta}=0$ and $\frac{\delta l}{\delta\lambda}=0$ we than obtain the Maximum likelihood estimate α, β and λ respectively. Information matrix of 3x3 will be obtained through

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\\ \hat{\lambda}\end{pmatrix}=\left[\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix}\begin{pmatrix}\hat{V}_{\alpha\alpha} & \hat{V}_{\alpha\beta} & \hat{V}_{\alpha\lambda}\\ \hat{V}_{\beta\alpha} & \hat{V}_{\beta\beta} & \hat{V}_{\beta\lambda}\\ \hat{V}_{\lambda\alpha} & \hat{V}_{\lambda\alpha} & \hat{V}_{\lambda\lambda}\end{pmatrix}\right]$$

$$V^{-1}=-E\begin{pmatrix}V_{\alpha\alpha} & V_{\alpha\beta} & V_{\alpha\lambda}\\ V_{\beta\alpha} & V_{\beta\beta} & V_{\beta\lambda}\\ V_{\lambda\alpha} & V_{\lambda\beta} & V_{\lambda\lambda}\end{pmatrix}$$

$$V_{\alpha\alpha}=\frac{\delta^{2}l}{\delta\alpha^{2}}\qquad V_{\beta\alpha}=V_{\alpha\beta}=\frac{\delta^{2}l}{\delta\alpha\beta}\qquad V_{\lambda\beta}=V_{\alpha\lambda}=\frac{\delta^{2}l}{\delta\alpha\lambda}$$

$$V_{\beta\beta}=\frac{\delta^{2}l}{\delta\beta^{2}}\qquad V_{\lambda\lambda}=\frac{\delta^{2}l}{\delta\lambda^{2}}\qquad V_{\lambda\beta}=V_{\beta\lambda}=\frac{\delta^{2}l}{\delta\beta\lambda}$$

Solution to the above equation inverse dispersion matrix yields the asymptotic variance and covariance of the maximum likelihood estimators $\hat{\alpha},\hat{\beta}$ *and* $\hat{\lambda}$.

4.1.1 Confidence Interval for the Parameters

The approximate confidence intervals for 100(α-1)% for α, β and λ is given respectively by

$$\hat{\alpha}\pm Z_{\frac{\alpha}{2}}\sqrt{V_{\alpha\alpha}}\,,\qquad \hat{\beta}\pm Z_{\frac{\alpha}{2}}\sqrt{V_{\beta\beta}}\quad\text{and}\quad \hat{\lambda}\pm Z_{\frac{\alpha}{2}}\sqrt{V_{\lambda\lambda}}$$

where the $Z_{\frac{\alpha}{2}}$ is the α^{th} percentiles of the standard normal distribution.

5. SIMULATION STUDY

In this section the simulation study is conducted using the quartile function of the TIIHLEE distribution in equation 47 with the help of R-statistics package. Tables 2 and 4; contains the mean, standard deviation and median of the 3-parameter TIIHLEE distribution for different parameter values. While Tables 3 and 5 contains values of Skewness and Kurtosis obtained with the same combination of parameters.

Different sample sizes are generated through the experiments at sample size n=20, 60 and 100. The generation of TIIHLEE distribution is very simple, if U has a uniform (0,1) random number, then

$$x = -\frac{1}{\beta} In\left(1-u^{\frac{1}{\alpha}}\right)\left(\frac{u}{2-u}\right)^{\frac{1}{\lambda}} \tag{47}$$

where α and λ are shape parameters while β is a scale parameter.

Table 1: The Mean, Median and Standard Deviation of TIIHLEE Distribution for β= 2, 4 and 6 for n=20

λ	α	β=2			β=4			β=6		
		Mean	**Median**	**SD(σ)**	**Mean**	**Median**	**SD(σ)**	**Mean**	**Median**	**SD(σ)**
0.5	0.5	0.1989	0.0152	0.4203	0.0681	0.0043	0.1525	0.0357	0.0053	0.0684
	0.8	0.3104	0.0438	0.6605	0.0482	0.0044	0.0692	0.1515	0.0409	0.2530
	1.5	0.3647	0.1005	0.5281	0.1238	0.0285	0.2850	0.0833	0.0134	0.1383
	2.5	0.3413	0.0873	0.5648	0.2789	0.0273	0.4476	0.0986	0.0373	0.1750
	5.0	0.4075	0.1611	0.5779	0.1331	0.0528	0.1808	0.1714	0.0687	0.2543
0.8	0.5	0.0545	0.0025	0.1172	0.0569	0.0062	0.1071	0.0594	0.0308	0.0758
	0.8	0.2447	0.1137	0.4610	0.1697	0.0330	0.2433	0.0741	0.0058	0.1249
	1.5	0.3357	0.1496	0.5279	0.1656	0.0603	0.2406	0.1010	0.0367	0.1394
	2.5	0.5494	0.4255	0.6271	0.2016	0.0634	0.3354	0.1399	0.0108	0.2018
	5.0	0.3046	0.1262	0.4796	0.3678	0.2126	0.4254	0.1371	0.0877	0.1239
1.5	0.5	0.2968	0.0677	0.6832	0.2026	0.0676	0.2798	0.0503	0.0132	0.0907
	0.8	0.3048	0.1676	0.3512	0.1782	0.0902	0.2314	0.1436	0.0764	0.1592
	1.5	0.3529	0.3526	0.2965	0.1409	0.0888	0.1397	0.1317	0.0472	0.1696
	2.5	0.6704	0.4085	0.7552	0.4558	0.4222	0.3849	0.2097	0.1334	0.1916
	5.0	0.3236	0.1492	0.3860	0.2595	0.2415	0.1991	0.3037	0.2544	0.2423
2.5	0.5	0.2944	0.1003	0.6585	0.1460	0.0606	0.2036	0.1525	0.0883	0.1948
	0.8	0.2100	0.1118	0.2160	0.2746	0.1913	0.3020	0.0901	0.0542	0.1295
	1.5	0.5283	0.1579	0.5945	0.1326	0.1043	0.1180	0.1139	0.0482	0.1253
	2.5	0.7175	0.5612	0.5693	0.3462	0.2713	0.2972	0.1772	0.1461	0.1434
	5.0	0.8504	0.6630	0.7205	0.4659	0.3970	0.3800	0.3038	0.2583	0.2169
5.0	0.5	0.2838	0.1611	0.3380	0.1007	0.0510	0.1120	0.0773	0.0222	0.1178
	0.8	0.3873	0.0958	0.5214	0.1798	0.0749	0.2594	0.1188	0.0453	0.1673
	1.5	1.0161	0.8203	1.0118	0.2101	0.1573	0.1928	0.1942	0.0984	0.2293
	2.5	0.7727	0.4607	0.6987	0.36480	0.2548	0.3394	0.2135	0.1546	0.1636
	5.0	1.0466	1.0256	0.6683	0.4777	0.4738	0.3700	0.3201	0.2352	0.2369

Source: Simulated data of TIIHLEE Distribution

Table 2
The Skewness and Kurtosis of TIIHLEE Distribution for β = 2, 4 and 6 for n=20

λ	α	β=2		β=4		β=6	
		Skewness	**Kurtosis**	**Skewness**	**Kurtosis**	**Skewness**	**Kurtosis**
0.5	0.5	2.7605	7.0995	2.4984	4.7120	2.4713	5.0467
	0.8	3.3230	10.7045	1.4158	1.0807	1.8829	2.3483
	1.5	1.5296	1.0886	3.2339	9.6584	1.7717	1.9833
	2.5	2.4444	6.0404	1.7785	2.3981	2.3062	3.9294
	5.0	1.4810	0.7401	1.5310	1.1522	1.7470	1.7764
0.8	0.5	2.4266	4.3013	1.8686	1.8945	1.4137	0.8688
	0.8	3.4090	11.2443	1.2780	0.1304	1.7608	1.9868
	1.5	2.1940	3.4301	2.3149	5.3151	1.3854	0.6494
	2.5	1.6010	2.1372	1.9330	2.0613	1.3826	0.7340
	5.0	2.1832	3.7032	1.3081	0.9087	1.4358	1.8095
1.5	0.5	2.9852	7.8715	1.3067	0.5216	2.2998	3.9465
	0.8	1.8693	3.8279	1.6048	1.4372	1.3556	0.8395
	1.5	0.5617	-0.6845	1.5540	2.3089	1.4908	1.3730
	2.5	2.1974	5.1908	0.4372	-1.1030	0.8392	-0.4987
	5.0	1.1651	-0.2433	0.9197	-0.1162	0.4043	-1.0866
2.5	0.5	3.3137	10.1083	1.7316	2.3636	1.6048	1.7137
	0.8	0.9914	-0.0622	1.8592	3.1977	2.1380	4.1346
	1.5	0.7873	-1.0130	0.9442	0.2347	1.1134	-0.3541
	2.5	0.5693	-1.1283	0.9424	-0.1199	1.1242	0.40688
	5.0	0.8493	-0.1838	0.9948	0.3070	1.1432	0.7627
5.0	0.5	1.5859	2.3822	1.1066	0.0247	2.0824	4.0687
	0.8	1.1901	0.1245	1.6618	1.3733	2.0176	3.0843
	1.5	2.6639	7.6528	1.2132	0.3007	2.1021	3.9069
	2.5	0.7732	-0.8997	2.0568	4.6337	0.7497	-0.5581
	5.0	0.8684	-0.0182	1.3353	1.4906	1.9000	3.3819

Source: Simulated data of TIIHLEE Distribution

Table 3
The Mean, Median and Standard Deviation of TIIHLEE Distribution for β= 2, 4 and 6 for n=60

λ	α	β=2			β=4			β=6		
		Mean	Median	SD(σ)	Mean	Median	SD(σ)	Mean	Median	SD(σ)
0.5	0.5	0.08423	0.0068	0.1648	0.0775	0.02016	0.1309	0.0752	0.0149	0.1278
	0.8	0.19903	0.0140	0.4479	0.1086	0.0180	0.2066	0.0676	0.0117	0.1173
	1.5	0.2151	0.0293	0.4423	0.2002	0.0299	0.3509	0.1006	0.0235	0.1806
	2.5	0.3149	0.0558	0.6801	0.1766	0.0387	0.2848	0.0878	0.0410	0.1440
	5.0	0.3115	0.0745	0.6694	0.1800	0.0710	0.2477	0.1456	0.0619	0.2287
0.8	0.5	0.2991	0.0536	0.5676	0.0837	0.0075	0.1618	0.0678	0.0067	0.1356
	0.8	0.2983	0.0518	0.4845	0.1330	0.0256	0.2227	0.0590	0.0144	0.0995
	1.5	0.3727	0.1054	0.5849	0.2005	0.0846	0.2589	0.0967	0.0533	0.1160
	2.5	0.4361	0.1826	0.6253	0.1836	0.1185	0.2055	0.1373	0.0743	0.1942
	5.0	0.4334	0.1585	0.6265	0.2570	0.1025	0.3471	0.1980	0.1152	0.2462
1.5	0.5	0.1938	0.0511	0.3036	0.1039	0.0317	0.2239	0.0869	0.0152	0.1539
	0.8	0.4307	0.1574	0.5894	0.12297	0.0536	0.2002	0.0927	0.0386	0.1323
	1.5	0.4810	0.2282	0.6149	0.2258	0.1218	0.2595	0.1221	0.0605	0.1508
	2.5	0.7448	0.4427	0.8611	0.3089	0.2574	0.2814	0.1524	0.1038	0.1690
	5.0	0.7680	0.5061	0.8800	0.3103	0.2546	0.2719	0.2129	0.1535	0.1897
2.5	0.5	0.1994	0.0523	0.3023	0.1513	0.0620	0.2373	0.0648	0.0211	0.1066
	0.8	0.3461	0.1833	0.4401	0.1149	0.0701	0.1288	0.1128	0.0679	0.1616
	1.5	0.4605	0.3217	0.4265	0.2836	0.1390	0.3284	0.1688	0.1170	0.1862
	2.5	0.5518	0.4162	0.5056	0.3076	0.1841	0.3340	0.2157	0.1385	0.2146
	5.0	0.8365	0.6974	0.6349	0.4909	0.4045	0.3887	0.2854	0.2346	0.2497
5.0	0.5	0.2144	0.0465	0.4137	0.11906	0.0413	0.1794	0.1004	0.0357	0.1415
	0.8	0.30909	0.2173	0.3803	0.1545	0.1086	0.1901	0.1030	0.0724	0.1267
	1.5	0.4749	0.3968	0.4453	0.2375	0.1984	0.2225	0.1583	0.1323	0.1484
	2.5	0.6325	0.5666	0.4919	0.3162	0.2833	0.2459	0.2108	0.1889	0.1639
	5.0	0.8676	0.8175	0.5484	0.4338	0.4088	0.2742	0.2892	0.2725	0.1828

Source: Simulated data of TIIHLEE Distribution

Table 4
The Skewness and Kurtosis of TIIHLEE Distribution for β = 2, 4 and 6 for n=100

λ	α	β=2		β=4		β=6	
		Skewness	**Kurtosis**	**Skewness**	**Kurtosis**	**Skewness**	**Kurtosis**
0.5	0.5	2.6522	6.8913	2.7909	9.8454	3.1338	12.1510
	0.8	2.9206	7.5651	2.6814	6.7944	2.5729	6.8354
	1.5	2.5596	6.0960	2.2126	4.2790	2.8232	8.8073
	2.5	4.4900	24.2552	2.1319	4.0126	2.8536	8.7059
	5.0	3.7787	15.0191	1.8697	3.0960	2.2283	4.2123
0.8	0.5	2.8030	7.9480	2.8309	8.3675	2.6712	7.0413
	0.8	1.9608	3.1489	2.0539	3.6068	2.1198	3.7680
	1.5	2.1851	4.3533	1.5233	1.8083	1.4757	1.3875
	2.5	2.2242	4.4239	1.7619	3.6404	2.1930	5.2340
	5.0	2.2864	5.3790	2.0579	4.0422	2.2199	5.1064
1.5	0.5	2.1765	4.4349	4.8624	27.1737	2.4631	0.7902
	0.8	1.6620	1.8107	3.0071	9.5212	2.0197	3.2161
	1.5	1.7293	3.0553	1.4559	1.4724	1.6026	1.8726
	2.5	1.7445	2.6305	1.0035	0.2863	1.7211	2.5854
	5.0	2.2933	6.3376	1.3676	2.1498	1.1703	1.2226
2.5	0.5	2.0054	3.6447	3.2458	13.3146	2.3983	5.5235
	0.8	1.9269	3.4403	2.3731	8.5481	2.2128	5.7122
	1.5	1.1315	0.5245	1.5902	2.0733	2.6418	9.5729
	2.5	1.4431	1.6562	1.9674	4.1451	1.5559	2.5248
	5.0	1.4212	2.9125	0.9914	0.2736	2.1122	7.1203
5.0	0.5	3.5894	15.228	2.9161	11.3122	1.6663	1.5679
	0.8	2.8351	11.0912	2.8351	11.0912	2.8350	11.0911
	1.5	2.2349	7.3552	2.2349	7.3552	2.2349	7.3552
	2.5	1.8990	5.5484	1.8990	5.5484	1.8990	5.5484
	5.0	1.5745	4.0175	1.5745	4.0175	1.5745	4.0175

Source: Simulated data of TIIHLEE Distribution

Table 5: The Mean, Median and Standard Deviation of TIIHLEE Distribution for β= 2, 4 and 6 for n=100

λ	α	β=2			β=4			β=6		
		Mean	Median	SD(σ)	Mean	Median	SD(σ)	Mean	Median	SD(σ)
0.5	0.5	0.2123	0.0159	0.4363	0.0817	0.0099	0.1597	0.0689	0.0091	0.1432
	0.8	0.1643	0.0162	0.4091	0.0823	0.0274	0.1202	0.0829	0.0179	0.1521
	1.5	0.2098	0.0346	0.3257	0.1417	0.0231	0.2834	0.1093	0.0206	0.2099
	2.5	0.2773	0.0578	0.5045	0.1848	0.0528	0.2769	0.1386	0.03971	0.2142
	5.0	0.4382	0.0940	0.6659	0.2229	0.1126	0.2908	0.1217	0.0500	0.1856
0.8	0.5	0.2259	0.0499	0.4268	0.1102	0.0107	0.2535	0.0668	0.0126	0.1090
	0.8	0.2968	0.0654	0.4934	0.1506	0.0265	0.2942	0.1097	0.0229	0.2139
	1.5	0.4683	0.1654	0.6033	0.1291	0.0514	0.1820	0.1065	0.0445	0.1514
	2.5	0.4110	0.2027	0.2027	0.1993	0.0875	0.2539	0.1537	0.0654	0.1966
	5.0	0.4974	0.2363	0.6272	0.2688	0.0994	0.3555	0.2159	0.1061	0.2507
1.5	0.5	0.2493	0.0492	0.5048	0.1087	0.0284	0.1847	0.0830	0.0347	0.1060
	0.8	0.2681	0.1144	0.4024	0.1254	0.0477	0.2156	0.0973	0.0467	0.1252
	1.5	0.47460	0.2679	0.5822	0.2617	0.1298	0.3513	0.1767	0.1087	0.2038
	2.5	0.6589	0.4162	0.6815	0.3000	0.1742	0.3318	0.2133	0.1337	0.2404
	5.0	0.6807	0.4031	0.7254	0.3431	0.2380	0.3024	0.2522	0.1737	0.2494
2.5	0.5	0.2412	0.1077	0.3102	0.1387	0.0443	0.2052	0.1056	0.0375	0.1585
	0.8	0.4220	0.2248	0.5760	0.1999	0.1000	0.2698	0.1194	0.0637	0.1661
	1.5	0.6399	0.3148	0.7672	0.2708	0.1612	0.2035	0.2035	0.1268	0.2267
	2.5	0.5829	0.3888	0.5447	0.3501	0.2387	0.3324	0.2122	0.1720	0.1692
	5.0	0.9057	0.7494	0.7569	0.4072	0.2658	0.3462	0.2807	0.1988	0.2273
5.0	0.5	0.2842	0.0756	0.4100	0.1194	0.0430	0.1678	0.0955	0.0309	0.1536
	0.8	0.3169	0.1758	0.3915	0.1852	0.1298	0.1953	0.1253	0.0674	0.1430
	1.5	0.6030	0.4009	0.5928	0.2792	0.2578	0.2077	0.1600	0.1139	0.1388
	2.5	0.7666	0.5978	0.6534	0.3061	0.2356	0.2646	0.2549	0.2234	0.1932
	5.0	1.0102	0.8665	0.5901	0.5215	0.4832	0.3370	0.36333	0.2794	0.2596

Source: Simulated data of TIIHLEE Distribution

Table 6
The Skewness and Kurtosis of TIIHLEE Distribution for β = 2, 4 and 6 for n=100

λ	α	β=2		β=4		β=6	
		Skewness	Kurtosis	Skewness	Kurtosis	Skewness	Kurtosis
0.5	0.5	2.8685	8.9665	6.3034	47.3527	3.4050	13.9613
	0.8	4.0304	17.604	1.8024	2.5208	3.2589	13.6198
	1.5	2.0912	4.8083	3.0731	10.4076	3.2059	12.1285
	2.5	2.6367	7.4831	2.0091	3.4827	1.9478	3.2316
	5.0	2.0684	4.1935	1.8383	3.1648	2.5220	6.9952
0.8	0.5	3.1775	11.2563	3.9926	18.8621	2.2427	4.9369
	0.8	1.9742	3.1150	2.6719	6.5145	3.2062	11.7088
	1.5	1.6101	1.9160	2.0418	3.8964	2.3870	6.1772
	2.5	4.0405	23.0941	1.5004	1.1130	1.9325	3.6083
	5.0	1.8858	4.2339	1.7569	2.4699	1.5321	1.7581
1.5	0.5	3.9116	18.2969	2.7573	8.8764	1.6374	2.1923
	0.8	2.8411	9.3340	4.1075	23.4492	1.9067	3.6540
	1.5	1.8512	3.1538	2.2736	5.2518	1.8209	3.3296
	2.5	1.6124	1.6124	1.7077	2.5634	2.4010	7.3863
	5.0	2.0032	4.6501	1.2536	0.9336	1.4182	2.0592
2.5	0.5	1.8638	2.5339	2.3203	7.2214	2.3505	5.5824
	0.8	2.0907	4.1234	2.5382	8.0472	3.2760	15.9463
	1.5	2.2096	6.5604	1.5155	2.0925	2.4407	9.8467
	2.5	1.8452	4.3835	1.4481	0.5627	1.2889	1.4312
	5.0	1.5338	2.7241	0.9294	-0.2828	1.0550	0.4980
5.0	0.5	1.9223	3.9925	1.8202	3.0641	2.3694	5.2655
	0.8	2.1443	4.4107	1.3729	1.7109	1.5855	1.9986
	1.5	1.8359	5.2441	0.7476	0.1816	1.0279	0.2928
	2.5	1.5240	2.6273	2.4213	8.5080	1.3524	2.3254
	5.0	1.0121	0.8251	0.8343	0.8544	1.0889	0.55220

Source: Simulated data of TIIHLEE Distribution

Table 1, 3 and 5 for sample 20, 60 and 100 respectively, it is observed that the mean, standard deviation and median are increasing functions of the scale parameters λ when the other parameters are held constant. Increasing the scale parameter λ increases the mean, standard deviation and median for fixed α and β. Also Table 3 and 5, Skewness and kurtosis remain constant as the scale parameter λ increases when the shape parameters are held constant. An increase in the shape parameters reduces the values of skewness and kurtosis when the scale parameter is held constant.

6. APPLICATIONS

To illustrate the importance and flexibility of the TIIHLEE distribution, two real data sets are demonstrated. We compare the fits of the TIIHLEE model with some models namely; Type II half logistic exponential (TIIHLE) by Elgarhy et al (2018), Half logistic Exponentiated Exponential (HLEE), Half Logistic Exponential (HLE), by Cordeiro *et al*(2015) Exponential distribution (E) by Nadarajah and Kotz (2006), Exponentiated

Exponential (EE) by Gupta and Kundu (2001), Marshal Olkin extended exponential (MOEE) by Srivastava et al. (2011) and Weibull distribution.

Tables 8 and 11 gives the MLEs of the unknown parameter(s) and Tables 9 and 12 gives the goodness of fit measures log-likelihood (*LL*), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Anderson-Darling (A^*), Cramér-von Mises (W^*) and Kolmogorov-Smirnov (K-S) for the fitted lifetime distribution.

The first data on the remission times (in months) of a random sample of 128 bladder cancer patients studied by Lee et al.(2003) reported by Cordeiro and Lemonte (2011) and Huang and Oluyede (2014).

0.080	0.200	0.400	0.500	0.510	0.810	0.900	1.050	1.190	1.260	1.350	1.400	1.460	1.760
2.020	2.020	2.070	2.09	2.230	2.260	2.460	2.540	2.620	2.640	2.690	2.690	2.750	2.830
2.870	3.020	3.25	3.310	3.360	3.360	3.480	3.520	3.570	3.640	3.700	3.820	3.880	4.180
4.230	4.260	4.330	4.340	4.400	4.500	4.510	4.870	4.980	5.060	5.090	5.170	5.320	5.320
5.340	5.410	5.410	5.490	5.620	5.710	5.850	6.540	6.760	6.930	6.94	6.970	7.090	7.260
7.280	7.320	7.390	7.590	7.620	7.630	7.660	7.870	7.930	8.260	8.370	8.530	8.650	8.660
9.020	9.220	9.470	9.740	10.06	10.34	10.66	10.75	11.25	11.64	11.79	11.98	12.02	12.03
12.07	12.63	13.11	13.29	13.80	14.24	14.76	14.77	14.83	15.96	16.62	17.14	18.10	19.13
21.73	22.69	23.63	25.74	25.82	26.31	32.15	34.26	36.66	43.01	46.12	79.05		

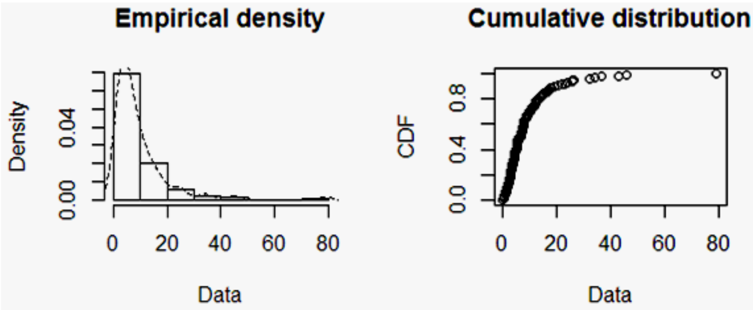

Fig 4: Histogram and CDF Plots of an Empirical Distribution for Bladder Cancer Patient

Table 7
Descriptive Statistics Bladder Cancer Patient

Min	Max	Mean	SD	Median	Q1	Q3	Skewness	Kurtosis
0.080	79.050	9.366	10.5083	6.395	3.348	11.840	3.2866	15.4831

Source: Cordeiro and Lemonte (2011)

Descriptive statistics in table 7 shows that the first data set on bladder cancer patient is over dispersed which is right skewed,

Correlation matrix of TIIHLEE Distribution

$$\begin{pmatrix} 1.000000000 & 0.004656698 & -0.999906980 \\ 0.004656698 & 1.000000000 & 0.005304919 \\ -0.999906980 & 0.005304919 & 1.000000000 \end{pmatrix}$$

The above correlation matrix indicates the pairs which have negative and positive correlation coefficient depending on the combination of the parameters.

The asymptotic variance covariance matrix for the MLEs of TIIHLEE Distribution

$$I_{ij}^{-1} = \begin{pmatrix} 1.961935e+01 & 0.0002500359 & -91.114098426 \\ 2.500359e-04 & 0.0001469481 & 0.001322953 \\ -9.111410e+01 & 0.0013229526 & 423.221041120 \end{pmatrix}$$

Table 8
Maximum Likelihood Estimates of Bladder Cancer Patient

Distribution	**Estimated Parameters**		
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
TIIHLEE	0.56054	0.0979	2.6034
TIIHLE	1.4586	0.0978	
EE	1.2186	8.2466	
E	6.0891		
W	0.0939		

Table 9
Goodness of Measure of Estimate for Bladder Cancer Patient

Distribution	**AIC**	**BIC**	**LL**	**A***	**W***	**K-S**
TIIHLEE	828.0746	836.6307	-411.0373	0.33438	0.05441	0.05088
TIIHLE	1003.520	1009.224	-449.7602	0.33622	0.05482	0.05482
EE	830.1552	835.8593	-413.0776	0.7110	0.1272	0.0724
E	1023.348	1026.200	-510.6739	11.2831	1.9939	0.1792
W	832.1738	837.87778	-414.0869	0.9593	0.1541	0.0701

Source: Cordeiro and Lemonte (2011)

Table 9 shows that the estimates of information criterion (Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC)), log-likelihood function (LL), Anderson Darling (A^*), Cramér-von Mises(W^*) and Kolmogorov-Smirnov (K-S) of the proposed TIIHLEE distribution provide consistently fit better as compared to other competitive models.

However, the confidence interval at 95% for the estimate $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ are (-2.1848, 3.3058), (0.0783, 0.1175) and (-37.7184, 42.9252) respectively. It can be observed that the confidence intervals for the parameters do not contain zero. This is an indication that all the parameters of the TIIHLEE distribution were significant at the 5% significance level.

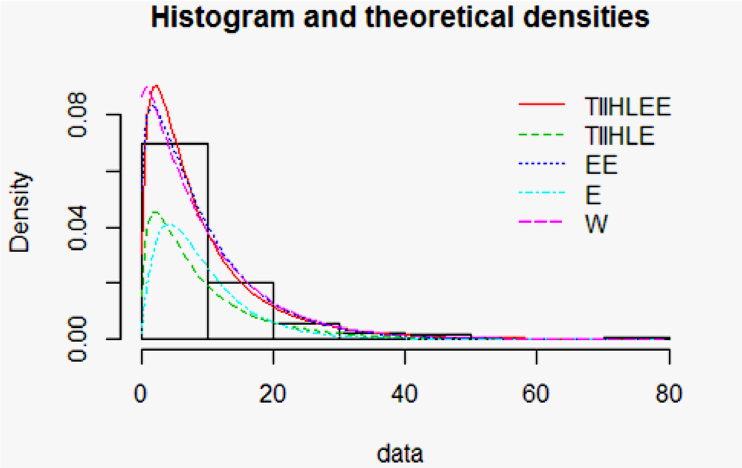

Fig 5: Fit Plot of TIIHLEE Model with other Models for Bladder Cancer Patient

Fig 5 shows the histogram of the bladder cancer data set along with fitted model. The theoretical density of TIIHLEE distribution has a better spread to the right than other models on the data set.

The second data set represents by Silva et al. (2010) the maximum annual flood discharges in units of 1000 cubic feet per second, of the North Saskachevan River at Edmonton, over a period of 48 years.

19.885	20.940	21.820	23.700	24.888	25.460	25.760	26.720	27.500	28.100	28.600	30.200
30.380	31.500	32.600	32.680	34.400	35.347	38.100	39.020	39.200	40.000	40.400	40.400
42.250	44.020	44.730	44.900	46.300	50.330	51.442	57.220	58.700	58.800	61.200	61.740
65.440	65.597	66.000	74.100	75.800	84.100	106.600	109.700	121.970	121.970	185.560	

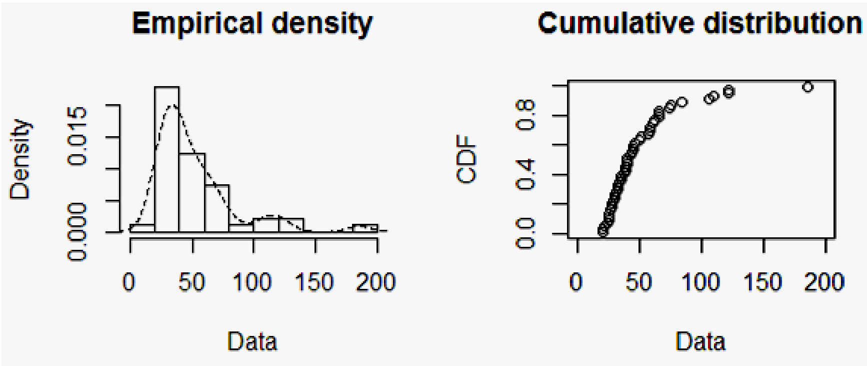

Fig 6: Histogram and CDF Plots of an Empirical Distribution for Maximum Annual Flood Discharge

Table 10
Descriptive Statistics Maximum Annual Flood Discharges in Units

Min	Max	Mean	SD	Median	Q1	Q3	Skewness	Kurtosis
19.8900	185.6000	51.5000	32.3768	40.4000	30.3400	61.3400	2.0686	4.9507

Source: Silva et al. (2010)

Table 10 gives a descriptive statistics indicate that the second data set on maximum annual discharge is under dispersed which is right skewed,

Correlation matrix of TIIHLEE Distribution

$$\begin{pmatrix} 1.0000000 & 0.1062620 & -0.9716528 \\ 0.1062620 & 1.0000000 & 0.1062697 \\ -0.9716528 & 0.1062697 & 1.0000000 \end{pmatrix}$$

The above correlation matrix indicates the pairs which have negative and positive correlation coefficient depending on the combination of the parameters.

Variance covariance matrix for the estimated parameters of TIIHLEE Distribution

$$I_{ij}^{-1} = \begin{pmatrix} 3.892484953 & 1.329686e\text{-}03 & -7.276383021 \\ 0.001329686 & 4.022672e\text{-}05 & 0.002558337 \\ -7.276383021 & 2.558337e\text{-}03 & 14.407279617 \end{pmatrix}$$

Table 11
Maximum Likelihood Estimates of Maximum Annual Flood Discharge in Unit

Distribution	Estimated Parameters		
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
TIIHLEE	1.7872	0.0407	3.4297
TIIHLE	6.2883	0.0413	
HLEE	5.7778	0.8870	
HLE	0.0115		
EE	5.8443	20.8215	
MOEE	11.1468	0.0547	

Table 12
Good of Measure for Maximum Annual Flood Discharge in Units

Distribution	AIC	BIC	LL	A*	W*	K-S
TIIHLEE	441.8781	447.4917	-217.9391	0.5610	0.0783	0.0958
TIIHLE	506.4079	510.1503	-251.2039	0.5730	0.0798	0.0964
HLEE	575.9462	579.6886	-285.9731	6.3672	1.2687	0.2830
HLE	486.6703	488.5415	-242.3351	17.5452	3.7762	0.4704
EE	442.8943	446.6367	-219.4471	0.8629	0.1301	0.1169
MOEE	456.8335	460.5759	-226.4167	1.5927	0.2008	0.1500

Source: Silva et al. (2010)

Table 12 shows that the estimates of information criterion (Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC)), log-likelihood function (LL), Anderson Darling (A^*), Cramér-von Mises (W^*) and Kolmogorov-Smirnov (K-S) of the proposed TIIHLEE distribution provide consistently fit better as compared to other competitive models.

The confidence interval at 95% for the estimate $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ are (-2.0798, 5.6542), (0.3946, 0.4194) and (-4.0099, 10.8693) respectively. It can be observed that the confidence intervals for the parameters do not contain zero. This is an indication that all the parameters of the TIIHLEE distribution were significant at the 5% significance level.

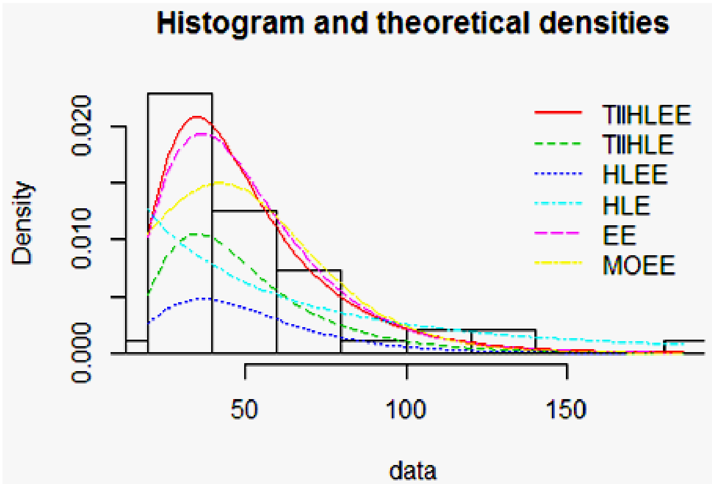

Fig 7: Fit Plot of TIIHLEE Model with other Models for Maximum Annual Flood Discharge

Fig 7 also shows the histogram of the Maximum Annual Flood Discharge data set along with fitted model. The theoretical density of TIIHLEE distribution has a better spread to the right than other models on the data set.

7. CONCLUSIONS

In this study, we proposed a modified two parameter which is added to type II half logistic family of distribution, called Type II Half Logistic Exponentiated Exponential (TIIHLEE). Some structural mathematical properties; Moment Incomplete moments, Probability Weighted Moment, Order Statistic, and Rényi entropy of the derived model are investigated. A simulation study is carried out to estimate the behavior of the shape and scale model parameters, also maximum likelihood estimators were investigator. The application of two real life dataset shows that the TIIHLEE strong and better fit than Type II half logistic exponential (TIIHLE), Half logistic Exponentiated Exponential (HLEE), Half Logistic Exponential (HLE), Exponential distribution (E), Exponentiated Exponential (EE), Marshal Olkin extended exponential (MOEE) and Weibull distribution.

However, the proposed distribution TIIHLEE hereby recommended as an alternative model in the area of theoretical and applied statistics.

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