Pak. J. Statist.
2019 Vol. 35(4), 321-348

WEIGHTED RAYLEIGH DISTRIBUTION REVISITED VIA INFORMATIVE AND NON-INFORMATIVE PRIORS

Sofi Mudasir[§]**, Uzma Jan** and **S.P. Ahmad**
Department of Statistics, University of Kashmir, Srinagar, India.
[§]Corresponding author Email: sofimudasir3806@gmail.com

ABSTRACT

In this manuscript, different statistical properties of the Weighted Rayleigh distribution have been derived. The model has been compared with the sub models for flexibility and efficiency using real life data sets. Further, the parameters of the model are estimated using the maximum likelihood approach. The variability of different priors as well as approximation techniques has been compared using posterior variance. The study depicts that Gumbel type II prior especially under normal approximation technique can be preferred.

KEYWORDS

Approximation techniques, Maximum Likelihood estimation, Posterior variance, Weighted Rayleigh Distribution.

1. INTRODUCTION

Rayleigh distribution is a lifetime distribution introduced by John William Rayleigh [19] and has been used in communications theory, physical sciences, engineering and medical imaging science. There are plethora of authors who contributed to this model, among them are Siddiqui [20] talked about the origin and properties of this model, Dey and Tanujit [6] obtain the Bayes estimate the scale parameter of Rayleigh distribution, Ahsanullah and Shakil [1] establish some new results on characterization of Rayleigh distribution, Asgharzadeh and Azizpour [4] derive the Bayes estimators of Rayleigh distribution based on hybrid censored sample, Ardianti [3] used classical and Bayesian methods to estimate the parameter of Rayleigh distribution. The probability density function (pdf) and cumulative distribution function (cdf) of Rayleigh distribution with scale parameter η is given by

$$f(u)=\frac{u}{\eta}\exp\left(-\frac{u^2}{2\eta}\right),\ u>0,\eta>0. \tag{1.1}$$

$$F(u)=1-\exp\left(-\frac{u^2}{2\eta}\right) \tag{1.2}$$

In the past few years, the trend of introducing an extra parameter to the existing distributions to model life time data gains a lot of attention. One of the techniques is to obtain the weighted version of the existing distribution in which the extra parameter (parameters) is assigned through weight function. The concept of weighted distributions was first introduced by Fisher [7] and studied by Rao [18] in a unified manner who pointed out that in many situations; it is not possible to select a sample with equal probability. The applications and usefulness of weighted distributions can be seen in paper by several authors including Gupta and Kundu [8], Jain et al. [9], Ajami and Jahanshahi [2], Patil and Rao [17], Sofi Mudasir and Ahmad [14,15], Jan et al. [10], and Dar et al. [5].

In the present study, we consider the weighted version of Rayleigh distribution and study its different properties. Then, the maximum likelihood method of estimation is used to estimate the model parameters. Also the ability of different priors and approximation techniques based on simulated and real life data sets is compared in terms of posterior variance.

2. MODEL, LIKELIHOOD FUNCTION AND MAXIMUM LIKELIHOOD ESTIMATION

The probability density function (pdf) of WR distribution (suggested by Ajami and Jahanshahi (2017)) is given by

$$f_w(u;\theta,\eta)=\frac{u^{\theta+1}}{2^{\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}+1\right)\eta^{\frac{\theta}{2}+1}}\exp\left(-\frac{u^2}{2\eta}\right),\ u>0,\eta>0,\theta>0 \qquad (2.1)$$

The cumulative distribution function (cdf) corresponding to (2.1) is given as

$$F_w(u;\theta,\eta)=\frac{\gamma\left(\frac{\theta}{2}+1,\frac{u^2}{2\eta}\right)}{\Gamma\left(\frac{\theta}{2}+1\right)} \qquad (2.2)$$

Also, the likelihood function is given as

$$L=\frac{1}{\left(2^{\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}+1\right)\right)^n\eta^{n\left(\frac{\theta}{2}+1\right)}}\lambda^{\theta+1}\exp\left(-\frac{t}{2\eta}\right) \qquad (2.3)$$

where $\lambda=\prod_{i=1}^{n}u_i$ and $t=\sum_{i=1}^{n}u_i^2$.

From eq. (2.3), the log-likelihood function is given by

$$l = -\frac{n\theta}{2}\log(2) - 2n\left(\frac{\theta}{2}+1\right)\log(\eta) - n\log\left(\Gamma\left(\frac{\theta}{2}+1\right)\right) + (\theta+1)\log(\lambda) - \frac{t}{2\eta} \tag{2.4}$$

Partial derivatives of eq. (2.4) w.r.t. the unknown parameters θ, η are

$$\frac{\partial l}{\partial \theta} = \frac{-n\log(2)}{2} - n\log(\eta) - n\psi\left(\frac{\theta}{2}+1\right) + \log(\lambda)$$

$$\frac{\partial l}{\partial \eta} = \frac{-n\theta}{\eta} - \frac{2n}{\eta} + \frac{t}{2\eta^2}$$

where $\psi(.)$ is the digamma function.

Now, the estimating equations for unknown parameters are

$$\frac{-n\log(2)}{2} - n\log(\eta) - n\psi\left(\frac{\theta}{2}+1\right) + \log(\lambda) = 0 \tag{2.5}$$

$$\frac{-n\theta}{\eta} - \frac{2n}{\eta} + \frac{t}{2\eta^2} = 0 \tag{2.6}$$

The maximum likelihood estimators of θ, η are obtained by solving equations (2.5) and (2.6) for θ and η.

The plots of pdf and cdf for different values of parameters are shown in Figure 1 and Figure 2.

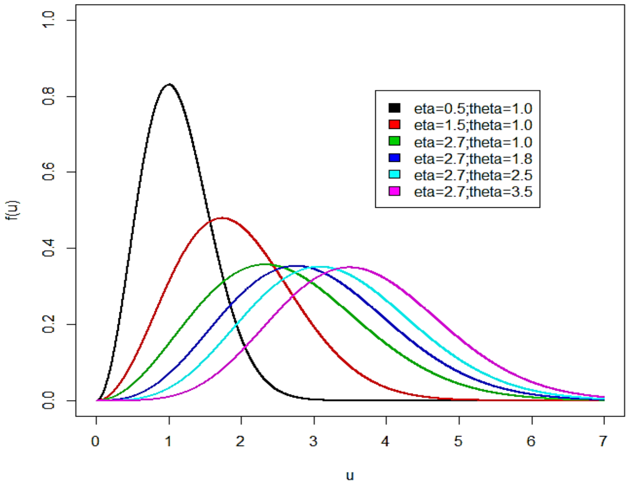

Figure 1: Probability Density Function

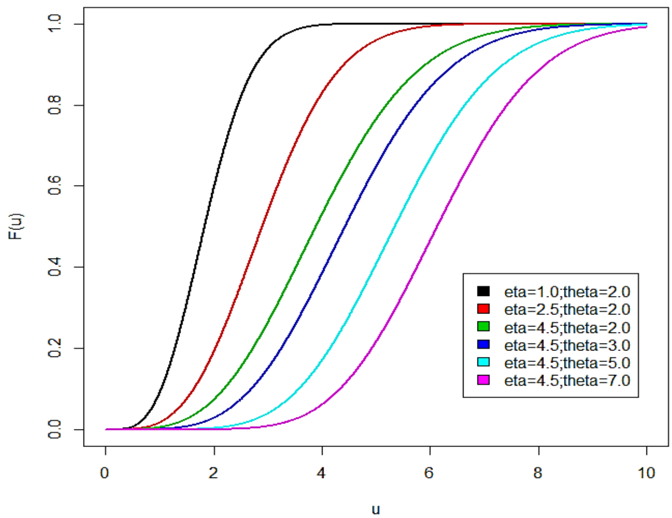

Figure 2: Cumulative Distribution Function

Table 1
Sub-Models of Weighted Rayleigh Distribution with Mean and Variance

Distribution	θ	η	Mean	Variance
Rayleigh	0	η	$(2\eta)^{\frac{1}{2}}\Gamma\left(\frac{3}{2}\right)$	$2\eta\left(1-\Gamma\left(\frac{3}{2}\right)\right)$
Length biased Rayleigh	1	η	$\frac{(2\eta)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}$	$3\eta-\frac{2\eta}{\left(\Gamma\left(\frac{3}{2}\right)\right)^{2}}$
Area biased Rayleigh	2	η	$(2\eta)^{\frac{1}{2}}\Gamma\left(\frac{5}{2}\right)$	$4\eta-\left(\Gamma\left(\frac{5}{2}\right)\right)^{2}2\eta$

3. PROPERTIES OF WEIGHTED RAYLEIGH DISTRIBUTION

3.1 Distributional Properties

Using eq. (2.1), the r^{th} moment of WR distribution about origin can be written as

$$\mu_r'=\frac{\rho_{\theta+r}}{\rho_\theta}(2\eta)^{\frac{r}{2}} \tag{3.1.1}$$

where $\rho_{\theta+s}=\Gamma\left(\frac{s+\theta}{2}+1\right)$

By using eq. (3.1.1), the mean and variance of WR distribution is given by

$$\text{Mean } = \frac{\rho_{\theta+1}}{\rho_\theta}(2\eta)^{\frac{1}{2}}$$

$$\text{Variance} = \frac{2\eta}{\rho_\theta^{\,2}}\left(\rho_\theta\rho_{\theta+2} - \rho_{\theta+1}^{\,2}\right)$$

The moment generating function of WR distribution cab be written as

$$M_U(t) == \sum_{r=0}^{\infty}\left(\frac{t^r}{r!}\right)\mu'_r$$

Using (3.1.1), the first four central moments are

$$\mu_1 = \frac{\rho_{\theta+1}}{\rho_\theta}(2\eta)^{\frac{1}{2}}$$

$$\mu_2 = \frac{\rho_\theta\rho_{\theta+2} - \rho_{\theta+1}^{\,2}}{\rho_\theta^{\,2}}2\eta$$

$$\mu_3 = \frac{\rho_\theta^{\,2}\rho_{\theta+3} - 3\rho_\theta\rho_{\theta+1}\rho_{\theta+2} + 2\rho_{\theta+1}^{\,3}}{\rho_\theta^{\,3}}(2\eta)^{\frac{3}{2}}$$

$$\mu_4 = \frac{\rho_\theta^{\,3}\rho_{\theta+4} - 4\rho_\theta^{\,2}\rho_{\theta+1}\rho_{\theta+3} + 6\rho_\theta\rho_{\theta+1}^{\,2}\rho_{\theta+2} - 3\rho_{\theta+1}^{\,4}}{\rho_\theta^{\,4}}(2\eta)^2$$

Hence

$$\text{Coefficient of skewness} = \frac{\left(\rho_\theta^{\,2}\rho_{\theta+3} - 3\rho_\theta\rho_{\theta+1}\rho_{\theta+2} + 2\rho_{\theta+1}^{\,3}\right)^2}{\left(\rho_\theta\rho_{\theta+2} - \rho_{\theta+1}^{\,2}\right)^3}$$

and

$$\text{Coefficient of kurtosis} = \frac{\rho_\theta^{\,3}\rho_{\theta+4} - 4\rho_\theta^{\,2}\rho_{\theta+1}\rho_{\theta+3} + 6\rho_\theta\rho_{\theta+1}^{\,2}\rho_{\theta+2} - 3\rho_{\theta+1}^{\,4}}{\left(\rho_\theta\rho_{\theta+2} - \rho_{\theta+1}^{\,2}\right)^2}$$

3.2 Ageing Properties

Using (2.1) and (2.2), the survival and hazard rate functions of WR distribution are given by

$$S(u) = \frac{\Gamma\left(\frac{\theta}{2}+1\right) - \gamma\left(\frac{\theta}{2}+1, \frac{u^2}{2\eta}\right)}{\Gamma\left(\frac{\theta}{2}+1\right)}$$

$$h(u)=\frac{u^{\theta+1}\exp\left(-\frac{u^2}{2\eta}\right)}{2^{\frac{\theta}{2}}\Gamma\left(\frac{\theta}{2}+1,\frac{u^2}{2\eta}\right)\eta^{\frac{\theta}{2}+1}}$$

The plots of survival and hazard rate functions for different values of parameters are given in Figure 3 and Figure 4 respectively.

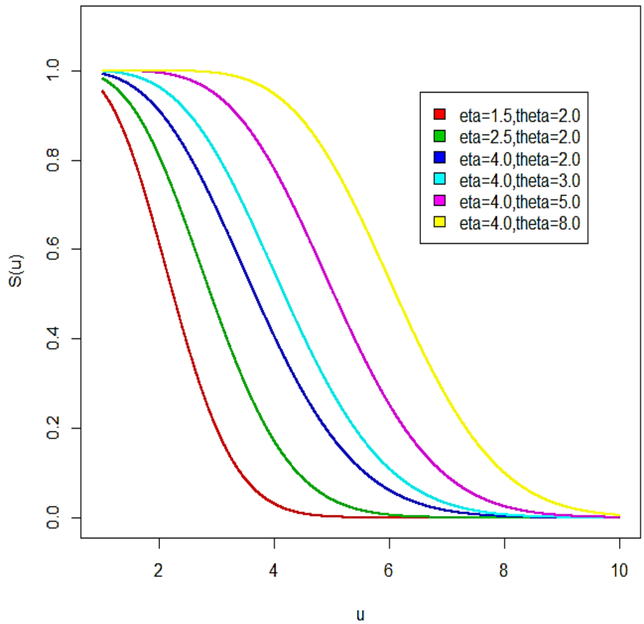

Figure 3: Survival Function

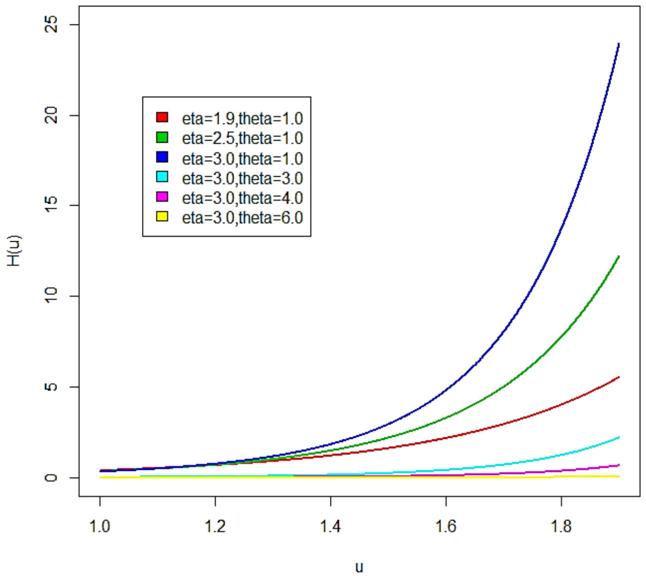

Figure 4: Hazard Function

4. POSTERIOR DISTRIBUTION UNDER THE ASSUMPTION OF DIFFERENT PRIORS

In this section, we derive the posterior distributions using informative and non-informative priors.

4.1 Posterior Distribution under Jeffrey's Prior

Jeffrey's prior for the scale parameter η is defined as

$$\pi_1(\eta) \propto \frac{1}{\eta}, \eta > 0 \tag{4.1.1}$$

By using the likelihood function (2.3) and the prior (4.1.1), we get the posterior distribution given as

$$P_1(\eta \mid \underline{u}) = k \frac{\lambda^{\theta+1} \exp\left(-\frac{t}{2\eta}\right)}{\left(2^{\frac{\theta}{2}} \Gamma\left(\frac{\theta}{2}+1\right)\right)^n \eta^{n\left(\frac{\theta}{2}+1\right)+1}}$$

where k is constant of proportionality and is given by

$$k = \frac{\left(\Gamma\left(\frac{\theta}{2}+1\right)\right)^n t^{\frac{n\theta}{2}+n}}{\Gamma\left(n\left(\frac{\theta}{2}+1\right)\right) \lambda^{\theta+1} 2^n}$$

Therefore,

$$P_1\left(\eta \mid \underline{u}\right)=\frac{\left(\frac{t}{2}\right)^{n\left(\frac{\theta}{2}+1\right)}\exp\left(-\frac{t}{2\eta}\right)}{\Gamma\left(n\left(\frac{\theta}{2}+1\right)\right)\eta^{n\left(\frac{\theta}{2}+1\right)+1}} \tag{4.1.2}$$

4.2 Posterior Distribution under Quasi Prior

Quasi prior is given by

$$\pi_2\left(\eta\right)\propto\frac{1}{\eta^d},d\geq 0 \tag{4.2.1}$$

By following the same path, the posterior distribution under Quasi prior is given as

$$P_2\left(\eta \mid \underline{u}\right)=\frac{\left(\frac{t}{2}\right)^{n\left(\frac{\theta}{2}+1\right)+d-1}\exp\left(-\frac{t}{2\eta}\right)}{\Gamma\left(n\left(\frac{\theta}{2}+1\right)+d-1\right)\eta^{n\left(\frac{\theta}{2}+1\right)+d}} \tag{4.2.2}$$

Remark 1:

Replacing $d=1$ in eq. (4.2.2), we get the posterior distribution under Jeffrey's prior.

Remark 2:

Replacing $d=3$ in eq. (4.2.2), we get the posterior distribution under Hartigan's prior.

4.3 Posterior Distribution under Gumbel Type II Prior

Suppose that the informative prior for the scale parameter η is Gumbel type II distribution which is given as

$$\pi_3\left(\eta\right)=\frac{a}{\eta^2}\exp\left(-\frac{a}{\eta}\right) \tag{4.3.1}$$

where $a>0$ is the hyper-parameter.

The posterior distribution for the likelihood function given in (2.3) and the prior (4.3.1) takes the form

$$P_3\left(\eta \mid \underline{u}\right)=\frac{\left(\frac{t}{2}+a\right)^{n\left(\frac{\theta}{2}+1\right)+1}\exp\left(-\frac{1}{\eta}\left(\frac{t}{2}+a\right)\right)}{\Gamma\left(n\left(\frac{\theta}{2}+1\right)+1\right)\eta^{n\left(\frac{\theta}{2}+1\right)+2}} \tag{4.3.2}$$

Remark 3:

Replacing $a = 1$ in eq. (4.3.2), we get the posterior distribution under inverse exponential prior.

4.4 Posterior Distribution under Levy Prior

The informative prior for the scale parameter η is assumed to be Levy distribution and is given by

$$\pi_4(\eta) = \frac{\sqrt{\left(\frac{b}{2\pi}\right)} \exp\left(-\frac{b}{2\eta}\right)}{\eta^{\frac{3}{2}}} \tag{4.4.1}$$

where $b > 0$ is the hyper-parameter.

Following the same path, we get the posterior distribution under the assumption of Levy prior as

$$P_4(\eta \mid \underline{u}) = \frac{\left(\frac{t+b}{2}\right)^{n\left(\frac{\theta}{2}+1\right)+\frac{1}{2}} \exp\left(-\frac{1}{\eta}\left(\frac{t+b}{2}\right)\right)}{\Gamma\left(n\left(\frac{\theta}{2}+1\right)+\frac{1}{2}\right)\eta^{n\left(\frac{\theta}{2}+1\right)+\frac{3}{2}}} \tag{4.4.2}$$

5. POSTERIOR MEAN AND POSTERIOR VARIANCE UNDER DIFFERENT PRIORS

5.1 Posterior Mean and Posterior Variance of η under Jeffrey's Prior

By using eq. (4.1.2), we have

$$E(\eta^r) = \frac{\left(\frac{t}{2}\right)^{n\left(\frac{\theta}{2}+1\right)}}{\Gamma\left(n\left(\frac{\theta}{2}+1\right)\right)} \int_0^\infty \exp\left(-\frac{t}{2\eta}\right)\eta^{r-n\left(\frac{\theta}{2}+1\right)-1} d\eta$$

By substituting $z = \frac{t}{2\eta}$, we get

$$E(\eta^r) = \frac{\Gamma\left(n\left(\frac{\theta}{2}+1\right)-r\right)}{\Gamma\left(n\left(\frac{\theta}{2}+1\right)\right)}\left(\frac{t}{2}\right)^r \tag{5.1.1}$$

Put r=1 in eq. (5.1.1), we get

$$E(\eta)=\frac{t}{2\left(n\left(\frac{\theta}{2}+1\right)-1\right)}$$

This is the posterior mean.

Put r=2 in eq. (5.1.1), we get

$$E(\eta^2)=\frac{t^2}{4\left(n\left(\frac{\theta}{2}+1\right)-1\right)\left(n\left(\frac{\theta}{2}+1\right)-2\right)}$$

Thus, the posterior variance is given by

$$V(\eta)=\frac{t^2}{4\left(n\left(\frac{\theta}{2}+1\right)-1\right)^2\left(n\left(\frac{\theta}{2}+1\right)-2\right)}$$

5.2 Posterior Mean and Posterior Variance of η under Quasi Prior

By using eq. (4.2.2), we have

$$E(\eta^r)=\frac{\left(\frac{t}{2}\right)^{n\left(\frac{\theta}{2}+1\right)+d-1}}{\Gamma\left(n\left(\frac{\theta}{2}+1\right)+d-1\right)}\int_0^\infty \exp\left(-\frac{t}{2\eta}\right)\eta^{r-n\left(\frac{\theta}{2}+1\right)-d}\,d\eta$$

After solving the above integral, we get

$$E(\eta^r)=\frac{\Gamma\left(n\left(\frac{\theta}{2}+1\right)+d-r-1\right)}{\Gamma\left(n\left(\frac{\theta}{2}+1\right)+d-1\right)}\left(\frac{t}{2}\right)^r \qquad (5.2.1)$$

If r=1 in eq. (5.2.1), we get

$$E(\eta)=\frac{t}{2\left(n\left(\frac{\theta}{2}+1\right)+d-2\right)}$$

This is the posterior mean.

If r=2 in eq. (5.2.1), we get

$$E(\eta^2)=\frac{t^2}{4\left(n\left(\frac{\theta}{2}+1\right)+d-2\right)\left(n\left(\frac{\theta}{2}+1\right)+d-3\right)}$$

Thus, the posterior variance is given by

$$V(\eta)=\frac{t^2}{4\left(n\left(\frac{\theta}{2}+1\right)+d-2\right)^2\left(n\left(\frac{\theta}{2}+1\right)+d-3\right)}$$

5.3 Posterior Mean and Posterior Variance of η under Gumbel Type II Prior

By using eq. (4.3.2), we have

$$E(\eta^r)=\frac{\left(\frac{t}{2}+a\right)^{n\left(\frac{\theta}{2}+1\right)+1}}{\Gamma\left(n\left(\frac{\theta}{2}+1\right)+1\right)}\int_0^\infty \exp\left(-\frac{1}{\eta}\left(\frac{t}{2}+a\right)\right)\eta^{r-n\left(\frac{\theta}{2}+1\right)-2}d\eta$$

After solving the above integral, we get

$$E(\eta^r)=\frac{\Gamma\left(n\left(\frac{\theta}{2}+1\right)+1-r\right)}{\Gamma\left(n\left(\frac{\theta}{2}+1\right)+1\right)}\left(\frac{t}{2}+a\right)^r \tag{5.3.1}$$

Put r=1 in eq. (5.3.1), we get

$$E(\eta)=\frac{t+2a}{2\left(n\left(\frac{\theta}{2}+1\right)\right)}$$

This is the posterior mean.

Put r=2 in eq. (5.3.1), we get

$$E(\eta^2)=\frac{(t+2a)^2}{4\left(n\left(\frac{\theta}{2}+1\right)\right)\left(n\left(\frac{\theta}{2}+1\right)-1\right)}$$

Now, the posterior variance is given by

$$V(\eta)=\frac{(t+2a)^2}{4\left(n\left(\frac{\theta}{2}+1\right)\right)^2\left(n\left(\frac{\theta}{2}+1\right)-1\right)}$$

5.4 Posterior Mean and Posterior Variance of η under Levy Prior

By using eq. (4.4.2), we have

$$E(\eta^r)=\frac{\left(\frac{t+b}{2}\right)^{n\left(\frac{\theta}{2}+1\right)+\frac{1}{2}}}{\Gamma\left(n(\frac{\theta}{2}+1)+\frac{1}{2}\right)}\int_0^\infty \exp\left(-\frac{1}{\eta}(\frac{t+b}{2})\right)\eta^{r-n\left(\frac{\theta}{2}+1\right)-\frac{3}{2}}d\eta$$

After solving the above integral, we get

$$E(\eta^r)=\frac{\Gamma\left(n\left(\frac{\theta}{2}+1\right)+\frac{1}{2}-r\right)}{\Gamma\left(n\left(\frac{\theta}{2}+1\right)+\frac{1}{2}\right)}\left(\frac{t+b}{2}\right)^r \tag{5.4.1}$$

Put r=1 in eq. (5.4.1), we get

$$E(\eta)=\frac{t+b}{2\left(n\left(\frac{\theta}{2}+1\right)-\frac{1}{2}\right)}$$

This is the posterior mean.

Put r=2 in eq. (5.4.1), we get

$$E(\eta^2)=\frac{(t+b)^2}{4\left(n\left(\frac{\theta}{2}+1\right)-\frac{1}{2}\right)\left(n\left(\frac{\theta}{2}+1\right)-\frac{3}{2}\right)}$$

Thus, the posterior variance is given by

$$V(\eta)=\frac{(t+b)^2}{4\left(n\left(\frac{\theta}{2}+1\right)-\frac{1}{2}\right)^2\left(n\left(\frac{\theta}{2}+1\right)-\frac{3}{2}\right)}$$

6. BAYESIAN APPROXIMATION TECHNIQUES OF POSTERIOR MODES

Bayesian paradigm gives a comprehensive model for updating the prior information in view of the current knowledge. Those who like the elegance of Bayesian outlook study important properties of the posterior and predictive distributions. If the resulting distribution is in closed form and difficult to characterize it, analytical or numerical approximation methods are often used for accuracy with less computational complicacy. Many authors have reviewed the approximation methods including Sultan and Ahmad [21] for generalized Power function distribution, Sultan et al. [22] for Dagum distribution and Jan and Ahmad [11] for inverse Lomax distribution.

6.1 Normal Approximation

In Bayesian approach, approximation techniques for large samples usually consider the normal approximation to the posterior distribution. If the posterior distribution is less skewed with sharp peak, the most convenient way is to approximate it by normal distribution. The Normal approximation for the posterior distribution $P(\eta\,|\,\underline{u})$ centered at mode is given as

$$P(\eta\,|\,\underline{u}) \sim N\left(\hat{\eta}, \left[I(\hat{\eta})\right]^{-1}\right), \tag{6.1.1}$$

where $I(\hat{\eta}) = -\frac{\partial^2}{\partial\theta^2}\log P(\eta\,|\,\underline{u})$

By using (4.1.2), we have

$$\log P_1(\eta\,|\,u)) = -\left(n\left(\frac{\theta}{2}+1\right)+1\right)\log\eta - \frac{t}{2\eta}$$

$$\Rightarrow \frac{\partial \log P_1(\eta\,|\,u)}{\partial\eta} = -\frac{\left(n\left(\frac{\theta}{2}+1\right)+1\right)}{\eta} + \frac{t}{2\eta^2}$$

Therefore,

$$\hat{\eta} = \frac{t}{2\left(n\left(\frac{\theta}{2}+1\right)+1\right)}$$

Now,

$$\frac{\partial^2 \log P_1(\eta\,|\,u)}{\partial\eta^2} = \frac{-4\left(n\left(\frac{\theta}{2}+1\right)+1\right)^3}{t^2}$$

$$\Rightarrow \left[I(\hat{\eta})\right]^{-1} = \frac{t^2}{4\left(n\left(\frac{\theta}{2}+1\right)+1\right)^3}$$

Thus, the posterior distribution under Jeffrey’s prior can be approximated as

$$P_1(\eta\,|\,\underline{u}) \sim \left(\frac{t}{2\left(n\left(\frac{\theta}{2}+1\right)+1\right)}, \frac{t^2}{4\left(n\left(\frac{\theta}{2}+1\right)+1\right)^3}\right)$$

Similarly, under Quasi Prior, the posterior distribution is given by (4.2.2) and can be approximated as

$$P_2\left(\eta\,|\,\underline{u}\right)\sim\left(\frac{t}{2\left(n\left(\frac{\theta}{2}+1\right)+d\right)},\frac{t^2}{4\left(n\left(\frac{\theta}{2}+1\right)+d\right)^3}\right)$$

Under Gumbel Prior, the posterior distribution is given as (4.3.2) and can be approximated as

$$P_3\left(\eta\,|\,\underline{u}\right)\sim\left(\frac{\left(2a+t\right)}{2\left(n\left(\frac{\theta}{2}+1\right)+2\right)},\frac{\left(2a+t\right)^2}{4\left(n\left(\frac{\theta}{2}+1\right)+2\right)^3}\right)$$

Under Levy Prior, the posterior distribution is given by (4.4.2) and it can be approximated by

$$P_4\left(\eta\,|\,\underline{u}\right)\sim\left(\frac{\left(t+b\right)}{2\left(n\left(\frac{\theta}{2}+1\right)+\frac{3}{2}\right)},\frac{\left(t+b\right)^2}{4\left(n\left(\frac{\theta}{2}+1\right)+\frac{3}{2}\right)^3}\right)$$

6.2 TK Approximation

Laplace's method proves to be an efficient procedure for solving the difficult integrals which arise in mathematics. For approximating the average values of functions of parameters and marginal densities, Laplace's method is generally used. The different manuscripts in literature which describe the method include Lindley [13], Tierney and Kadane [23]. Tierney and Kadane presented the Laplace method for computing $E\left(h(\eta)\,|\,\underline{u}\right)$ as

$$E\left(h(\eta)\,|\,\underline{u}\right)\cong\frac{\hat{\varphi}^*\exp\left(-nh^*\left(\hat{\eta}^*\right)\right)}{\hat{\varphi}\exp\left(-nh\left(\hat{\eta}\right)\right)}, \qquad (6.2.1)$$

where $-nh\left(\hat{\eta}\right)=\log P(\eta\,|\,\underline{u})$; $-nh*\left(\hat{\eta}^*\right)=\log P(\eta\,|\,\underline{u})+\log h(\eta)$; $\hat{\varphi}^2=-\left[-nh''\left(\hat{\eta}\right)\right]^{-1}$; $\hat{\varphi}^{*2}=-\left[-nh''*\left(\hat{\eta}^*\right)\right]^{-1}$

Under Jeffrey's prior, the posterior distribution is given by the equation (4.1.2). By using this equation, we have

$$-nh(\eta) = \log P_1(\eta \mid \underline{u})$$
$$= -\left(n\left(\frac{\theta}{2}+1\right)+1\right)\log\eta - \frac{t}{2\eta}$$
$$\Rightarrow\ -nh'(\eta) = -\frac{\left(n\left(\frac{\theta}{2}+1\right)+1\right)}{\eta} + \frac{t}{2\eta^2}$$

Therefore,

$$\hat{\eta} = \frac{t}{2\left(n\left(\frac{\theta}{2}+1\right)+1\right)}$$

Also,

$$-nh''(\hat{\eta}) = -\frac{4\left(n\left(\frac{\theta}{2}+1\right)+1\right)^3}{t^2}$$

Therefore,

$$\hat{\varphi} = \frac{t}{2\left(n\left(\frac{\theta}{2}+1\right)+1\right)^{\frac{3}{2}}}$$

Now,

$$-nh^*\left(\eta^*\right) = \log P_1(\eta \mid \underline{u}) + \log h(\eta)$$
$$= -n\left(\frac{\theta}{2}+1\right)\log\eta^* - \frac{t}{2\eta^*}$$
$$\Rightarrow\ -nh'^*\left(\eta^*\right) = -\frac{n\left(\frac{\theta}{2}+1\right)}{\eta^*} + \frac{t}{2\eta^{*2}}$$

Therefore,

$$\hat{\eta}^* = \frac{t}{2n\left(\frac{\theta}{2}+1\right)}$$

Also,

$$-nh''^{*}\left(\hat{\eta}^{*}\right)=-\frac{4\left(n\left(\frac{\theta}{2}+1\right)\right)^{3}}{t^{2}}$$

Therefore,

$$\hat{\varphi}^{*}=\frac{t}{2\left(n\left(\frac{\theta}{2}+1\right)\right)^{\frac{3}{2}}}$$

Now, by using eq. (6.2.1), we have

$$E\left(\eta\,|\,\underline{u}\right)==\left(\frac{n\left(\frac{\theta}{2}+1\right)+1}{n\left(\frac{\theta}{2}+1\right)}\right)^{-\left(n\left(\frac{\theta}{2}+1\right)-\frac{3}{2}\right)}\left(\frac{2\left(n\left(\frac{\theta}{2}+1\right)+1\right)}{t}\right)^{-1}\exp\left(1\right)$$

Similarly,

$$E\left(\eta^{2}\,|\,\underline{u}\right)\cong\frac{\hat{\varphi}^{**}\exp\left(-nh^{**}\left(\hat{\eta}^{**}\right)\right)}{\hat{\varphi}\exp\left(-nh\left(\hat{\eta}\right)\right)}\text{, where }-nh^{**}\left(\hat{\eta}^{**}\right)=\log\left(\eta^{2}\right)-nh\left(\eta\right)$$

$$\Rightarrow\ E\left(\eta^{2}\,|\,\underline{u}\right)=\left(\frac{n\left(\frac{\theta}{2}+1\right)+1}{n\left(\frac{\theta}{2}+1\right)-1}\right)^{-\left(n\left(\frac{\theta}{2}+1\right)-\frac{5}{2}\right)}\left(\frac{2\left(n\left(\frac{\theta}{2}+1\right)+1\right)}{t}\right)^{-2}\exp\left(2\right)$$

Therefore,

$$V\left(\eta\,|\,\underline{u}\right)=\left(\frac{n\left(\frac{\theta}{2}+1\right)+1}{n\left(\frac{\theta}{2}+1\right)-1}\right)^{-\left(n\left(\frac{\theta}{2}+1\right)-\frac{5}{2}\right)}\left(\frac{2\left(n\left(\frac{\theta}{2}+1\right)+1\right)}{t}\right)^{-2}\exp\left(2\right)$$
$$-\left(\left(\frac{n\left(\frac{\theta}{2}+1\right)+1}{n\left(\frac{\theta}{2}+1\right)}\right)^{-\left(n\left(\frac{\theta}{2}+1\right)-\frac{3}{2}\right)}\left(\frac{2\left(n\left(\frac{\theta}{2}+1\right)+1\right)}{t}\right)^{-1}\exp\left(1\right)\right)^{2}.$$

Under Quasi Prior, the posterior distribution is given by (4.2.2). By following the same path, we have

$$E(\eta\mid\underline{u})=\left(\frac{n\left(\frac{\theta}{2}+1\right)+d}{n\left(\frac{\theta}{2}+1\right)+d-1}\right)^{-\left(n\left(\frac{\theta}{2}+1\right)+d-\frac{5}{2}\right)}\left(\frac{2\left(n\left(\frac{\theta}{2}+1\right)+d\right)}{t}\right)^{-1}\exp(1)$$

$$E\left(\eta^2\mid\underline{u}\right)=\left(\frac{n\left(\frac{\theta}{2}+1\right)+d}{n\left(\frac{\theta}{2}+1\right)+d-2}\right)^{-\left(n\left(\frac{\theta}{2}+1\right)+d-\frac{7}{2}\right)}\left(\frac{2\left(n\left(\frac{\theta}{2}+1\right)+d\right)}{t}\right)^{-2}\exp(2)$$

Therefore,

$$V(\eta\mid\underline{u})=\left(\frac{n\left(\frac{\theta}{2}+1\right)+d}{n\left(\frac{\theta}{2}+1\right)+d-2}\right)^{-\left(n\left(\frac{\theta}{2}+1\right)+d-\frac{7}{2}\right)}\left(\frac{2\left(n\left(\frac{\theta}{2}+1\right)+d\right)}{t}\right)^{-2}\exp(2)$$
$$-\left(\left(\frac{n\left(\frac{\theta}{2}+1\right)+d}{n\left(\frac{\theta}{2}+1\right)+d-1}\right)^{-\left(n\left(\frac{\theta}{2}+1\right)+d-\frac{5}{2}\right)}\left(\frac{2\left(n\left(\frac{\theta}{2}+1\right)+d\right)}{t}\right)^{-1}\exp(1)\right)^2$$

Similarly, under Gumbel Type II Prior with the posterior distribution given in equation (4.3.2), we have

$$E(\eta\mid\underline{u})=\left(\frac{n\left(\frac{\theta}{2}+1\right)+2}{n\left(\frac{\theta}{2}+1\right)+1}\right)^{-\left(n\left(\frac{\theta}{2}+1\right)-\frac{1}{2}\right)}\left(\frac{2\left(n\left(\frac{\theta}{2}+1\right)+2\right)}{(2a+t)}\right)^{-1}\exp(1)$$

$$E\left(\eta^2\mid\underline{u}\right)=\left(\frac{n\left(\frac{\theta}{2}+1\right)+2}{n\left(\frac{\theta}{2}+1\right)}\right)^{-\left(n\left(\frac{\theta}{2}+1\right)-\frac{3}{2}\right)}\left(\frac{2\left(n\left(\frac{\theta}{2}+1\right)+2\right)}{(2a+t)}\right)^{-2}\exp(2)$$

Therefore,

$$V(\eta\,|\,\underline{u})=\left(\frac{n\left(\frac{\theta}{2}+1\right)+2}{n\left(\frac{\theta}{2}+1\right)}\right)^{-\left(n\left(\frac{\theta}{2}+1\right)-\frac{3}{2}\right)}\left(\frac{2\left(n\left(\frac{\theta}{2}+1\right)+2\right)}{(2a+t)}\right)^{-2}\exp(2)$$

$$-\left(\left(\frac{n\left(\frac{\theta}{2}+1\right)+2}{n\left(\frac{\theta}{2}+1\right)+1}\right)^{-\left(n\left(\frac{\theta}{2}+1\right)-\frac{1}{2}\right)}\left(\frac{2\left(n\left(\frac{\theta}{2}+1\right)+2\right)}{(2a+t)}\right)^{-1}\exp(1)\right)^{2}$$

Under Levy Prior, the posterior distribution is given by equation (4.4.2). By following the same procedure, we have

$$E(\eta\,|\,\underline{u})=\left(\frac{n\left(\frac{\theta}{2}+1\right)+\frac{3}{2}}{n\left(\frac{\theta}{2}+1\right)+\frac{1}{2}}\right)^{-\left(n\left(\frac{\theta}{2}+1\right)-1\right)}\left(\frac{2\left(n\left(\frac{\theta}{2}+1\right)+\frac{3}{2}\right)}{(t+b)}\right)^{-1}\exp(1)$$

$$E\left(\eta^{2}\,|\,\underline{u}\right)=\left(\frac{n\left(\frac{\theta}{2}+1\right)+\frac{3}{2}}{n\left(\frac{\theta}{2}+1\right)-\frac{1}{2}}\right)^{-\left(n\left(\frac{\theta}{2}+1\right)-2\right)}\left(\frac{2\left(n\left(\frac{\theta}{2}+1\right)+\frac{3}{2}\right)}{(t+b)}\right)^{-2}\exp(2)$$

Therefore,

$$V(\eta\,|\,\underline{u})=\left(\frac{n\left(\frac{\theta}{2}+1\right)+\frac{3}{2}}{n\left(\frac{\theta}{2}+1\right)-\frac{1}{2}}\right)^{-\left(n\left(\frac{\theta}{2}+1\right)-2\right)}\left(\frac{2\left(n\left(\frac{\theta}{2}+1\right)+\frac{3}{2}\right)}{(t+b)}\right)^{-2}\exp(2)$$

$$-\left(\left(\frac{n\left(\frac{\theta}{2}+1\right)+\frac{3}{2}}{n\left(\frac{\theta}{2}+1\right)+\frac{1}{2}}\right)^{-\left(n\left(\frac{\theta}{2}+1\right)-1\right)}\left(\frac{2\left(n\left(\frac{\theta}{2}+1\right)+\frac{3}{2}\right)}{(t+b)}\right)^{-1}\exp(1)\right)^{2}$$

7. ESTIMATES AND MODEL COMPARISON

In this section, we consider two real life data sets to estimate the parameters and establish the superiority of WR distribution over its sub models. The data set 1 represent fracture toughness of Alumina (Al_2O_3) (in the units of MPa m1=2), Nadarajah & Kotz [16] and the data set 2 is from an accelerated life test of 59 conductors, failure times are in hours, and there are no censored observations Lawless [12].

Data Set 1:

5.5, 5, 4.9, 6.4, 5.1, 5.2, 5.2, 5, 4.7, 4, 4.5, 4.2, 4.1, 4.56, 5.01, 4.7, 3.13,3.12, 2.68, 2.77, 2.7, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04,2.08, 2.13, 3.8, 3.73, 3.71, 3.28, 3.9, 4, 3.8, 4.1, 3.9, 4.05, 4, 3.95, 4, 4.5, 4.5,4.2, 4.55, 4.65, 4.1, 4.25, 4.3, 4.5, 4.7, 5.15, 4.3, 4.5, 4.9, 5, 5.35, 5.15, 5.25, 5.8,5.85, 5.9, 5.75, 6.25, 6.05, 5.9, 3.6, 4.1, 4.5, 5.3, 4.85, 5.3, 5.45, 5.1, 5.3, 5.2,5.3, 5.25, 4.75, 4.5, 4.2, 4, 4.15, 4.25, 4.3, 3.75, 3.95, 3.51, 4.13, 5.4, 5, 2.1, 4.6,3.2, 2.5, 4.1, 3.5, 3.2, 3.3, 4.6, 4.3, 4.3, 4.5, 5.5, 4.6, 4.9, 4.3, 3, 3.4, 3.7, 4.4,4.9, 4.9, 5.0.

Data Set 2:

2.997, 4.137, 4.288, 4.531, 4.700, 4.706, 5.009, 5.381, 5.434, 5.459, 5.589, 5.640, 5.807, 5.923, 6.033, 6.071, 6.087, 6.129, 6.352, 6.369, 6.476, 6.492, 6.515, 6.522, 6.538, 6.545, 6.573, 6.725, 6.869, 6.923, 6.948, 6.956, 6.958, 7.024, 7.224, 7.365, 7.398, 7.459, 7.489, 7.495, 7.496, 7.543, 7.683, 7.937, 7.945, 7.974, 8.120, 8.336, 8.532, 8.591, 8.687, 8.799, 9.218, 9.254, 9.289, 9.663, 10.092, 10.491, 11.038.

The descriptive statistics for data set 1 and data set 2 are presented in Tables 2 and 3 respectively. The values of the parameters of weighted Rayleigh distribution (WRD), Rayleigh distribution (RD), length biased Rayleigh distribution (LBRD) and area biased Rayleigh distribution (ABRD) have been calculated and are reported in Table 4 and Table 5. Also, for comparing WRD with other used models, we use the concept of Akaike information criteria (AIC), Bayesian information criteria (BIC) and corrected Akaike information criteria (AICC). The model which possesses the least values of AIC and BIC is the best one. The calculated values of AIC and BIC are reported in Tables 4 and 5.

Table 2
Descriptive Statistics for Data Set 1

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Skewness	Kurtosis
1.680	3.850	4.380	4.325	5.000	6.810	-0.416714	3.093454

Table 3
Descriptive Statistics for Data Set 2

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Skewness	Kurtosis
2.997	6.052	6.923	6.980	7.941	11.038	0.193172	3.087389

Table 4
Maximum Likelihood Estimates, (Standard Error) and Statistics for Model Selection using Data Set 1

Model	$\hat{\eta}$	$\hat{\theta}$	$-l$	AIC	BIC	AICC	D	P-value
WRD	2.2561094 (0.2988713)	6.7484742 (1.0936554)	173.4827	350.9655	356.5237	351.0689	0.11115	0.10570
RD	9.8687555 (0.9046673)	-	221.0345	444.0690	446.8481	444.1031	0.29855	1.226e-09
LBRD	6.5791703 (0.4924385)	-	200.8449	403.6897	406.4689	403.7239	0.24251	1.667e-06
ABRD	4.9343779 (0.3198482)	-	189.1835	380.3669	383.1460	380.4011	0.20378	0.0001021

Table 5
Maximum Likelihood Estimates, (Standard Error) and Statistics for Model Selection using Data Set 2

Model	$\hat{\eta}$	$\hat{\theta}$	$-l$	AIC	BIC	AICC	D	P-value
WRD	5.3047623 (0.9957555)	7.6672242 (1.7219719)	111.3657	226.7313	230.8864	226.9456	0.062566	0.9639
RD	25.641160 (3.338194)	-	137.4123	276.8247	278.9022	276.8949	0.31278	1.257e-05
LBRD	17.094108 (1.817083)	-	127.0589	256.1177	258.1952	256.1879	0.24306	0.001488
ABRD	12.82058 (1.18023)	-	120.9336	243.8672	245.9448	243.9374	0.19302	0.02121

Since for both data sets, the lowest values of AIC, BIC, AICC, Kolmogorov-Simirnov distances (D) and highest p-value correspond to WR distribution. Hence it is concluded that the WR distribution provides the best fit as compared to its sub-models.

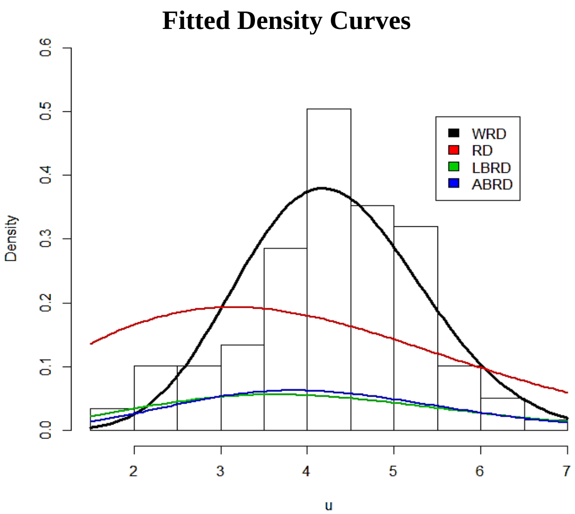

Figure 5: Plots for the Histogram and Estimated Density using Data Set 1

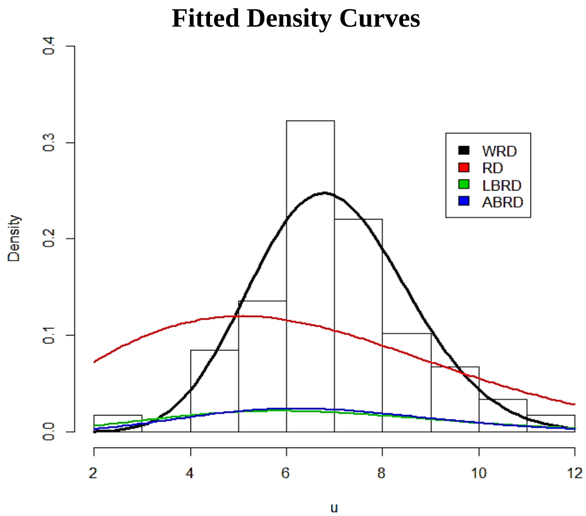

Figure 6: Plots for the Histogram and Estimated Density using Data Set 2

Histograms for the two data sets and estimated densities of WR, Rayleigh, LBR and ABR distributions are plotted in Figure 5 and Figure 6 respectively. From these figures, it is revealed that WR distribution fits the both data sets in better way as compared to Rayleigh, LBR and ABR distributions.

8. ILLUSTRATIVE EXAMPLE

This section presents the performance of different priors based on a simulation study and real life data application.

8.1 Simulation Study

In this sub-section, random samples of size 25 (small), 50 (medium) and 100 (large) are generated from WR distribution with $\theta = 3$ and $\eta = 2$ and repeated the process for 100000 times. The values of the weighted parameter θ =1.026, 3.435, 5.765 are obtained through R-software. In our simulation study, we have used $a = 0.5, 1.5, 2.5,$ $b = 1.0, 2.5, 3.0$ and $d = 0.3, 0.7, 1.0.$ The results of the simulation study are presented in Tables 6,7 and 8.

Table 6
Posterior Mean and Posterior Variance under Jeffrey's, Quasi, Gumbel Type II and Levy Priors using Generated Data

n	θ	*a*	*b*	*d*	Jeffrey's Prior		Quasi Prior		Gumbel Type II Prior		Levy Prior	
					Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance
	1.026	0.5	1.0	0.3	2.899532	0.234676	2.955717	0.248719	2.836095	**0.218423**	2.874086	0.227402
25	3.435	1.5	2.5	0.7	1.595149	0.038589	1.602330	0.039116	1.593748	**0.037947**	1.601858	0.038622
	5.765	2.5	3.0	1.0	1.111519	**0.012996**	1.111519	0.012996	1.125824	0.013194	1.121297	0.013157
	1.026	0.5	1.0	0.3	3.625567	0.178476	3.659886	0.183616	3.584251	**0.172094**	3.608098	0.175568
50	3.435	1.5	2.5	0.7	2.006662	0.030078	2.011136	0.030280	2.002933	**0.029744**	2.008484	0.030020
	5.765	2.5	3.0	1.0	1.401417	**0.010222**	1.401417	0.010222	1.407076	0.010251	1.405545	0.010256
	1.026	0.5	1.0	0.3	3.240988	0.070355	3.256153	0.071349	3.222872	**0.069107**	3.233558	0.069799
100	3.435	1.5	2.5	0.7	1.799152	0.011999	1.801148	0.012039	1.798052	**0.011940**	1.800444	0.011994
	5.765	2.5	3.0	1.0	1.257897	**0.004097**	1.257897	0.004097	1.261096	0.004106	1.260143	0.004106

Table 7
Posterior Mean and Posterior Variance under Jeffrey's, Quasi, Gumbel Type II and Levy Priors using Generated Data under Normal Approximation Technique

n	θ	*a*	*b*	*d*	Jeffrey's Prior		Quasi Prior		Gumbel Type II Prior		Levy Prior	
					Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance
	1.026	0.5	1.0	0.3	2.75016	0.194808	2.800663	0.20573	2.69366	**0.18219**	2.72791	0.18923
25	3.435	1.5	2.5	0.7	1.54887	0.034799	1.555641	0.03525	1.54817	**0.03427**	1.55572	0.03485
	5.765	2.5	3.0	1.0	1.08884	**0.012090**	1.088849	0.01209	1.10309	0.01228	1.09854	0.01224
	1.026	0.5	1.0	0.3	3.53096	0.16265	3.56351	0.16719	3.49193	**0.15703**	3.51456	0.16010
50	3.435	1.5	2.5	0.7	1.97734	0.02856	1.98168	0.028753	1.97387	**0.02825**	1.97924	0.02851
	5.765	2.5	3.0	1.0	1.38705	**0.00985**	1.38705	0.00985	1.39272	0.00989	1.39117	0.00989
	1.026	0.5	1.0	0.3	3.19842	0.06716	3.21319	0.068104	3.18082	**0.06599**	3.19123	0.066649
100	3.435	1.5	2.5	0.7	1.78596	0.01169	1.78792	0.011733	1.78491	**0.01163**	1.78726	0.01169
	5.765	2.5	3.0	1.0	1.257897	**0.004097**	1.257897	0.004097	1.261096	0.004106	1.260143	0.004106

Table 8
Posterior Mean and Posterior Variance under Jeffrey's, Quasi, Gumbel Type II and Levy Priors using Generated Data under T-K Approximation

n	θ	*a*	*b*	*d*	Jeffrey's Prior		Quasi Prior		Gumbel Type II Prior		Levy Prior	
					Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance
25	1.026	0.5	1.0	0.3	2.897341	0.233963	2.898956	0.566549	2.834060	**0.217793**	2.871971	0.226728
	3.435	1.5	2.5	0.7	1.594775	0.038553	1.594913	0.061577	1.593385	**0.037911**	1.601488	0.03858
	5.765	2.5	3.0	1.0	1.11139	**0.012990**	1.111391	0.012990	1.12569	0.013188	1.121170	0.013150
50	1.026	0.5	1.0	0.3	3.624881	0.178340	3.625389	0.42963	3.583591	**0.171967**	3.607425	0.175436
	3.435	1.5	2.5	0.7	2.006545	0.030070	2.006588	0.048065	2.002818	**0.029737**	2.008367	0.030013
	5.765	2.5	3.0	1.0	1.401376	**0.010221**	1.401376	0.010221	1.407036	0.010250	1.405504	0.010254
100	1.026	0.5	1.0	0.3	3.240835	0.070341	3.240948	0.169109	3.222721	**0.069094**	3.233406	0.069785
	3.435	1.5	2.5	0.7	1.799126	0.011999	1.799136	0.019188	1.798025	**0.011940**	1.800418	0.011994
	5.765	2.5	3.0	1.0	1.257888	**0.004096**	1.257888	0.004096	1.261087	0.004106	1.260134	0.004105

8.2 Real life data application

Here we consider data set 1 and data set 2 for comparing the performance of different priors and the results are presented in Tables 9, 10,11,12,13 and 14 respectively.

Table 9
Posterior Mean and Posterior Variance under Jeffrey's, Quasi, Gumbel Type II and Levy Priors using Data Set 1

θ	*a*	*b*	*d*	Jeffrey's Prior		Quasi Prior		Gumbel Type II Prior		Levy Prior	
				Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance
2.149	0.5	1.0	0.3	4.776522	0.0931742	4.790160	0.09397573	4.759198	**0.0921234**	4.768857	0.0926862
	1.5	2.5	0.7			4.782357	0.09351662	4.763249	**0.0922803**	4.771902	0.0928046
	2.5	3.0	1.0			4.776522	0.09317425	4.767300	**0.0924373**	4.772916	0.0928440
3.976	0.5	1.0	0.3	3.312111	0.0310264	3.318663	0.03121110	3.304203	**0.0307914**	3.308855	0.0309217
	1.5	2.5	0.7			3.314916	0.03110540	3.307015	**0.0308438**	3.310968	0.0309612
	2.5	3.0	1.0			3.312111	0.03102644	3.309827	**0.0308963**	3.311672	0.0309744
5.998	0.5	1.0	0.3	2.473002	0.0129056	2.476653	0.01296293	2.468856	**0.0128353**	2.471453	0.0128759
	1.5	2.5	0.7			2.474566	0.01293016	2.470958	**0.0128572**	2.473031	0.0128923
	2.5	3.0	1.0			2.473002	0.01290565	2.473059	**0.0128790**	2.473557	0.0128978

Table 10
Posterior Mean and Posterior Variance under Jeffrey's, Quasi, Gumbel Type II and Levy Priors using Data Set 2

θ	*a*	*b*	*d*	Jeffrey's Prior		Quasi Prior		Gumbel Type II Prior		Levy Prior	
				Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance
	0.5	1.0	0.3			10.44793	0.7591201	10.33010	**0.7334237**	10.36548	0.7410023
2.966	1.5	2.5	0.7	10.397661	0.7481911	10.41914	0.7528491	10.33693	**0.7343933**	10.37061	0.7417370
	2.5	3.0	1.0			10.39766	0.7481911	10.34375	**0.7353636**	10.37233	0.7419819
	0.5	1.0	0.3			8.905697	0.4696542	8.820370	**0.4561061**	8.846150	0.4601249
3.816	1.5	2.5	0.7	8.869149	0.4638844	8.884776	0.4663455	8.826199	**0.4567091**	8.850534	0.4605811
	2.5	3.0	1.0			8.869149	0.4638844	8.832027	**0.4573125**	8.851995	0.4607332
	0.5	1.0	0.3			7.264430	0.2546276	7.207991	**0.2486469**	7.225198	0.2504347
5.117	1.5	2.5	0.7	7.240094	0.2520730	7.250504	0.2531636	7.212754	**0.2489756**	7.228779	0.2506830
	2.5	3.0	1.0			7.240094	0.2520730	7.217517	**0.2493045**	7.229972	0.2507658

Table11
Posterior Mean and Posterior Variance under Jeffrey's, Quasi, Gumbel Type II and Levy Priors using Data Set 1 using Normal Approximation Techniques

θ	*a*	*b*	*d*	Jeffrey's Prior		Quasi Prior		Gumbel Type II Prior		Levy Prior	
				Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance
	0.5	1.0	0.3			4.75139	0.09133	4.72095	**0.08955**	4.73045	0.09009
2.149	1.5	2.5	0.7	4.73798	0.09056	4.74372	0.09089	4.72497	**0.08970**	4.73347	0.09021
	2.5	3.0	1.0			4.73798	0.09056	4.72898	**0.08986**	4.73448	0.09025
	0.5	1.0	0.3			3.30001	0.03060	3.28572	**0.03019**	3.29032	0.030319
3.976	1.5	2.5	0.7	3.29353	0.03042	3.29630	0.03049	3.28851	**0.03024**	3.29242	0.03035
	2.5	3.0	1.0			3.29353	0.03042	3.29131	**0.03029**	3.29312	0.03037
	0.5	1.0	0.3			2.46625	0.01277	2.45852	**0.01264**	2.46109	0.01268
5.998	1.5	2.5	0.7	2.46263	0.01271	2.46418	0.01274	2.46061	**0.01266**	2.46267	0.01270
	2.5	3.0	1.0			2.46263	0.01271	2.46270	**0.01269**	2.46319	0.01270

Table12
Posterior Mean and Posterior Variance under Jeffrey's, Quasi, Gumbel Type II and Levy Priors using Data Set 2 using Normal Approximation Techniques

θ	*a*	*b*	*d*	Jeffrey's Prior		Quasi Prior		Gumbel Type II Prior		Levy Prior	
				Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance
2.966	0.5	1.0	0.3	10.2566	0.71323	10.30558	0.72348	10.19097	**0.69938**	10.22540	0.70649
	1.5	2.5	0.7			10.27758	0.71760	10.19770	**0.70030**	10.23047	0.70719
	2.5	3.0	1.0			10.25667	0.71323	10.20444	**0.70123**	10.23216	0.70742
3.816	0.5	1.0	0.3	8.76636	0.44531	8.80206	0.45077	8.71873	**0.43795**	8.74392	0.44175
	1.5	2.5	0.7			8.78162	0.44764	8.72449	**0.43853**	8.74825	0.44219
	2.5	3.0	1.0			8.76636	0.44531	8.73025	**0.43911**	8.74970	0.44234
5.117	0.5	1.0	0.3	7.17145	0.24379	7.19532	0.24624	7.13997	**0.24052**	7.15685	0.24223
	1.5	2.5	0.7			7.18166	0.24484	7.14469	**0.24084**	7.16040	0.24247
	2.5	3.0	1.0			7.17145	0.24379	7.14941	**0.24115**	7.16158	0.24255

Table13
Posterior Mean and Posterior Variance under Jeffrey's, Quasi, Gumbel Type II and Levy Priors using Data Set 1 using T-K Approximation Techniques

θ	*a*	*b*	*d*	Jeffrey's Prior		Quasi Prior		Gumbel Type II Prior		Levy Prior	
				Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance
2.149	0.5	1.0	0.3	4.77643	0.09316	4.77650	0.22382	4.75911	**0.09211**	4.76877	0.09267
	1.5	2.5	0.7			4.77646	0.14898	4.76316	**0 .09227**	4.77181	0.09279
	2.5	3.0	1.0			4.77643	0.09316	4.76721	**0.09243**	4.77283	0.09283
3.976	0.5	1.0	0.3	3.31208	0.03102	3.31210	0.07451	3.30417	**0.03079**	3.30882	0.03092
	1.5	2.5	0.7			3.31209	0.04962	3.30698	**0.03084**	3.31093	0.03096
	2.5	3.0	1.0			3.31208	0.03102	3.30979	**0.03089**	3.31164	0.03097
5.998	0.5	1.0	0.3	2.47299	0.01290	2.472998	0.03098	2.46884	**0.01283**	2.47144	0.01287
	1.5	2.5	0.7			2.472997	0.02064	2.47094	**0.01285**	2.47301	0.01289
	2.5	3.0	1.0			2.472991	0.01290	2.47304	**0.01287**	2.47354	0.01289

Table 14
Posterior Mean and Posterior Variance under Jeffrey's, Quasi, Gumbel Type II and Levy Priors using Data Set 2 using T-K Approximation Techniques

θ	*a*	*b*	*d*	Jeffrey's Prior		Quasi Prior		Gumbel Type II Prior		Levy Prior	
				Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance	Posterior Mean	Posterior Variance
2.966	0.5	1.0	0.3	10.39714	0.74803	10.39753	1.79848	10.32959	**0.73327**	10.36496	0.74085
	1.5	2.5	0.7			10.39733	1.19575	10.33641	**0.73424**	10.37009	0.74158
	2.5	3.0	1.0			10.39714	0.74803	10.34324	**0.73521**	10.37181	0.74183
3.816	0.5	1.0	0.3	8.86882	0.46381	8.86906	1.11482	8.82004	**0.45603**	8.84582	0.46005
	1.5	2.5	0.7			8.86894	0.74151	8.82587	**0.456642**	8.85021	0.46051
	2.5	3.0	1.0			8.86882	0.46381	8.83170	**0.45724**	8.85167	0.46066
5.117	0.5	1.0	0.3	7.23991	0.25204	7.24004	0.60564	7.20781	**0.24862**	7.22502	0.25041
	1.5	2.5	0.7			7.23998	0.40300	7.21257	**0.24895**	7.22860	0.25065
	2.5	3.0	1.0			7.23991	0.25204	7.21734	**0.24928**	7.22979	0.25074

9. CONCLUSION

The extension of Rayleigh model that is, weighted Rayleigh distribution has been compared with its sub models for flexibility and efficiency. Different distributional and ageing properties of the Weighted Rayleigh distribution have been computed. It can be clearly seen from Tables 4 and 5 that the generalized model fits perfectly to the two data sets also with lesser values of AIC, AICC and BIC than its sub-models. This can also be justified by observing the fitted density curves of the two data sets. The estimation of parameters of WR distribution is done by the method of maximum likelihood. Further, the posterior variance of the scale parameter η under different informative and non-informative priors using different techniques has been ascertained. Tables 6, 7, 8, 9, 10, 11, 12, 13 and 14 represents posterior mean and posterior variance under Jeffrey's, Quasi, Gumbel type II and Levy priors using generated and real life data sets. From these tables, it is observed that the variability is less under the Gumbel Type II Prior especially under normal approximation method and thus can be preferred as compared to other priors used in the study.

10. REFERENCES

1. Ahsanullah, M. and Shakil, M. (2013). Characterizations of Rayleigh distribution based on order statistics and record values. *Bulletin of the Malaysian Mathematical Sciences Society*, 36(3), 625-635.
2. Ajami, M. and Jahanshahi, S.M.A. (2017). Parameter Estimation in Weighted Rayleigh Distribution. *Journal of Modern Applied Statistical Methods*, 16(2), 256-276.

3. Ardianti, F. (2018). Estimating parameter of Rayleigh distribution by using Maximum Likelihood method and Bayes method. In *IOP Conference Series: Materials Science and Engineering, IOP Publishing,* 300(1), 012-036.
4. Asgharzadeh, A. and Azizpour, M. (2016). Bayesian inference for Rayleigh distribution under hybrid censoring. *International Journal of System Assurance Engineering and Management*, 7(3), 239-249.
5. Dar, A.A., Ahmed, A. and Reshi, J.A. (2018). Characterization and Estimation of Weighted Maxwell-Boltzmann Distribution. *Applied Mathematics & Information Sciences*, 12(1), 193-202.
6. Dey, S. and Dey, T. (2011). Rayleigh distribution revisited via extension of Jeffrey's prior information and a new loss function. *Revstat–Statistical Journal*, 9(3), 213-226.
7. Fisher, R.A. (1934). The Effects of Methods of Ascertainment upon the Estimation of Frequencies, *Annals of Eugenics*, 6, 13-25.
8. Gupta, R.D. and Kundu, D. (2009). A new class of weighted exponential distributions. *Statistics*, 43(6), 621-634.
9. Jain, K., Singla, N. and Gupta, R. (2014). A weighted version of gamma distribution. *Discussiones Mathematicae Probability and Statistics*, 34, 89-111.
10. Jan U., Fatima, K and Ahmad S.P. (2017). On Weighted Ailamujia Distribution and its Applications to Lifetime Data. *Journal of Statistics Applications and Probability,* 6(3), 619-633.
11. Jan, U. and Ahmad S.P. (2017). Bayesian Analysis of Inverse Lomax Distribution Using Approximation Techniques. *Mathematical theory and Modelling* 7(7), 1-12.
12. Lawless, J.F. (2003). Statistical models and methods for lifetime data (Vol. 362). John Wiley & Sons.
13. Lindley, D.V. (1980). Approximate Bayes Methods. Bayesian Statistics eds. J.M. Bernado, M.H. Degroot, D.V. Lindley and A.M.F. Smith Valencia, Spain: University Press.
14. Mudasir, S. and Ahmad, S.P. (2015). Structural properties of length biased Nakagami distribution. *International Journal of Modern Mathematical Sciences*, 13(3), 217-227.
15. Mudasir, S. and Ahmad, S.P. (2017). Parameter estimation of weighted Erlang distribution using R-software. *Mathematical theory and modeling*, 7, 1-21.
16. Nadarajah, S. and Kotz, S. (2006). The beta exponential distribution. *Reliability Engineering & System Safety*, 91(6), 689-697.
17. Patil, G.P. and Rao, C.R. (1978). Weighted distributions and size-biased sampling with applications to wildlife populations and human families. *Biometrics*, 34, 179-189.
18. Rao, C.R. (1965). On discrete distributions arising out of methods of ascertainment, in G.P. Patil, ed. *Classical and Contagious Discrete Distributions*, Pergamon Press and Statistical Publishing Society, Calcutta, India.
19. Rayleigh, J. (1880). On the resultant of a large number of vibrations of the same pitch and of arbitrary phase, *Philosophical Magazine*, 10, 73-78.

20. Siddiqui, M.M. (1962). Some problems connected with Rayleigh distributions. *Journal of Research of the National Bureau of Standards*, 66(2), 167-174.
21. Sultan, H. and Ahmad, S.P. (2015). Bayesian Approximations for Shape Parameter of Generalized Power Function Distribution. *Journal of Reliability and Statistical Studies*, 10(1), 149-159.
22. Sultan, H., Jan, U. and Ahmad, S.P. (2018). Bayesian Normal and T K Approximations for the shape Parameter of Type 1 Dagum Distribution. *International Journal of Mathematical Sciences and Computing*, 3, 13-22.
23. Tierney, L. and Kadane, J.B. (1986). Accurate approximations for posterior moments and marginal densities. *Journal of the American Statistical Association*, 81(393), 82-86.