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BAYESIAN PROCEDURE MODIFIED BY FOURIER SERIES RESIDUAL TO FITTING THE LOGISTIC GROWTH MODEL

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ABSTRACT

The logistic growth model is used in multiple scientific applications, and it has been applied in the Bayesian procedure to estimate this model. It has also been employed in this paper by using Fourier series residuals. This was done to obtain the Bayesian procedure Modified by Fourier series Residuals to fit the Logistic Growth Model as a special case of non-linear models. To achieve this aim it is necessary to use a simulation technique to generate the samples and perform the required comparisons using various sample sizes (10, 18, 26, 34, and 42) and models depending on the standard deviation (0.5, 2.5). The results obtained have supported the modified method, which is as a result of the Absolute Percentage Errors (APE), in addition to the Mean Absolute Percentage Error (MAPE) and the Root Mean Squares Error (RMSE). We have also applied the new method on data that represented the cost of implementation of the mountain pass La Braguía in the North of Spain from June 2012 to July 2013.

AMS Subject Classification: 62F15, 62P20.

KEYWORDS

Autocatalytic; Logistic Growth Curve; Fourier Series Residual; Nonlinear Model; Simulation technique.

1. INTRODUCTION

The logistic growth model is usually used to describe the phenomenon of slow growth, followed by rapid growth, which in turn is followed by slow growth again to an asymptotic maximum, to take the shape of an S-curve. The reason is that projects start slowly when the resources necessarily need to be set up, and then projects start to accelerate once all resources have been acquired (José et al., 2015).

Fekedulegn, Mac Siurtain, and Colbert (1999) compare the growth of some models as the parameters of these models have been estimated using the iterative Marquardt non-linear regression; those models have been applied to high spruce in Norway. David (2001) aimed to demonstrate that the curve Logistic can be derived directly as a simple consequence of the more familiar differential equation model for exponential decay and that the curve itself is nothing more than a familiar friend in disguise. The disguise is removed by abandoning our fixation on the reference point (0; P0), representing the initial population at time zero, in favour of a much more natural choice. Guo, Song and Ye (2005) apply the grey Verhulst model on time series error corrected for the port throughput forecasting, especially when the throughput increases according to the curve with S type, and it can yield more accurate results than the traditional model in the prediction about port throughput in the saturation stage. José et al. (2012) present methods of determining critical points in logistic functions that separate the early stages of growth from the asymptotic phase, with the aim of establishing a stopping critical point in the growth and on this basis determining differences in treatments. Dmitry and Roland (2012) used a single logistic curve and analysis of its components focusing on the coherence between model, data and interpretation. They also discussed the directions that lead to the improvement of these techniques. The new concept of applying the component logistic models for an unambiguous definition of the system is suggested and tested on two examples from completely different areas. Gleb (2012) presented a new method for analysing growth curves representing the logistic growth with changing carrying capacity. The presented method is intended for discovering such changes by analysing growth curves. The application of this method to the curves displaying the growth of CO2 emissions detects the pattern of changes in the limits to growth that can be interpreted as the invention of technologies that increase the availability of fossil fuels. Shaghayegh, Hamidreza and Shirin (2014) applied a Grey Verhulst model in studying global ICT development. Also, a new Grey Verhulst model and Fourier residual Grey Verhulst model were suggested to improve forecasting accuracy. Data of the world fixed-telephone subscriptions, which is one of the ICT indicators from 2001 to 2010, was used as a forecasted example. Finally, we increased the precision by employing the Grey Markov model, which is the mixture of the GM(1,1) and a Markov model.

Those have been encouraged to modify the estimates obtained from the use of the Bayesian procedure and have been dealing with the logistic model that is one of the nonlinear complex models; thus it is possible to expand and apply the results obtained from these models with other simpler models.

To achieve the aim, this paper is organised as follows. In Section 2, we present the logistic growth curve and Fourier Series Residual. In addition to that, we explain the Bayesian procedure Modified by Fourier Series Residuals (FB). In Section 3, we adopt simulation techniques to generate the samples and perform the required comparisons, with different sample sizes and using a variety of models. This is to determine the possible cases; note that this method is used in the case of phenomena, which includes a few observations, and in Section 4 we illustrate the results of our method via analysis of the simulated dataset. In section 5 we have used the data that represented the cost of implementing the mountain pass La Braguía in the North of Spain from June 2012 to July 2013. We conclude in Section 7.

2. LOGISTIC GROWTH CURVE

The growth rate of the Logistic growth curve can be defined as follows (Grewal, 1990):

$$\frac{\partial f(X)}{\partial X} = \frac{\lambda}{\alpha} f(X)\left[\alpha - f\left(X\right)\right] \tag{1}$$

where

α: The upper limit of the size of the population (or individual) after a period $(X \to \infty)$
λ : Relative Growth Rate
γ : Influence point
and $0 < f(X) < \alpha$

From the Eq. (1), by doing an integration with respect to (X) we get:

$$f(X) = \frac{\alpha}{1 + e^{-\lambda(X-\gamma)}} \quad , \; 0 < X < \infty \tag{2}$$

The Eq. (2) represents the logistics growth curve, also called Autocatalytic, which can be rewritten as follows:

$$f(X) = \frac{\alpha}{1 + \eta e^{-\lambda X}} + \underline{\varepsilon} \tag{3}$$

where $\eta > 0$.

This curve is used to study the population growth in a specific area or species, a private species going to decrease in its number, or the growth phenomenon in biology.

3. FOURIER SERIES RESIDUAL

Assuming the actual data of the phenomenon being studied is:

$$Y = (Y_1, Y_2, ..., Y_n) \tag{4}$$

where (Y) a nonnegative series and n is sample size.

The estimated values of the series in the specified method are:

$$\hat{Y} = (\hat{Y}_2, \hat{Y}_3, ..., \hat{Y}_n) \tag{5}$$

In addition, from it can be obtained residuals series described in the formula $\varepsilon = (\varepsilon_2, \varepsilon_3, ..., \varepsilon_n)'$, since:

$$\varepsilon_k = Y_k - \hat{Y}_k \quad , \; k = 2, 3, ..., n \tag{6}$$

And residuals series can be modified by using Fourier series as follows (Erdal, Baris and Okyay, 2010):

$$\varepsilon_k \cong \frac{1}{2}\alpha_0 + \sum_{i=1}^{z}\left[\alpha_i \cos\left(\frac{2\pi i}{T}k\right) + \beta_i \sin\left(\frac{2\pi i}{T}k\right)\right] \; , \; k = 2, 3, ..., n \tag{7}$$

where $T = n - 1$ and $z = \left[\frac{T}{2}\right] - 1$ which only takes integer numbers (Guo, Song and Ye, 2005).

The Eq. (7) can be rewritten as follows:

$$\varepsilon = PC \tag{8}$$

where P and C represent the known matrices as follows:

$$\mathrm{P} = \begin{bmatrix} \frac{1}{2} & \cos\left(\frac{2\pi\,1}{\mathrm{T}}\,2\right) \sin\left(\frac{2\pi\,1}{\mathrm{T}}\,2\right) & \cdots \cos\left(\frac{2\pi\,z}{\mathrm{T}}\,2\right) & \sin\left(\frac{2\pi\,z}{\mathrm{T}}\,2\right) \\ \frac{1}{2} & \cos\left(\frac{2\pi\,1}{\mathrm{T}}\,3\right) \sin\left(\frac{2\pi\,1}{\mathrm{T}}\,3\right) & \cdots \cos\left(\frac{2\pi\,z}{\mathrm{T}}\,3\right) & \sin\left(\frac{2\pi\,z}{\mathrm{T}}\,3\right) \\ \vdots & \vdots \quad \vdots & \vdots \quad \vdots & \vdots \\ \frac{1}{2} & \cos\left(\frac{2\pi\,1}{\mathrm{T}}\,\mathrm{n}\right) \sin\left(\frac{2\pi\,1}{\mathrm{T}}\,\mathrm{n}\right) & \cdots \cos\left(\frac{2\pi\,z}{\mathrm{T}}\,\mathrm{n}\right) & \sin\left(\frac{2\pi\,z}{\mathrm{T}}\,\mathrm{n}\right) \end{bmatrix} \tag{9}$$

$$C = \left(\alpha_0, \alpha_1, \beta_1, \alpha_2, \beta_2, ..., \alpha_z, \beta_z\right)' \tag{10}$$

By using the Ordinary Least Squares method to solve Eq. (8) the following equation can be obtained:

$$C = \left(P'P\right)^{-1} P'\varepsilon \tag{11}$$

Substituting for (α_i) and (β_i) from Eq. (7) into Eq. (11) yields the modified residuals values:

$$\hat{\varepsilon} = \left(\hat{\varepsilon}_2, \hat{\varepsilon}_3, ..., \hat{\varepsilon}_n\right)'. \tag{12}$$

4. BAYESIAN PROCEDURE MODIFIED BY FOURIER SERIES RESIDUALS (FB)

Following from the logistic growth curve shown in Eq. (3) and adding a random error term result as in the following formula:

$$f(X) = \frac{\alpha}{1 + \eta e^{-\lambda X}} + \underline{\varepsilon} \tag{13}$$

It is possible to re-formulate Eq. (13) as follows:

$$\underline{Y} = f(\underline{X};\ \underline{\theta}) + \underline{\varepsilon} \tag{14}$$

where $\underline{X} = \left(\underline{X}_1, \underline{X}_2,\ ...\ ,\ \underline{X}_k\right)'$, $\underline{\theta} = \left(\alpha,\ \eta,\ \lambda\right)$, and $\underline{\varepsilon}$ represented the random errors of rank (n×1).

And

$$f(\underline{X};\ \underline{\theta}) = \frac{\alpha}{1 + \eta e^{-\lambda \underline{t}}}\ ,\ E\left(\underline{Y}\right) = f(\underline{X};\ \underline{\theta})$$

It is assumed that

$$\underline{\varepsilon} \sim N(\underline{0}, \sigma^2 I_n)$$

where the observation (i) for the dependent variable can be written as follows:

$$Y_i = f(\underline{X}_i;\ \underline{\theta}) + \varepsilon_i \tag{15}$$

And $f(\underline{X}_i;\ \underline{\theta})$ is the nonlinear response function, which is nonlinear in its parameters (Kutner et al., 2004).

Rewriting the non-linear model (14) by using Taylor's series to $f(\underline{X},\underline{\theta})$ at the first derivative, we obtain (Sohn and Kang, 1996):

$$f(\underline{X}_i;\underline{\theta}) = f(\underline{X}_i;\underline{\theta}^{(0)}) + \sum_{j=1}^{p}\left[\frac{\partial f(\underline{X}_i;\underline{\theta})}{\partial\theta_j}\right]_{\underline{\theta}=\underline{\theta}^{(0)}}(\theta_j - \underline{\theta}_j^{(0)}) \tag{16}$$

Briefly

$$Y_i - f_i^{(0)} = \sum_{j=1}^{p} D_{ij}^{(0)}\beta_j^{(0)} + \varepsilon_i \tag{17}$$

where $f_i^{(0)} = f(\underline{X}_i;\underline{\theta}^{(0)})$, $\beta_j^{(0)} = \theta_j - \theta_j^{(0)}$ and $D_{ij}^{(0)} = \left[\frac{\partial f(\underline{X}_i;\underline{\theta})}{\partial\theta_j}\right]_{\underline{\theta}=\underline{\theta}^{(0)}}$

Then, the ordinary least squares method can be used to estimate the parameters $\left(\beta_j^{(0)}\right)$, where $\left(j = 1, 2, \ldots, p\right)$.

As it $\left(\hat{\underline{\beta}}^{(0)}\right)$ represents a vector of least squares estimation of the parameters $\left(\underline{\beta}^{(0)}\right)$ of rank (p×1).

Assuming $\underline{Y}^* = \underline{Y} - \underline{f}^{(0)}$ then:

$$\underline{Y}^* = D^{(0)}\underline{\beta}^{(0)} + \underline{\varepsilon} \tag{18}$$

where $\left(D^{(0)}\right)$ is a matrix of rank (n×p).

Using Bayes method, linear model parameters can be estimated (18), and assuming that the vector of random errors $(\underline{\varepsilon})$ is distributed as Normal distribution with mean (0), and variance-covariance matrix $\left(\sigma^2 I_n\right)$, with the independent variables $\left(D^{,S}\right)$ distributed independently of $(\underline{\varepsilon})$, then the probability distribution function of $\left(\underline{Y}^*\right)$ writes proportionally as follows:

$$L\left(\underline{Y}^* / \underline{\beta}, \sigma\right) \propto \frac{1}{\sigma^n}\exp\left\{-\frac{1}{2\sigma^2}\left[v\hat{\sigma}^2 + \left(\beta - \hat{\underline{\beta}}\right)' D' D\left(\underline{\beta} - \hat{\underline{\beta}}\right)\right]\right\} \tag{19}$$

where

$$v = n - p\,,\ \hat{\beta} = (D'D)^{-1} D' \underline{Y}^*,\quad \hat{\sigma}^2 = \frac{\left(\underline{Y}^* - D\hat{\underline{\beta}}\right)'\left(\underline{Y}^* - D\hat{\underline{\beta}}\right)}{v}$$

To obtain a posterior probability density function through which Bayesian can be estimated, and assuming simple information exists about the parameter (σ) that is the lower and upper limits (Halpern, 1973), and (β) distributed Multivariate Normal distribution, then natural conjugate probability density function can be used follows:

$$f(\beta,\sigma) \propto \frac{1}{\sigma^{(p+1)}}\ \exp\left\{-\frac{1}{2\sigma^2}\left[(\beta-\bar{\beta})'\,Q^{-1}\ \left(\beta-\bar{\beta}\right)\right]\right\} \tag{20}$$

The values of $(\bar{\beta})$ based on the default values for the parameters vector (β), and matrix values (Q) shall be as follows: $\sigma^2 Q = \sigma^2 \left(D'D\right)^{-1}$

Multiplying the Eq. (20) with the likelihood function described by Eq. (19), we get the joint posterior probability density function for parameters, namely:

$$f\left(\beta,\sigma/Y^*\right) \propto \frac{1}{\sigma^{(n+p+1)}}\ \exp\left\{-\frac{1}{2\sigma^2}\left[(\beta-\bar{\beta})'\,Q^{-1}\left(\beta-\bar{\beta}\right)+\left(Y^*-Z\beta\right)'\left(Y^*-Z\beta\right)\right]\right\} \tag{21}$$

With the integration of Eq. (21) with respect to the limits (σ), we get the joint posterior probability density function for the parameters (β), that is:

$$f\left(\beta/Y^*\right) \propto \left[1+\left(\beta-E\right)'\,\frac{\beta^*}{M}\left(\beta-E\right)\right]^{-\frac{(n+p)}{2}} \tag{22}$$

where $\beta^* = \left(D'D + Q^{-1}\right)$, $M = \left(Y^{*'}Y^* + \bar{\beta}'\,Q^{-1}\ \bar{\beta}\right) - E'\ \beta^*\,E$.

Eq. (22) is the multivariate (t) distribution with (n) degree of freedom and with assuming a quadratic loss function then (E) represented the Bayesian estimated for the parameters (β), namely:

$$E = \left(D'D + Q^{-1}\right)^{-1}\left(Q^{-1}\ \bar{\beta} + D'\ \mathrm{Y}^*\right) \tag{23}$$

The estimate we get from Eq. (23) is added to the initial estimate to obtain the Bayesian estimation for the parameters (θ) of the multiple non-linear model shown in the Eq. (14), and depending on the repetition we get to the final estimate of the parameters (θ) and it can get the predicted values $\hat{Y}$.

We can employ the Residuals series resulting from estimating the logistic growth model by the Bayesian procedure in Eq. (11), which can then be used to compensate for outputs in Eq. (7) to obtain the modified Residuals series values and predictive values:

$$\hat{Y}f_1 = \hat{Y}_1$$
$$\hat{Y}f_k = \hat{Y}_k + \hat{\varepsilon}_k \ , \ k = 2,3,\ldots,n . \tag{24}$$

5. SIMULATION

This paper used a logistic growth model to generate the data, depending on the initial values of the parameters, which were assumed according to the cost of implementing the mountain pass La Braguía in the North of Spain from June 2012 to July 2013 (José et al., 2015). Five samples were generated to simulate this model (using sizes 10, 18, 26, 34 and 42 observations), and these observations represent the growth size. Eight models were generated, as shown in Table 1, depending on the logistic growth curve shown in Eq. (3). A random error term was also generated for this model, considering that it distributes normal, with mean zero, and standard deviation taking two values (0.5, 2.5).

Table 1
Assume Values for the Parameters of the Logistic Growth Model

Model		**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Logistic	α	352.9	352.9	352.9	352.9	352.5	352.5	352.5	352.5
	η	12.8	12.8	12.5	12.5	12.8	12.8	12.5	12.5
	λ	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6
	σ	0.5	2.5	0.5	2.5	0.5	2.5	0.5	2.5

6. RESULTS

Depending on the run size, 1000 experiments for each model of the virtual models have compared the estimated logistic growth model parameters by using ordinary least squares (OLS), Bayesian procedure (BNC) and Bayesian procedure Modified by Fourier Series Residuals (FB). The following are the results of the different criteria that have been adopted in this paper to compare the methods.

We used the Mean Absolute Percentage Error (MAPE) (Shaghayegh, Hamidreza and Shirin, 2014) and Root Mean Squares Error (RMSE) (Chai and Draxler, 2014) to compare the methods.

Table 2 shows MAPE according to each method for all eight generated models. The results in the table show that 37 out of 40 values prefer BF. When comparing BNC with OLS (shown in Table 2), the results have shown that 3 out of 40 values prefer OLS.

With regards to Table 3 (including RMSE) the results have given preference to BF compared to other methods for all RMSE values.

Table 2
Mean Absolute Percentage Error (MAPE)

N	Model	OLS	BNC	BF	Best
10	1	0.1033	0.1187	0.1024	BF
	2	0.1308	0.2369	0.1141	BF
	3	0.1070	0.1160	0.1012	BF
	4	0.1407	0.2543	0.1352	BF
	5	0.1064	0.1164	0.1076	OLS
	6	0.1420	0.2599	0.1337	BF
	7	0.1060	0.1173	0.1068	OLS
	8	0.1240	0.2572	0.1295	OLS
18	1	0.0607	0.0642	0.0575	BF
	2	0.0908	0.1213	0.0697	BF
	3	0.0626	0.0649	0.0559	BF
	4	0.0781	0.1285	0.0640	BF
	5	0.0626	0.0671	0.0603	BF
	6	0.0873	0.1361	0.0693	BF
	7	0.0615	0.0629	0.0568	BF
	8	0.0774	0.1312	0.0752	BF
26	1	0.0436	0.0465	0.0400	BF
	2	0.0700	0.1057	0.0503	BF
	3	0.0426	0.0462	0.0388	BF
	4	0.0721	0.1106	0.0431	BF
	5	0.0450	0.0463	0.0394	BF
	6	0.0572	0.0902	0.0454	BF
	7	0.0432	0.0458	0.0399	BF
	8	0.0549	0.0983	0.0480	BF
34	1	0.0335	0.0360	0.0298	BF
	2	0.0529	0.0799	0.0355	BF
	3	0.0332	0.0358	0.0308	BF
	4	0.0446	0.0810	0.0327	BF
	5	0.0341	0.0361	0.0299	BF
	6	0.0537	0.0851	0.0324	BF
	7	0.0326	0.0359	0.0303	BF
	8	0.0530	0.0877	0.0316	BF
42	1	0.0298	0.0326	0.0245	BF
	2	0.0433	0.0665	0.0312	BF
	3	0.0277	0.0304	0.0250	BF
	4	0.0451	0.0722	0.0291	BF
	5	0.0280	0.0302	0.0241	BF
	6	0.0435	0.0648	0.0282	BF
	7	0.0271	0.0297	0.0249	BF
	8	0.0440	0.0723	0.0250	BF

Table 3
Root Mean Squares Error (RMSE)

N	Model	OLS	BNC	BF	Best
10	**1**	0.4457	0.4160	0.1986	**BF**
	2	2.0238	2.1365	0.9569	**BF**
	3	0.4444	0.4187	0.1972	**BF**
	4	2.0099	2.1370	0.9757	**BF**
	5	0.4492	0.4161	0.1983	**BF**
	6	1.9931	2.1141	0.9416	**BF**
	7	0.4490	0.4186	0.1962	**BF**
	8	2.0475	2.1596	0.9749	**BF**
18	**1**	0.4635	0.4547	0.1487	**BF**
	2	2.2541	2.3123	0.7351	**BF**
	3	0.4659	0.4569	0.1473	**BF**
	4	2.2457	2.3166	0.7223	**BF**
	5	0.4670	0.4573	0.1473	**BF**
	6	2.2507	2.3219	0.7377	**BF**
	7	0.4644	0.4533	0.1461	**BF**
	8	2.2642	2.3365	0.7421	**BF**
26	**1**	0.4741	0.4697	0.1192	**BF**
	2	2.3283	2.3807	0.6164	**BF**
	3	0.4784	0.4740	0.1254	**BF**
	4	2.3234	2.3733	0.6192	**BF**
	5	0.4756	0.4701	0.1225	**BF**
	6	2.3229	2.3719	0.6015	**BF**
	7	0.4748	0.4709	0.1230	**BF**
	8	2.3206	2.3727	0.6121	**BF**
34	**1**	0.4788	0.4768	0.1054	**BF**
	2	2.3708	2.4052	0.5275	**BF**
	3	0.4818	0.4801	0.1079	**BF**
	4	2.3688	2.4088	0.5359	**BF**
	5	0.4799	0.4773	0.1086	**BF**
	6	2.3833	2.4215	0.5436	**BF**
	7	0.4794	0.4773	0.1079	**BF**
	8	2.3616	2.3976	0.5384	**BF**
42	**1**	0.4859	0.4842	0.0975	**BF**
	2	2.3889	2.4187	0.4856	**BF**
	3	0.4825	0.4814	0.0954	**BF**
	4	2.3921	2.4224	0.4783	**BF**
	5	0.4838	0.4822	0.0955	**BF**
	6	2.4094	2.4344	0.4890	**BF**
	7	0.4829	0.4815	0.0983	**BF**
	8	2.4028	2.4341	0.4788	**BF**

7. APPLICATION

The section aims to use the three methods (OLS, BNC and BF) based on actual data, which could benefit the specialists in management and economics to monitoring and reviewing the gap between actual and predictive observations of the logistic growth model depending on the nature of the data used. To achieve this aim, we have used the data that represented the cost of implementing the mountain pass La Braguía in the North of Spain from June 2012 to July 2013 (José et al., 2015). Table 4 shows the Actual cost, the expected cost and the errors of estimation corresponding.

Table 4
Actual Cost of the Implementation the Mountain Pass and Predictive Values with the Error Corresponding

Actual Values	OLS		BNC		BF	
	Forecasting	Error	Forecasting	Error	Forecasting	Error
25.567	27.7853	-2.2183	22.0972	3.4698	25.567	0
66.293	50.535	15.758	44.9698	21.3232	65.8598	-0.4332
78.293	86.6119	-8.3189	84.5569	-6.2639	78.7262	0.4332
124.073	136.0083	-11.9353	140.4981	-16.4251	123.6398	-0.4332
191.367	191.5991	-0.2321	200.4921	-9.1251	191.8002	0.4332
259.845	241.8983	17.9467	248.4836	11.3614	259.4118	-0.4332
285.612	279.1849	6.4271	278.6524	6.9596	286.0452	0.4332
290.843	302.9406	-12.0976	294.8436	-4.0006	290.4098	-0.4332
303.489	316.6477	-13.1587	302.8016	0.6874	303.9222	0.4332
316.431	324.1082	-7.6772	306.5439	9.8871	315.9978	-0.4332
320.69	328.0402	-7.3502	308.2671	12.4229	321.1232	0.4332
336.756	330.0773	6.6787	309.0529	27.7031	336.3228	-0.4332
349.379	331.1235	18.2555	309.4096	39.9694	349.8122	0.4332

It is clear from Table 4 that the BF is the best as it gave the least errors. Figure 1 shows the actual and predictive values depending on the method used, as shown approaching very large predictive values using BF from the actual values compared to the other two methods, so the predictive values and errors resulting from it according to the BF were better than the method of the gray prediction reached by the José (Joséet al., 2015).

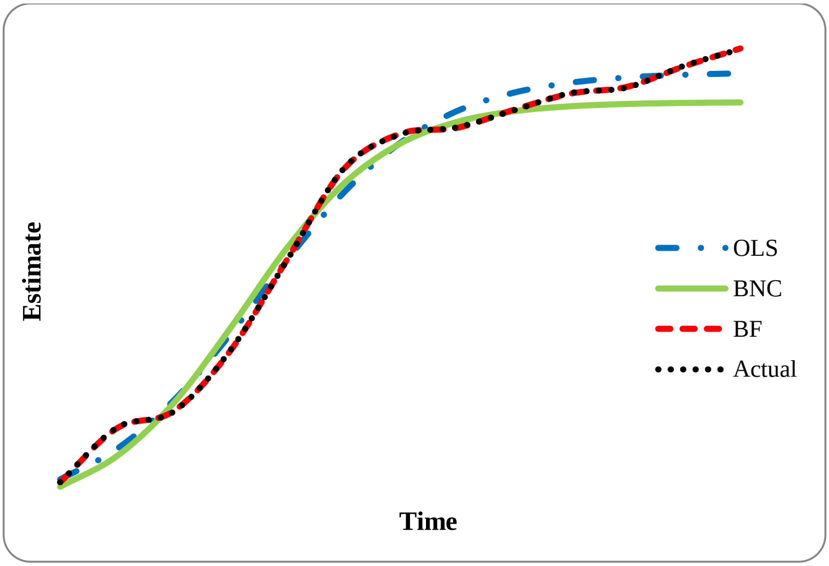

Figure 1: Actual and Predictive Values According to the Method

Comparing the Absolute Relative Error (APE) (Erdal, Baris and Okyay, 2010) as in Figure 2, it is shown that the values of APE resulting from the use of BF are less than the other. This gives another indication of the preference of the suggested method.

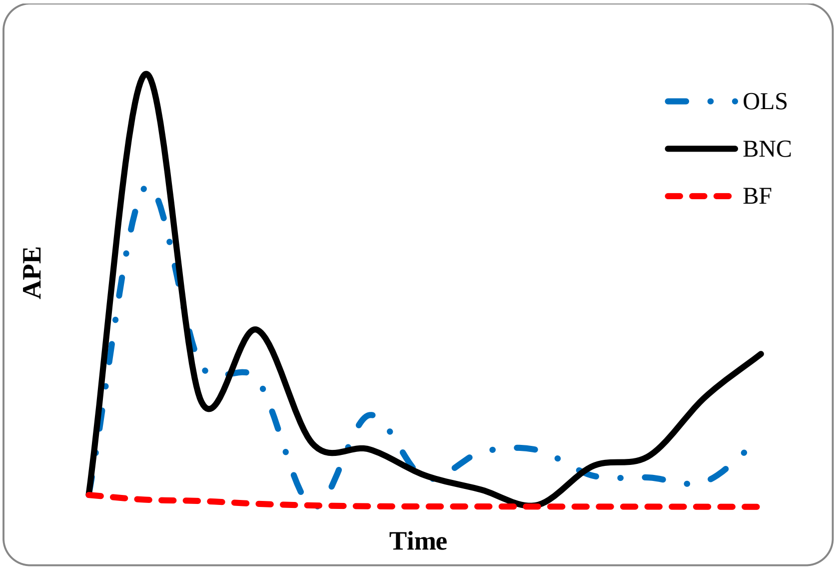

Figure 2: Absolute Relative Error (APE)

Table 5 shows that the BF is best and most accurate, as it showed the lower MAPE and RMSE.

Table 5
Prediction Accuracy Criteria

Method	MAPE	RMSE
OLS	74.7152	11.2261
BNC	94.2485	15.0174
BF	3.9169	0.4162

8. CONCLUSION

Depending on the results of simulations and when analysing the APE, MAPE and RMSE, the Bayesian procedure Modified by Fourier Series Residuals (FB) reached more efficient predictions for the logistic growth model compared to the other methods, regardless of the sample size or standard deviation value. Also, the ordinary least squares method (OLS) gave predictions for the logistic growth model better than the Bayesian method (BNC).

On the other side and through the cost implementation mountain pass in the North of Spain for the period from June data 2012 to July 2013, it has been found that Bayesian procedure Modified by Fourier Series Residuals (FB) gave the best results compared to other methods. It is also better than the A residual Grey forecasting method.

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