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ON THE PRODUCT AND QUOTIENT OF PARETO AND RAYLEIGH RANDOM VARIABLES

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ABSTRACT

The distributions of products and ratios of random variables are of interest in many areas of the sciences. In this paper, we find analytically the probability distributions of the product XY and the ratio X/Y, when X and Y are two independent random variables following Pareto and Rayleigh distributions respectively.

KEYWORDS

Product Distribution, Ratio Distribution, Pareto Distribution, Rayleigh Distribution, Error function, probability density function, Moment of order r, Survival function, Hazard function.

1. INTRODUCTION

For given random variables X and Y, the distributions of the product XY and the ratio X/Y are of interest in many areas of the sciences. Examples of XY include traditional portfolio selection models, relationship between attitudes and behavior, number of cancer cells in tumor biology and stream flow in hydrology. The distributions of ratio of random variables are of interest in many areas of the sciences, engineering, physics, number theory, order statistics, economics, biology, genetics, medicine, hydrology, psychology, classification, and ranking and selection, see Ali, Nadarajah and Woo (2005), Ali, Pal and Woo (2007), Shcolnick (1985) and Bhargava and Khatri (1981). Examples include safety factor in engineering, mass to energy ratios in nuclear physics, target to control precipitation in meteorology, inventory ratios in economics and Mendelian inheritance ratios in genetics, see Ali, Nadarajah and Woo (2005), Ali, Pal and Woo (2007). Also ratio distribution involving two Gaussian random variables are used in computing error and outage probabilities, see Bu-Salih (1983). It has many applications especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures and the aging of concrete pressure vessels, see Feldstein (1971), Gradshteyn and Ryzhik (2000). An important example of ratios of random variables is the stress strength model in the context of reliability. It describes the life of a component which has a random strength Y and is subjected to random stress X. The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever $Y > X$. Thus, $\Pr(X < Y)$ is a measure of component reliability see Feldstein (1971), Gradshteyn and Ryzhik (2000). The distributions of XY and X/Y have been studied by several

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authors, especially when X and Y are independent random variables and come from the same family. With respect to products of random variables, see Sakamoto (1943), Nadarajah (2005), for uniform family, Harter (1951), Steece, B.M. (1976), Wallgren (1980), Nadarajah and Choi (2006) for Student's t family, Springer and Thompson (1970, Nadarajah and Kotz (2005) for normal family, Stuart (1962), Nadarajah (2007) and Podolski, Nadarajah and Kotz (2006), Podolski (1972) for gamma family, Steece (1976), Nadarajah (2006), Bhargava and Khatri (1981), Harter (1951), Tang and Gupta (1984), Nadarajah (2010) for beta family, AbuSalih (1983), Gradshteyn and Ryzhik (2000), for power function family, and Malik and Trudel (1986), Idrizi (2014) for exponential family (for a comprehensive review of known results, see also Rathie and Rohrer (1987), Modi and Joshi (2012). With respect to ratios of random variables, see Marsaglia (1965), Stuart (1962), and Korhonen and Narula (1989), Tang and Gupta (1984) for normal family, Press (1969), Wallgren (1980) for Student's t family, Basu and Lochner (1971), Grubel (1968), for Weibull family, Shcolnick (1985) for stable family, Hawkins and Han (1986), Hawkins and Han (1986) for non-central chi-squared family, Provost (1989) for gamma family, and Pham-Gia (2000), Lomnicki (1967) for beta family. In latest years, the study of product and ratio when X and Y belong to different families has been a great interest in many areas of the sciences. For example, distributions of the product and ratio, when X and Y Are Gamma and Weibull independent random variables respectively, have been studied by Nadarajah and Kotz (2006), Park (2010). The distributions of the product and ratio, when X and Y are independent random variables with Pareto and Gamma distributions respectively, have been studied by Nadarajah (2010), Pham-Gia (2000). The Ratio of Pareto and Kumaraswamy Random Variables have been studied by Idrizi (2014), Ali, Pal and Woo (2007). In the applications mentioned before, it is quite possible that X and Y could arise from different but similar distributions. In this paper, we find analytically the probability distributions of the product XY and the ratio X/Y, when X and Y are two independent random variables following Pareto and Rayleigh distributions respectively. with probability density functions (p.d.f.s)

$$f_X(x) = \frac{ca^c}{x^{c+1}} \tag{1}$$

$$f_Y(y) = \frac{y}{b^2} e^{\frac{-y^2}{2b^2}} \tag{2}$$

respectively, for $a \le x < \infty, a > 0, c > 0, 0 < y < \infty, b > 0$. The calculations of this paper involve several important lemmas

LEMMA 1

For $p > 0$, we have

$$\int_u^{+\infty} \frac{e^{-p^2x^2}}{x^{2n}} dx = \frac{\Gamma(-\frac{2n-1}{2}, p^2u^2)}{2} p^{2n-1} \tag{3}$$

Proof:

By definition we know that $\Gamma(\mathrm{s}, \mathrm{x}) = \int_x^{+\infty} t^{s-1} e^{-t} dt$

if we substitute $t = p^2x^2$ in (3) we get

$$\frac{p^{2n}}{2p^2}\int_{p^2u^2}^{+\infty}\frac{e^{-t}}{t^n}\frac{p}{t^{1/2}}dt = \frac{p^{2n-1}}{2}\int_{p^2u^2}^{+\infty}t^{-(\frac{2n+1}{2})}e^{-t}dt = \frac{\Gamma(-\frac{2n-1}{2},p^2u^2)}{2}p^{2n-1} \tag{4}$$

LEMMA 2

For $p > 0, \mathrm{n} = 0,1, \dots$, we have

$$\int_0^{+\infty} x^{2n}e^{-px^2}dx = \frac{(2n-1)!!}{2(2p)^n}\sqrt{\frac{\pi}{p}} \tag{5}$$

Proof:

First of all, we have

$$\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi} \tag{6}$$

Let $X = \sqrt{p}x$

$$\int_0^{+\infty} x^{2n}e^{-px^2}dx = \frac{1}{(\sqrt{p})^{2n}\sqrt{p}}\int_0^{+\infty}X^{2n}e^{-X^2}dX = \frac{1}{p^n\sqrt{p}}\int_0^{+\infty}X^{2n}e^{-X^2}dX \tag{7}$$

Let $a_n = \int_0^{+\infty} X^{2n}e^{-X^2}dX$, prove that $a_n = \frac{(2n)!\sqrt{\pi}}{2^{2n}n!2}$

Integration by parts implies:

$$a_n = \frac{(2n-1)}{2}\int_0^{+\infty}X^{2n-2}e^{-X^2}dX = \frac{(2n-1)2n}{2^2n}a_{n-1} \tag{8}$$

And since $a_0 = \frac{\sqrt{\pi}}{2}$ we conclude

$$a_n = \int_0^{+\infty}X^{2n}e^{-X^2}dX = \frac{(2n)!}{2^{2n}n!}\frac{\sqrt{\pi}}{2}$$

and

$$\int_0^{+\infty} x^{2n}e^{-px^2}dx = \frac{(2n)!}{p^n\sqrt{p}2^{2n}n!}\frac{\sqrt{\pi}}{2} \tag{9}$$

And since $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi} = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$

$$\int_0^{+\infty} x^{2n}e^{-px^2}dx = \frac{\Gamma(n+\frac{1}{2})}{2p^n\sqrt{p}} = \frac{(2n-1)!!}{2(2p)^n}\sqrt{\frac{\pi}{p}} \tag{10}$$

LEMMA 3

For $p > 0$

$$\int_0^{+\infty} x^{2n+1} e^{-px^2} dx = \frac{n!}{2p^{n+1}} \tag{11}$$

Proof:

If we substitute $u = px^2$ we get

$$\int_0^{+\infty} x^{2n+1} e^{-px^2} dx = \frac{1}{2p^{n+1}} \int_0^{+\infty} u^n e^{-u} du \tag{12}$$

To get the result we have to prove

$$\int_0^{+\infty} u^n e^{-u} du = n! \tag{13}$$

For $n = 0$ integration by parts yields:

$$\int_0^{+\infty} e^{-u} du = 1 = 0! \tag{14}$$

Assume the statement holds for some arbitrary natural number $n \geq 0$,

And prove that the statement holds for $n + 1$.

Integration by parts implies:

$$\int_0^{+\infty} u^{n+1} e^{-u} du = (n+1) \int_0^{+\infty} e^{-u} u^n du = (n+1)n! = (n+1)! \tag{15}$$

2. DISTRIBUTION OF THE PRODUCT *XY*

Theorem 2.1.

Suppose X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$ the cumulative distribution function *c.d.f.* of $Z = XY$ can be expressed as:

<u>Case 1:</u>

If $c + 1 = 2n$,

$$F_Z(z) = 1 - \left(\frac{a}{z}\right)^c \frac{1}{b^2} \left[\frac{c!!}{2(\frac{1}{b^2})^{\frac{c+1}{2}}} \sqrt{2b^2\pi} \right] \tag{16}$$

<u>Case 2:</u>

If $c + 1 = 2n + 1$

$$F_Z(z) = 1 - \frac{1}{b^2}\left[\frac{\left(\frac{c}{2}\right)!}{2\left(\frac{1}{2b^2}\right)^{\frac{c+2}{2}}}\right] \tag{17}$$

Proof:

The c.d.f. corresponding to (1) is $F_X(x) = 1 - (\frac{a}{x})^c$ Thus, one can write the c.d.f. of XY as:

$$\begin{aligned}\Pr(XY \le z) &= \int_0^{+\infty} F_X\left(\frac{z}{y}\right) f_Y(y)dy \\ &= \int_0^{+\infty}\left[1 - \left(\frac{ay}{z}\right)^c\right]\frac{y}{b^2}e^{-\frac{y^2}{2b^2}}dy \\ &= 1 - \left(\frac{a}{z}\right)^c \frac{1}{b^2}\int_0^{+\infty} y^{c+1}e^{-\frac{y^2}{2b^2}}dy\end{aligned} \tag{18}$$

where, $z > 0, a > 0, c > 0, y > 0, b > 0$.

The proof of Theorem 2.1 easily follows by using Lemma 2 when $c + 1 = 2n$, and Lemma 3 when $c + 1 = 2n + 1$ in the integral above.

Corollary 2.2.

Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$, the probability density function p.d.f. of $Z = XY$ can be expressed as:

Case 1:

If $c + 1 = 2n$,

$$f_Z(z) = \frac{ca^c}{z^{c+1}}\frac{1}{b^2}\left[\frac{c!!}{2\left(\frac{1}{b^2}\right)^{\frac{c+1}{2}}}\sqrt{2b^2\pi}\right] \tag{19}$$

Case 2:

If $c + 1 = 2n + 1$

$$f_Z(z) = \frac{ca^c}{z^{c+1}}\frac{1}{b^2}\left[\frac{\left(\frac{c}{2}\right)!}{2\left(\frac{1}{2b^2}\right)^{\frac{c+2}{2}}}\right] \tag{20}$$

For $z > 0$.

Proof:

The probability density function $f_Z(z)$ in (20) and (19) easily follows by differentiation.

Corollary 2.3

Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $c > r$ and $z \geq \alpha, \alpha > 0$. the moment of order r of $Z = XY$ can be expressed as:

Case 1:

If $c + 1 = 2n$,

$$E[Z^r] = ca^c \frac{\alpha^{r-c}}{c-r} \frac{1}{b^2} \left[\frac{c!!}{2\left(\frac{1}{b^2}\right)^{\frac{c+1}{2}}} \sqrt{2b^2\pi} \right] \tag{21}$$

Case 2:

If $c + 1 = 2n + 1$

$$E[Z^r] = ca^c \frac{\alpha^{r-c}}{c-r} \frac{1}{b^2} \left[\frac{\left(\frac{c}{2}\right)!}{2\left(\frac{1}{2b^2}\right)^{\frac{c+2}{2}}} \right] \tag{22}$$

Proof:

Case 1:

If $c + 1 = 2n$,

$$\begin{aligned} E[Z^r] &= \int_{-\infty}^{+\infty} z^r f_Z(z) dz = \int_{\alpha}^{+\infty} z^r \frac{ca^c}{z^{c+1}} \frac{1}{b^2} \left[\frac{c!!}{2\left(\frac{1}{b^2}\right)^{\frac{c+1}{2}}} \sqrt{2b^2\pi} \right] dz \\ &= ca^c \frac{\alpha^{r-c}}{c-r} \frac{1}{b^2} \left[\frac{c!!}{2\left(\frac{1}{b^2}\right)^{\frac{c+1}{2}}} \sqrt{2b^2\pi} \right] \end{aligned} \tag{23}$$

For $c > r, z \geq \alpha, \alpha > 0$

Same proof for case 2.

Corollary 2.4

Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $c > 1, \alpha > 0$. the Expected value of $Z = XY$ is obtained for $r = 1$ and it can be expressed as:

for $r = 1$

Case 1:

If $c + 1 = 2n$,

$$E[Z] = ca^c \frac{\alpha^{1-c}}{c-1} \frac{1}{b^2} \left[\frac{c!!}{2\left(\frac{1}{b^2}\right)^{\frac{c+1}{2}}} \sqrt{2b^2\pi} \right] \tag{24}$$

Case 2:

If $c + 1 = 2n + 1$,

$$E[Z] = ca^c \frac{\alpha^{1-c}}{c-1} \frac{1}{b^2} \left[\frac{\left(\frac{c}{2}\right)!}{2\left(\frac{1}{2b^2}\right)^{\frac{c+2}{2}}} \right] \tag{25}$$

For $c > 1, \alpha > 0$.

Corollary 2.5

Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $c > 2, \alpha > 0$. the Variance of $Z = XY$ can be expressed as:

Case 1:

If $c + 1 = 2n$,

$$\sigma^2 = ca^c \frac{\alpha^{2-c}}{c-2} \frac{1}{b^2} \left[\frac{c!!}{2\left(\frac{1}{b^2}\right)^{\frac{c+1}{2}}} \sqrt{2b^2\pi} \right] - \left[ca^c \frac{\alpha^{1-c}}{c-1} \frac{1}{b^2} \left[\frac{c!!}{2\left(\frac{1}{b^2}\right)^{\frac{c+1}{2}}} \sqrt{2b^2\pi} \right] \right]^2 \tag{26}$$

Case 2:

If $c + 1 = 2n + 1$,

$$\sigma^2 = ca^c \frac{\alpha^{2-c}}{c-2} \frac{1}{b^2} \left[\frac{\left(\frac{c}{2}\right)!}{2\left(\frac{1}{2b^2}\right)^{\frac{c+2}{2}}} \right] - \left[ca^c \frac{\alpha^{1-c}}{c-1} \frac{1}{b^2} \left[\frac{\left(\frac{c}{2}\right)!}{2\left(\frac{1}{2b^2}\right)^{\frac{c+2}{2}}} \right] \right]^2 \tag{27}$$

For $c > 2, \alpha > 0$.

Proof:

For $r = 2$,

Case 1:

If $c + 1 = 2n$,

$$E[Z^2] = c\alpha^c \frac{\alpha^{2-c}}{c-2} \frac{1}{b^2} \left[\frac{c!!}{2\left(\frac{1}{b^2}\right)^{\frac{c+1}{2}}} \sqrt{2b^2\pi} \right] \tag{28}$$

Case 2:

If $c + 1 = 2n + 1$,

$$E[Z^2] = c\alpha^c \frac{\alpha^{2-c}}{c-2} \frac{1}{b^2} \left[\frac{\left(\frac{c}{2}\right)!}{2\left(\frac{1}{2b^2}\right)^{\frac{c+2}{2}}} \right] \tag{29}$$

The variance of $Z = XY$ is easy follows from:

$$\sigma^2 = E[Z^2] - E[Z]^2 \tag{30}$$

Corollary 2.6

Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $z > 0$. The Survival function of $Z = XY$ can be expressed as:

Case 1:

If $c + 1 = 2n$,

$$S_Z(z) = 1 - F_Z(z) = \left(\frac{a}{z}\right)^c \frac{1}{b^2} \left[\frac{c!!}{2\left(\frac{1}{b^2}\right)^{\frac{c+1}{2}}} \sqrt{2b^2\pi} \right] \tag{31}$$

Case 2:

If $c + 1 = 2n + 1$,

$$S_Z(z) = 1 - F_Z(z) = \left(\frac{a}{z}\right)^c \frac{1}{b^2} \left[\frac{\left(\frac{c}{2}\right)!}{2\left(\frac{1}{2b^2}\right)^{\frac{c+2}{2}}} \right] \tag{32}$$

Corollary 2.7

Let X and Y are independent and distributed according to (1) and (2), respectively. Then for > 0 the Hazard function of $Z = XY$ can be expressed as:

$$h_Z(z) = \frac{f_Z(z)}{S_Z(z)} = \frac{c}{z} \tag{33}$$

For case 1 and case 2.

3. DISTRIBUTION OF THE RATIO X/Y

Theorem 3.1

Suppose $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent and distributed according to (1) and (2), respectively. Then for $\boldsymbol{w > 0, u > 0}$. the cumulative distribution function c.d.f. of $\boldsymbol{W = X/Y}$ can be expressed as:

$$F_W(w) = 1 - \frac{a^c}{w^c b^2}\left[\frac{\Gamma(-\frac{c-2}{2}, \frac{u^2}{2b^2})}{2}\left(\frac{1}{\sqrt{2}b}\right)^{c-2}\right] \tag{34}$$

For $w > 0, u > 0$.

Proof:

The c.d.f. corresponding to (1) is $F_X(x) = 1 - (\frac{a}{x})^c$. Thus, one can write the c.d.f. of X/Y as:

$$\begin{aligned} F_W(w) = \Pr\left(\frac{X}{Y} \leq w\right) &= \int_0^{+\infty} F_X(wy) f_Y(y) dy \\ &= \int_u^{+\infty}\left[1 - \left(\frac{a}{yw}\right)^c\right] f_Y(y) dy \\ &= 1 - \frac{a^c}{w^c b^2}\int_u^{+\infty}\frac{e^{\frac{-y^2}{2b^2}}}{y^{c-1}} dy \\ &= 1 - \frac{a^c}{w^c b^2}\left[\frac{\Gamma\left(-\frac{c-2}{2}, \frac{u^2}{2b^2}\right)}{2}\left(\frac{1}{\sqrt{2}b}\right)^{c-2}\right] \end{aligned} \tag{35}$$

where, $w > 0, a > 0, c > 0, b > 0, y \geq u, u > 0$. The proof of Theorem 3.1 easily follows by using Lemma 1 in the integral above.

Corollary 3.2

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent and distributed according to (1) and (2), respectively. Then for $\boldsymbol{w > 0, u > 0}$. The probability density function p.d.f. of $\boldsymbol{W = X/Y}$ can be expressed as:

$$f_W(w) = \frac{ca^c}{b^2 w^{c+1}}\left[\frac{\Gamma\left(-\frac{c-2}{2}, \frac{u^2}{2b^2}\right)}{2}\left(\frac{1}{\sqrt{2}b}\right)^{c-2}\right] \tag{36}$$

For $w > 0, u > 0$.

Proof:

The probability density function $f_W(w)$ in (36) easily follows by differentiation.

Corollary 3.3

Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $c > r, a > 0, u > 0, l = \left[\frac{a}{u}\right] + 1$ and $w \geq l$. the moment of order r of $W = X/Y$ can be expressed as:

$$E[W^r] = \frac{ca^c}{b^2}\frac{l^{r-c}}{c-r}\left[\frac{\Gamma\left(-\frac{c-2}{2}, \frac{u^2}{2b^2}\right)}{2}\left(\frac{1}{\sqrt{2}b}\right)^{c-2}\right] \tag{37}$$

For $c > r, a > 0, u > 0, l = \left[\frac{a}{u}\right] + 1$ and $w \geq l$.

Proof:

$$\begin{aligned} E[W^r] &= \int_{-\infty}^{+\infty} w^r f_W(w) dw \\ &= \frac{ca^c}{b^2}\left[\frac{\Gamma\left(-\frac{c-2}{2}, \frac{u^2}{2b^2}\right)}{2}\left(\frac{1}{\sqrt{2}b}\right)^{c-2}\right]\int_l^{+\infty} \frac{w^r}{w^{c+1}} dw \\ &= \frac{ca^c}{b^2}\frac{l^{r-c}}{c-r}\left[\frac{\Gamma\left(-\frac{c-2}{2}, \frac{u^2}{2b^2}\right)}{2}\left(\frac{1}{\sqrt{2}b}\right)^{c-2}\right] \end{aligned} \tag{38}$$

For $c > r, a > 0, u > 0, l = \left[\frac{a}{u}\right] + 1$ and $w \geq l$.

Corollary 3.4

Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $c > 1, a > 0, u > 0, l = \left[\frac{a}{u}\right] + 1$ and $w \geq l$. The Expected value of $W = X/Y$ is obtained for $r = 1$ and it can be expressed as:

$$E[W] = \frac{ca^c}{b^2}\frac{l^{1-c}}{c-1}\left[\frac{\Gamma\left(-\frac{c-2}{2}, \frac{u^2}{2b^2}\right)}{2}\left(\frac{1}{\sqrt{2}b}\right)^{c-2}\right] \tag{39}$$

For $c > 1, a > 0, u > 0, l = \left[\frac{a}{u}\right] + 1$ and $w \geq l$.

Corollary 3.5

Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $c > 2, a > 0, u > 0, l = \left[\frac{a}{u}\right] + 1$ and $w \geq l$. The Variance of $W = X/Y$ can be expressed as:

$$\sigma^2 = \frac{ca^c}{b^2}\frac{l^{2-c}}{c-2}\left[\frac{\Gamma\left(-\frac{c-2}{2},\frac{u^2}{2b^2}\right)}{2}\left(\frac{1}{\sqrt{2}b}\right)^{c-2}\right] - \left[\frac{ca^c}{b^2}\frac{l^{1-c}}{c-1}\left[\frac{\Gamma\left(-\frac{c-2}{2},\frac{u^2}{2b^2}\right)}{2}\left(\frac{1}{\sqrt{2}b}\right)^{c-2}\right]\right]^2$$

For $c > 2, a > 0, u > 0, l = \left[\frac{a}{u}\right] + 1$ and $w \geq l$.

Proof:

For $r = 2$

$$E[W^2] = \frac{ca^c}{b^2}\frac{l^{2-c}}{c-2}\left[\frac{\Gamma\left(-\frac{c-2}{2},\frac{u^2}{2b^2}\right)}{2}\left(\frac{1}{\sqrt{2}b}\right)^{c-2}\right] \tag{40}$$

The Variance of $W = X/Y$ is easy follows:

$$\sigma^2 = E[W^2] - E[W]^2 \tag{41}$$

Corollary 3.6

Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $w > 0$. The Survival of $W = X/Y$ can be expressed as:

$$S_W(w) = 1 - F_W(w) = \frac{a^c}{w^c b^2}\left[\frac{\Gamma\left(-\frac{c-2}{2},\frac{u^2}{2b^2}\right)}{2}\left(\frac{1}{\sqrt{2}b}\right)^{c-2}\right] \tag{42}$$

For $w > 0$.

Corollary 3.7

Let X and Y are independent and distributed according to (1) and (2), respectively. Then for $w > 0$. The Hazard function of $W = X/Y$ can be expressed as:

$$h_W(w) = \frac{f_W(w)}{S_W(w)} = \frac{c}{w} \tag{43}$$

4. APPLICATIONS AND SIMULATIONS

In this section, we present an application to show our result. suppose an electric circuit with two amplifiers in series, X_1 is a random variable follows Pareto distribution with parameter $c = 1$ and $a = 1$, and X_2 is a random variable follows Rayleigh distribution with parameter $b = 1$, then the total amplification gain is $Z = X_1.X_2$ and by using our result their pdf is $f_Z(z) = \frac{6}{z^3}$.

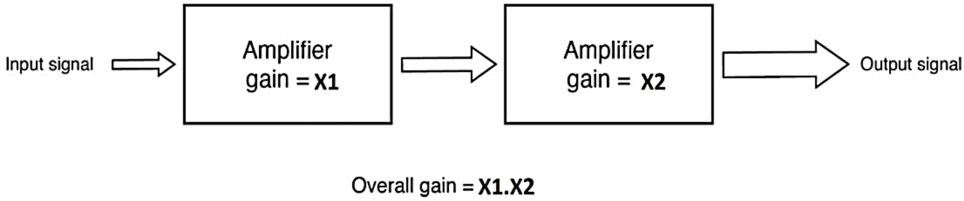

Figure 1: An Electric Circuit with Two Amplifiers in Series

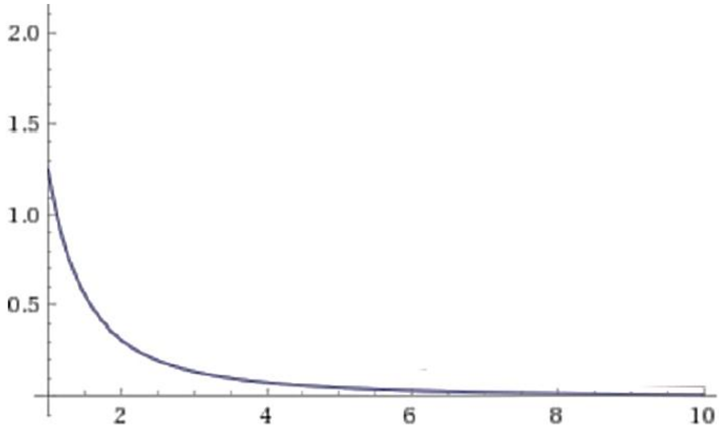

Figure 2: Plots of the pdf (19) for $b = 1$, and $a = 1, c = 1$

Another example involves the distribution of ratio of two independent variables.

Let's consider the below PERT network. A PERT chart, sometimes called a PERT diagram, is a project management tool used to schedule, organize and coordinate tasks within a project. It provides a graphical representation of a project's timeline that allows project managers to break down each individual task in the project for analysis.

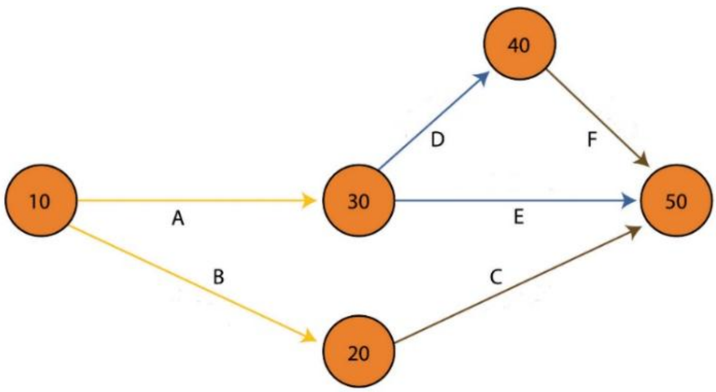

Figure 3: PERT Network

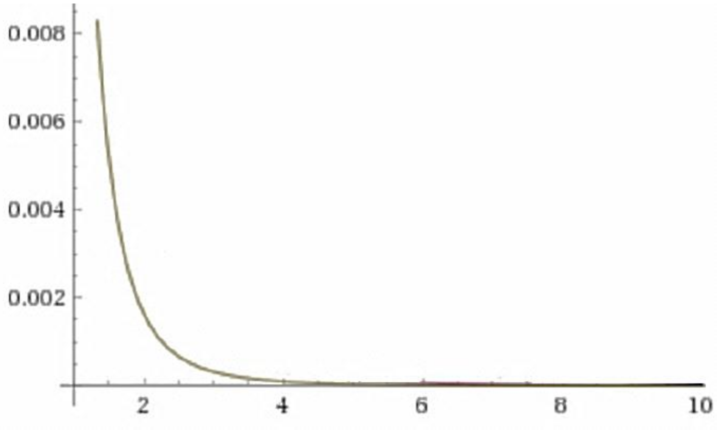

Figure 4: Plots of the pdf (36) for $b = 1$, and $a = 1, c = 3$

In the above network, we are interesting of the feasibility of starting the series of activities, say A and B, on the same date may be investigated by considering the random variable $Z = \frac{A}{B}$. This idea suggests that through examination of such probabilities as $\Pr\left(\frac{A}{B} > k\right)$ and $\Pr\left(k' < \frac{A}{B} < k\right)$, the need for rescheduling $A \; or \; B$ may be determined. For instance, if the time to accomplish task A is a random variable follows Pareto distribution with parameter $c = 1$ and $a = 1$, and task B is a random variable follows Rayleigh distribution with parameter $b = 1$, then by using our result their pdf is $f_Z(z) = \frac{2\sqrt{2}\Gamma\left(\frac{1,1}{2}\right)}{\sqrt{\pi}z^2}$ for $u = 1$.

5. CONCLUSION

This paper has derived The analytical expressions of the PDF, CDF, the rth moment function, the variance, the survival function, and the hazard function, for the distributions of XY and X/Y when X and Y are Preto and Rayleigh random variables distributed independently of each other. We illustrate our results in two figures of the pdf for the distributions of product and ratio.

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