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ON GENERALIZED R-NORM FUZZY INFORMATION MEASURE, DIRECTED DIVERGENCE AND THEIR PROPERTIES

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ABSTRACT

The given article describes a generalized parametric R-norm fuzzy information measure which gives the generality of previously verified R-norm measures. The corresponding R-norm fuzzy directed divergence measure is also developed. We study the important properties and state the particular cases. In addition, we take mathematical data to establish the results of the properties and show the validity of these measures. Further, we analyze the nature of the proposed measures at several values of parameters. Also, the application of the proposed R-norm will be discussed with the help of techniques namely pattern recognition and Multi criteria decision making.

KEYWORDS

Membership Function, R-norm Fuzzy Information Measure, R-norm Fuzzy Directed Divergence Measure, Pattern Recognition, Multi-criteria Decision Measure.

AMS Classification: 94A17, 94A24.

1. INTRODUCTION

R-norm information measure is a function taking real values defined on the set of probability distributions ∇_n such that $\nabla_n \to \Re^+ \left(\Re^+ = \{R : R > 0(\neq 1)\}\right)$ which was introduced by Boekee and Lubbe (1980) as:

$$H_R(P) = \frac{R}{R-1}\left[1 - \left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right]; R > 0(\neq 1)$$

Hooda (2004) proposed the related fuzzy measure of Boekee and Lubbe's (1980) measure as:

$$H_R(\mathrm{M}) = \frac{R}{R-1}\sum_{i=1}^{n}\left[1 - \left(\upsilon_{\mathrm{M}}^R(z_i) + \left(1 - \upsilon_{\mathrm{M}}(z_i)\right)^R\right)^{\frac{1}{R}}\right]; R > 0(\neq 1)$$

Several authors have worked in this field to outdo the limitations of this measure. For this purpose generalizations of this measure have been proposed by Hooda and Ram (1998), Hooda and Bajaj (2008), Kumar and Choudhary (2012), Tomar and Ohlan (2014), Saima et al. (2018), Saima et al. (2019), Safeena et al. (communicated), etc.

Besides, a lot of attention is given to evaluate the difference in two probability distributions, known as Divergence Measures. Kullback and Leibler (1951) described the first divergence measure of the two probability distributions P from the probability distribution Q, represented as $D(P:Q)$ and given by $D(P:Q)=\sum_{i=1}^{n} p_i \log \frac{p_i}{q_i}$. Also, the measure of directed divergence was given by Kullback(1959) as

$$J(P:Q)=\sum_{i=1}^{n}(p_i-q_i)\log\frac{p_i}{q_i}$$

Afterwards, Bhandari and Pal (1993) developed the corresponding fuzzy measure of Kullback and Leibler (1951) and also gave the fuzzy symmetric divergence measure. Since then, a number of authors have given F.D.Ms which include Montes et al. (2002), Hooda (2004), Safeena et al. (2018), Saima et al. (2018) etc.

Hooda and Bajaj (2008) developed the R-norm F.D.M and defined it as

$$I_R(\mathrm{M},\mathrm{N})=\frac{R}{R-1}\sum_{i=1}^{n}\left\{\left[\upsilon_{\mathrm{M}}^{R}(z_i)\upsilon_{\mathrm{N}}^{1-R}(z_i)+\left(1-\upsilon_{\mathrm{M}}(z_i)\right)^{R}\left(1-\upsilon_{\mathrm{N}}(z_i)\right)^{1-R}\right]^{\frac{1}{R}}-1\right\}$$

In the next sections, a new generalized R.F.I.M. is presented along with its R.F.D.D.M.

2. GENERALIZED TWO PARAMETRIC R-NORM FUZZY INFORMATION MEASURE

We propound the generalized two parametric R.F.I.M. as:

$$H_R^{\gamma,\delta}(\mathrm{M})=\frac{R+\gamma-\delta}{R-\delta}\sum_{i=1}^{n}\left\{1-\left[\upsilon_{\mathrm{M}}^{\frac{R+\gamma-\delta}{\gamma}}(z_i)+\left(1-\upsilon_{\mathrm{M}}(z_i)\right)^{\frac{R+\gamma-\delta}{\gamma}}\right]^{\frac{\gamma}{R+\gamma-\delta}}\right\}$$

$$R\neq\delta,\ R>0(\neq 1),0<(\gamma,\delta)\leq 1 \qquad (1)$$

The generalized R.F.I.M. given in (1) is valid if it satisfies the four properties, viz. sharpness, maximality, resolution and symmetry.

Sharpness:

$H_R^{\gamma,\delta}(\mathrm{M})=0,$ iff $\upsilon_{\mathrm{M}}(z_i)=0$ *or* 1; $\forall\ i=1,2,...,n$ i.e., $H_R^{\gamma,\delta}(\mathrm{M})$ gives smallest value iff M is a crisp set

Proof:

If $H_R^{\gamma,\delta}(\mathrm{M})=0,$ i.e.,

$$H_R^{\gamma,\delta}(\mathrm{M})=\frac{R+\gamma-\delta}{R-\delta}\sum_{i=1}^{n}\left\{1-\left[\upsilon_{\mathrm{M}}^{\frac{R+\gamma-\delta}{\gamma}}(z_i)+\left(1-\upsilon_{\mathrm{M}}(z_i)\right)^{\frac{R+\gamma-\delta}{\gamma}}\right]^{\frac{\gamma}{R+\gamma-\delta}}\right\}=0$$

We can write

$$\upsilon_{\mathrm{M}}^{\frac{R+\gamma-\delta}{\gamma}}(z_i)+\left(1-\upsilon_{\mathrm{M}}(z_i)\right)^{\frac{R+\gamma-\delta}{\gamma}}=1 \tag{2}$$

Since $\frac{R+\gamma-\delta}{R-\delta}>0$ therefore (2) is satisfied in case $\upsilon_{\mathrm{M}}(z_i)=0$ or $1\ \forall i=1,2,\ldots n$.

Conversely, if we take M as a crisp set then either $\upsilon_{\mathrm{M}}(z_i)=0$ or 1

$$\Rightarrow \upsilon_{\mathrm{M}}^{\frac{R+\gamma-\delta}{\gamma}}(z_i)+\left(1-\upsilon_{\mathrm{M}}(z_i)\right)^{\frac{R+\gamma-\delta}{\gamma}}=1 \text{ for } \frac{R+\gamma-\delta}{R-\delta}>0 \text{ for which } H_R^{\gamma,\delta}(\mathrm{M})=0$$

Hence, $H_R^{\gamma,\delta}(\mathrm{M})=0$ if and only if M is crisp set.

Maximality:

$H_R^{\gamma,\delta}(\mathrm{M})$ provides maximum value if and only if M is most fuzzy set.

We have

$$H_R^{\gamma,\delta}(\mathrm{M})=\frac{R+\gamma-\delta}{R-\delta}\sum_{i=1}^{n}\left\{1-\left[\upsilon_{\mathrm{M}}^{\frac{R+\gamma-\delta}{\gamma}}(z_i)+\left(1-\upsilon_{\mathrm{M}}(z_i)\right)^{\frac{R+\gamma-\delta}{\gamma}}\right]^{\frac{\gamma}{R+\gamma-\delta}}\right\}$$

$$R\neq\delta,\ R>0(\neq 1),0<(\gamma,\delta)\leq 1 \tag{3}$$

Now, differentiating equation (3) with respect to $\upsilon_{\mathrm{M}}(z_i)$, we get

$$\frac{\partial H_R^{\gamma,\delta}(\mathrm{M})}{\partial\upsilon_{\mathrm{M}}(z_i)}=-\frac{R+\gamma-\delta}{R-\delta}\sum_{i=1}^{n}\left[\left\{\upsilon_{\mathrm{M}}^{\frac{R+\gamma-\delta}{\gamma}}(z_i)+\left(1-\upsilon_{\mathrm{M}}(z_i)\right)^{\frac{R+\gamma-\delta}{\gamma}}\right\}^{\frac{\delta-R}{R+\gamma-\delta}} \left\{\upsilon_{\mathrm{M}}^{\frac{R-\delta}{\gamma}}(z_i)+\left(1-\upsilon_{\mathrm{M}}(z_i)\right)^{\frac{R-\delta}{\gamma}}\right\}\right]$$

For $0\leq\upsilon_{\mathrm{M}}(z_i)<0.5$, then $\frac{\partial H_R^{\gamma,\delta}(\mathrm{M})}{\partial\upsilon_{\mathrm{M}}(z_i)}>0$ i.e., $H_R^{\gamma,\delta}(\mathrm{M})$ is an increasing function of $\upsilon_{\mathrm{M}}(z_i)$.

Similarly, for $0.5<\upsilon_{\mathrm{M}}\left(z_i\right)\leq 1$ we get $\dfrac{\partial H_R^{\gamma,\delta}\left(\mathrm{M}\right)}{\partial \upsilon_{\mathrm{M}}\left(z_i\right)}<0$. This implies $H_R^{\gamma,\delta}\left(\mathrm{M}\right)$ is a decreasing function of $\upsilon_{\mathrm{M}}\left(z_i\right)$.

Also, when $\upsilon_{\mathrm{M}}\left(z_i\right)=0.5$, $\dfrac{\partial H_R^{\gamma,\delta}\left(\mathrm{M}\right)}{\partial \upsilon_{\mathrm{M}}\left(z_i\right)}=0$.

The above results are established arithmetically in the below tables.

Table 2.1
To Check Maximality Property at Different Values of Parameters

R	γ	δ	$\upsilon_{\mathrm{M}}\left(z_i\right)_{[0,0.5)}$	$\dfrac{\partial H_R^{\gamma,\delta}\left(\mathrm{M}\right)}{\partial \upsilon_{\mathrm{M}}\left(z_i\right)}$	$\upsilon_{\mathrm{M}}\left(z_i\right)_{(0.5,1]}$	$\dfrac{\partial H_R^{\gamma,\delta}\left(\mathrm{M}\right)}{\partial \upsilon_{\mathrm{M}}\left(z_i\right)}$
0.80	0.91	0.31	0.10	7.3248	0.52	-7.3603
15			0.12	5.6536	0.57	-5.9132
0.45	0.35	0.66	0.19	9.7238	0.69	-inf
45			0.23	6.0350	0.76	-6.0472
1.5	0.5	0.5	0.36	6.6924	0.89	-6.0880
			0.49		1.00	

From Table 2.1 we find that $H_R^{\gamma,\delta}\left(\mathrm{M}\right)$ is increasing in the interval $0\leq\upsilon_{\mathrm{M}}\left(z_i\right)<0.5$ and decreasing in the interval $0.5<\upsilon_{\mathrm{M}}\left(z_i\right)\leq 1$.

Resolution:

$H_R^{\gamma,\delta}\left(\mathrm{M}\right)\geq H_R^{\gamma,\delta}\left(\mathrm{M}^*\right)$, where M^* denotes the sharpened form of M.

Proof:

As maximality shows $H_R^{\gamma,\delta}\left(\mathrm{M}\right)$ is increasing and decreasing function of $\upsilon_{\mathrm{M}}\left(z_i\right)$ in $0\leq\upsilon_{\mathrm{M}}\left(z_i\right)<0.5$ and $0.5<\upsilon_{\mathrm{M}}\left(z_i\right)\leq 1$ respectively, therefore

$$\upsilon_{\mathrm{M}^*}\left(z_i\right)\leq\upsilon_{\mathrm{M}}\left(z_i\right)\Rightarrow H_R^{\gamma,\delta}\left(\mathrm{M}\right)\geq H_R^{\gamma,\delta}\left(\mathrm{M}^*\right) \text{ in } 0\leq\upsilon_{\mathrm{M}}\left(z_i\right)<0.5$$

$$\upsilon_{\mathrm{M}^*}\left(z_i\right)\geq\upsilon_{\mathrm{M}}\left(z_i\right)\Rightarrow H_R^{\gamma,\delta}\left(\mathrm{M}\right)\geq H_R^{\gamma,\delta}\left(\mathrm{M}^*\right) \text{ in } 0.5<\upsilon_{\mathrm{M}}\left(z_i\right)\leq 1$$

This can be shown in the following tables as:

Table 2.2

Comparison of $H_R^{\gamma,\delta}(\mathrm{M})$ with the Sharpened Version in the Range $0 \le \upsilon_{\mathrm{M}}(z_i) < 0.5$

R	γ	δ	$\upsilon_{\mathrm{M}}(z_i)$	$H_R^{\gamma,\delta}(\mathrm{M})$	$\upsilon_{\mathrm{M}^*}(z_i)$	$H_R^{\gamma,\delta}(\mathrm{M}^*)$
0.80	0.91	0.31	0.23	2.4840	0.20	1.8879
15			0.12	1.5586	0.00	1.1574
0.45	0.22	0.78	0.36	5.9577	0.32	4.5936
45			0.49	1.4976	0.41	1.0986
1.5	0.5	0.5	0.09	1.9688	0.02	1.5030
			0.19		0.14	

Table 2.2(a)

Comparison of $H_R^{\gamma,\delta}(\mathrm{M})$ with the Sharpened Version in the Range $0.5 < \upsilon_{\mathrm{M}}(z_i) \le 1$

R	γ	δ	$\upsilon_{\mathrm{M}}(z_i)$	$H_R^{\gamma,\delta}(\mathrm{M})$	$\upsilon_{\mathrm{M}^*}(z_i)$	$H_R^{\gamma,\delta}(\mathrm{M}^*)$
0.80	0.91	0.31	0.89	2.4355	0.92	2.2844
15			0.69	1.6280	0.71	1.4755
0.45	0.22	0.78	0.60	5.3943	0.62	5.1994
45			0.76	1.5692	0.80	1.4009
1.5	0.5	0.5	1.0	2.0191	1.0	1.8699
			0.52		0.56	

From the results of table (2.2) and (2.2(a)), we find that measure defined in (1) holds the property of resolution numerically as well.

Symmetry:

$H_R^{\gamma,\delta}(\mathrm{M}) = H_R^{\gamma,\delta}(\mathrm{M}')$ where $\mathrm{M}' = 1 - \mathrm{M}$

Proof:

Let $\upsilon_{\mathrm{M}}(z_i) = \{0.23, 0.65, 0.91, 0.43, 0.39, 0.82\}$, $\gamma = 0.91, \delta = 0.31, \& R = 0.80$

For the above given values, $H_R^{\gamma,\delta}(\mathrm{M}) = 2.7693$ and $H_R^{\gamma,\delta}(1-\mathrm{M}) = 2.7693$.

Thus, the property holds.

The above results confirm the validity of the proposed measure as the four basic properties are satisfied.

2.1 Particular Cases:

- If $\gamma=1$ the R.F.I.M. given in (1) reduces to the measure given by Tomar and Anshu (2014).
- If $\gamma=1$ and $\beta=1$ measure (1) reduces to the R.F.I.M. given by Hooda (2004).
- If $\gamma=1, \beta=1$ and $R\to 1$ measure (1) tends Fuzzy information measure introduced by De Luca and Termini (1972).

3. MONOTONIC BEHAVIOUR OF THE R-NORM FUZZY INFORMATION MEASURE

In this section, we check the monotonic nature of the R.F.I.M. defined in (1) by considering different values of the parameters R, α and β given the value of membership functions:

$$\upsilon_{M_1}(z)=\{0.23, 0.65, 0.91, 0.43, 0.39, 0.82\},$$
$$\mu_{M_2}(z)=\{0.76, 0.45, 0.12, 0.83, 0.63, 0.23\},$$
$$\mu_{M_3}(z)=\{0.56, 0.32, 0.74, 0.18, 0.66, 0.41\},$$
$$\mu_{M_4}(z)=\{0.43, 0.22, 0.81, 0.33, 0.72, 0.10\}.$$

Table 3.1
Behaviour of $H_R^{\gamma,\delta}(M)$ for Fixed $\gamma=0.91$ and $\delta=0.31$ for Different Membership Functions

R	$H_R^{0.91,0.31}(M_1)$	$H_R^{0.91,0.31}(M_2)$	$H_R^{0.91,0.31}(M_3)$	$H_R^{0.91,0.31}(M_4)$
0.89	2.7078	2.6324	3.0787	2.6237
6	1.9209	1.8118	2.2394	1.7859
20	1.7471	1.6527	2.0400	1.6215
34	1.7151	1.6226	2.0026	1.5918
57	1.6968	1.6053	1.9813	1.5748
79	1.6893	1.5982	1.9725	1.5679
108	1.6841	1.5933	1.9664	1.5630
130	1.6817	1.5910	1.9636	1.5608
135	1.6812	1.5906	1.9631	1.5604
140	1.6802	1.5902	1.9627	1.5600

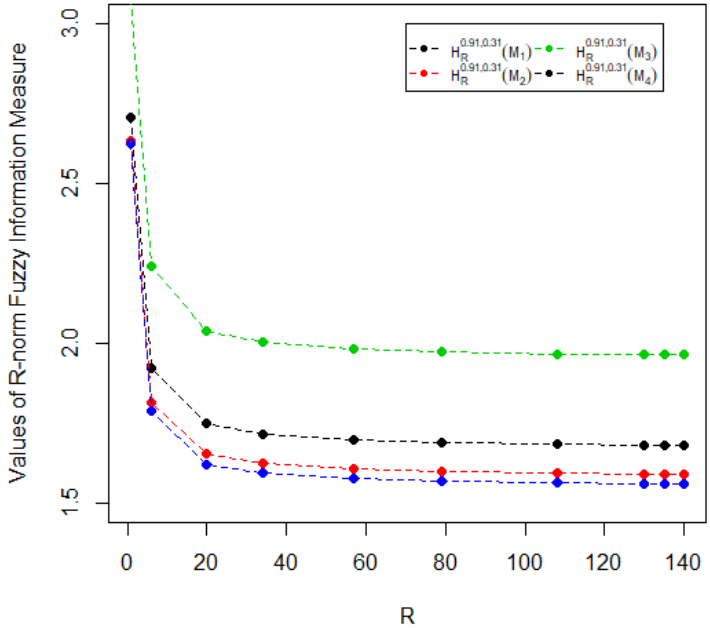

Figure 3.1

The above Table 3.1 show there is inverse relation between R and R.F.I.M. (1). Thus, it is obvious that R.F.I.M. has monotonic decreasing nature with respect to R. The same behaviour is conveyed by the Figure 3.1.

Table 3.2

Behaviour of $H_R^{\gamma,\delta}(\mathrm{M})$ with respect to γ and $\upsilon_{M_j}(z_i)$ $\forall i=1,2,\ldots 6,\ \ j=1,2,3,4$.

γ	$H_{0.56}^{\gamma,0.31}(\mathrm{M})$				$H_{12}^{\gamma,0.82}(\mathrm{M})$			
	M_1	M_2	M_3	M_4	M_1	M_2	M_3	M_4
0.1	2.17	2.06	2.53	2.05	1.68	1.59	1.96	1.56
0.2	2.44	2.34	2.81	2.34	1.69	1.60	1.98	1.57
0.3	2.60	2.52	2.97	2.51	1.71	1.62	2.00	1.59
0.4	2.71	2.64	3.08	2.63	1.72	1.63	2.01	1.60
0.5	2.79	2.72	3.16	2.71	1.74	1.65	2.03	1.61
0.6	2.85	2.78	3.22	2.77	1.75	1.66	2.05	1.63
0.7	2.90	2.83	3.27	2.82	1.77	1.67	2.07	1.64
0.8	2.94	2.87	3.31	2.86	1.78	1.69	2.08	1.66
0.9	2.97	2.91	3.34	2.90	1.80	1.70	2.10	1.67
1	3.00	2.94	3.36	2.93	1.81	1.71	2.12	1.68

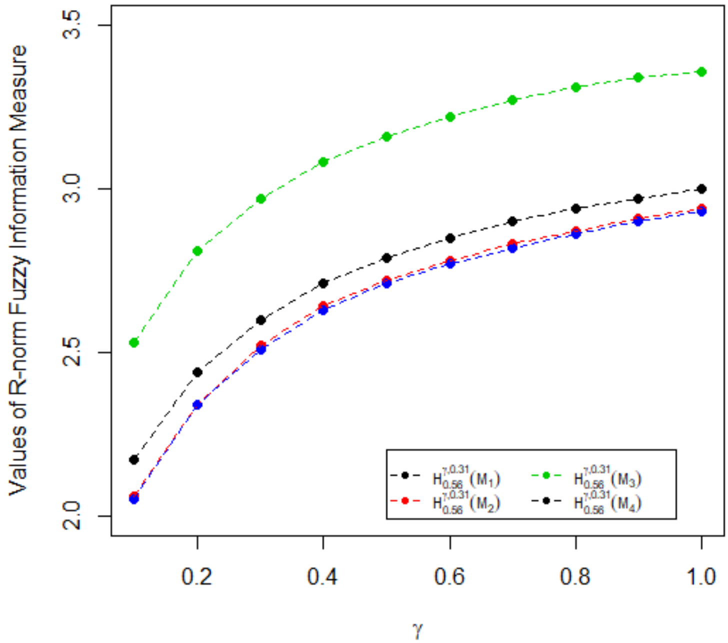

Figure 3.2

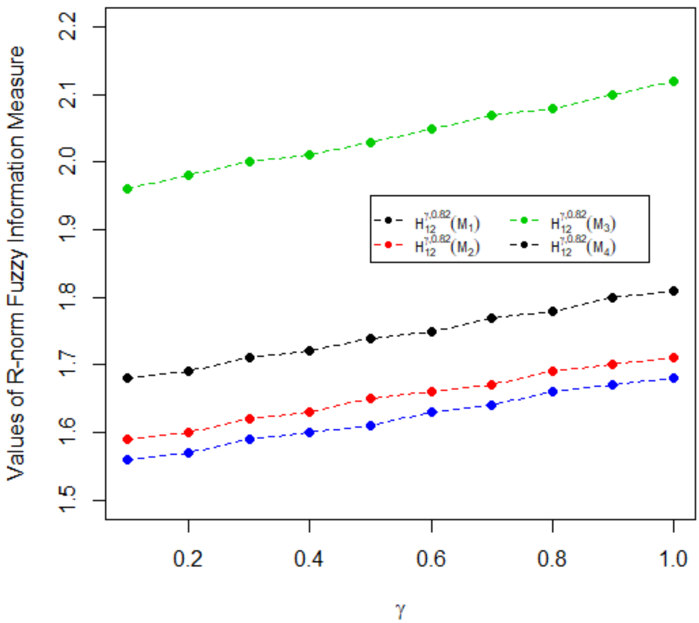

Figure 3.2(a)

The results of Table 3.2 indicate that as we increase the value of γ, the value of R.F.I.M. (1) also increases. This relation between the parameter γ and the measure (1) is displayed in the Figures 3.2 & 3.2(a).

Table 3.3

Behaviour of $H_R^{\gamma,\delta}(\mathrm{M})$ with respect to δ and $\upsilon_{M_j}(z_i)\ \forall i=1,2,...6,\ \ j=1,2,3,4$

δ	$H_{0.56}^{0.91,\delta}(\mathrm{M})$				$H_{15}^{0.45,\delta}(\mathrm{M})$			
	M_1	M_2	M_3	M_4	M_1	M_2	M_3	M_4
0.1	2.79	2.72	3.16	2.71	1.7204	1.6173	2.0088	1.5968
0.2	2.87	2.80	3.24	2.79	1.7207	1.6177	2.0092	1.5971
0.3	2.96	2.90	3.33	2.89	1.7211	1.6180	2.0096	1.5974
0.4	3.08	3.02	3.44	3.01	1.7214	1.6183	2.0100	1.5977
0.5	3.21	3.16	3.57	3.15	1.7218	1.6187	2.0105	1.5981
0.6	3.38	3.33	3.74	3.32	1.7221	1.6190	2.0109	1.5984
0.7	3.60	3.55	3.96	3.54	1.7225	1.6193	2.0113	1.5987
0.8	3.89	3.84	4.25	3.83	1.7229	1.6197	2.0117	1.5991
0.9	4.30	4.26	4.68	4.25	1.7232	1.6200	2.0122	1.5994
1	4.96	4.92	5.36	4.90	1.7236	1.6204	2.0126	1.5998

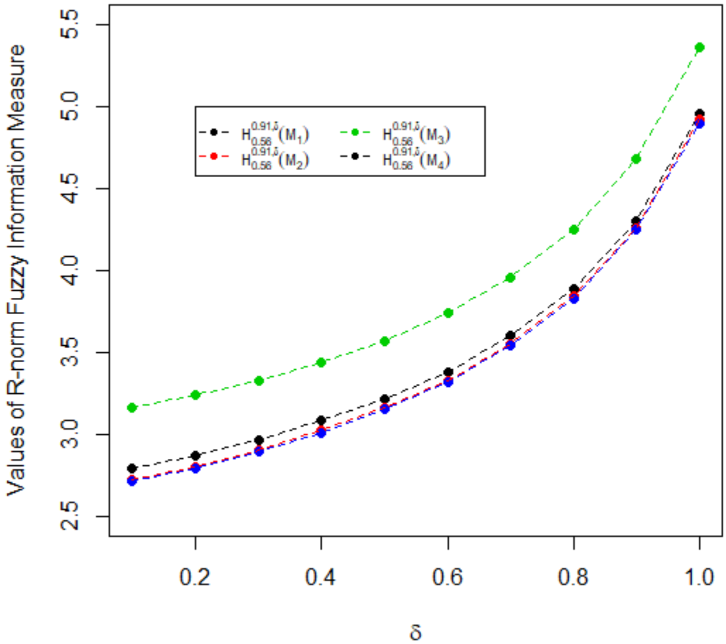

Figure 2.3

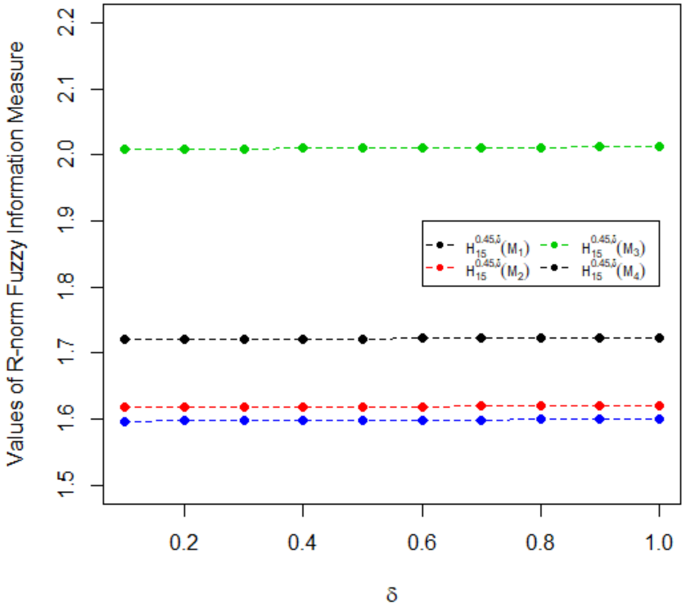

Figure 3.3(a)

The results of Table 3.3 again indicate that as the value of β is increased; the value of R.F.I.M. also increases. This increasing relationship between the parameter β and the measure (1) is displayed in the Figures 3.3 & 3.3(a).

4. DIRECTED DIVERGENCE OF THE PROPOSED R-NORM FUZZY INFORMATION MEASURE

In this section, we express the divergence form of the measure defined in (1) as:

$$I_R^{\gamma,\delta}(\mathrm{M},\mathrm{N})=\frac{R+\gamma-\delta}{R-\delta}\sum_{i=1}^{n}\left\{\left[\begin{array}{l}\upsilon_{\mathrm{M}}^{\frac{R+\gamma-\delta}{\gamma}}(z_i)\,\upsilon_{\mathrm{N}}^{1-\frac{R+\gamma-\delta}{\gamma}}(z_i)\\ +\left(1-\upsilon_{\mathrm{M}}(z_i)\right)^{\frac{R+\gamma-\delta}{\gamma}}\left(1-\upsilon_{\mathrm{N}}(z_i)\right)^{1-\frac{R+\gamma-\delta}{\gamma}}\end{array}\right]^{\frac{\gamma}{R+\gamma-\delta}}-1\right\} \tag{4}$$

$$R>0(\neq 1),(R\neq\delta),\ 0<(\gamma,\delta)\leq 1,\ (R+\gamma)>\delta$$

We re-write (4) by setting $\kappa=\dfrac{R+\gamma-\delta}{R-\delta}$ and $\theta=\dfrac{R+\gamma-\delta}{\gamma}$

$$I_R^{\gamma,\delta}(\mathrm{M},\mathrm{N})=\kappa\sum_{i=1}^{n}\left\{\left[\upsilon_{\mathrm{M}}^{\theta}(z_i)\,\upsilon_{B}^{1-\theta}(z_i)+\left(1-\upsilon_{\mathrm{M}}(z)\right)^{\theta}\left(1-\upsilon_{\mathrm{N}}(z_i)\right)^{1-\theta}\right]^{\frac{1}{\theta}}-1\right\} \tag{5}$$

Further, we define the fuzzy symmetric divergence measure as

$$J_R^{\gamma,\delta}(\mathrm{M},\mathrm{N}) = I_R^{\gamma,\delta}(\mathrm{M},\mathrm{N}) + I_R^{\gamma,\delta}(\mathrm{N},\mathrm{M}) \tag{6}$$

For $\gamma = 1$ and $\delta = 1$ measure (5) reduces to the R.F.D.D.M. given by Hooda and Bajaj (2008).

The measure defined in (5) is valid if it satisfies the following properties:

1. $I_R^{\gamma,\delta}(\mathrm{M},\mathrm{N}) > 0$.
2. $I_R^{\gamma,\delta}(\mathrm{M},\mathrm{N}) = 0$ if $\upsilon_\mathrm{M}(z_i) = \upsilon_\mathrm{N}(z_i)$.
3. $I_R^{\gamma,\delta}(\mathrm{M},\mathrm{N})$ is convex if $\dfrac{\partial^2 I_R^{\gamma,\delta}(\mathrm{M},\mathrm{N})}{\partial \upsilon_\mathrm{M}^2(z_i)} > 0$ & $\dfrac{\partial^2 I_R^{\gamma,\delta}(\mathrm{M},\mathrm{N})}{\partial \upsilon_\mathrm{N}^2(z_i)} > 0$ for the given limits of $R \neq \beta,\ R > 0(\neq 1), 0 < (\alpha,\beta) \leq 1$.
4. $I_R^{\gamma,\delta}(\mathrm{M},\mathrm{N}) \neq I_R^{\gamma,\delta}(\mathrm{N},\mathrm{M})$.
5. $I_R^{\gamma,\delta}(\mathrm{M},\mathrm{N})$ gives the same value for $\upsilon_\mathrm{M}(z_i)$ and $1-\upsilon_\mathrm{M}(z_i)$. The same holds for $\upsilon_\mathrm{N}(z_i)$.

Let $\sum_{i=1}^{n} \upsilon_\mathrm{M}(z_i) = t$ & $\sum_{i=1}^{n} \upsilon_\mathrm{N}(z_i) = v$, then

$$\sum_{i=1}^{n} \left(\frac{\upsilon_\mathrm{M}(z_i)}{t}\right)^{\theta} \left(\frac{\upsilon_\mathrm{N}(z_i)}{v}\right)^{1-\theta} - 1 \geq 0$$

$$\Rightarrow \sum_{i=1}^{n} \upsilon_\mathrm{M}(z_i)^{\theta}\, \upsilon_\mathrm{N}(z_i)^{1-\theta} \geq t^{\theta} v^{1-\theta} \tag{7}$$

Likewise, we can write

$$\sum_{i=1}^{n} \left(1-\upsilon_\mathrm{M}(z_i)\right)^{\theta} \left(1-\upsilon_\mathrm{N}(z_i)\right)^{1-\theta} \geq (n-t)^{\theta} (n-v)^{1-\theta}. \tag{8}$$

Taking sum of (7) and (8), we obtain the below expression

$$\sum_{i=1}^{n} \left\{ \upsilon_\mathrm{M}^{\theta}(z_i)\, \upsilon_\mathrm{N}^{1-\theta}(x_i) + \left(1-\upsilon_\mathrm{M}(z_i)\right)^{\theta} \left(1-\upsilon_\mathrm{N}(z_i)\right)^{1-\theta} \right\} \geq t^{\theta} v^{1-\theta} + (n-t)^{\theta} (n-v)^{1-\theta}. \tag{9}$$

Since $R \neq \delta$, thus two cases arise. Either $R < \delta$ or $R > \delta$.

Case I: $R < \delta$

Suppose $\left\{ \upsilon_\mathrm{M}^{\theta}(z_i)\, \upsilon_\mathrm{N}^{1-\theta}(x_i) + \left(1-\upsilon_\mathrm{M}(z_i)\right)^{\theta} \left(1-\upsilon_\mathrm{N}(z_i)\right)^{1-\theta} \right\} = y_i$, then $y_i < 1$ and $\dfrac{1}{\theta} > 1$.

$$\Rightarrow y_i - n > y_i^{\frac{1}{\theta}} - n.$$

$$\Rightarrow \kappa\sum_{i=1}^{n}\{y_i-1\}\le\kappa\sum_{i=1}^{n}\left\{y_i^{\frac{1}{\theta}}-1\right\} \text{ where } \kappa<0.$$

$$\Rightarrow I_R^{\gamma,\delta}(\mathrm{M},\mathrm{N})\ge\kappa\left[t^{\theta}v^{1-\theta}+(n-t)^{\theta}(n-v)^{1-\theta}-n\right]$$

Further, we take $\psi(t)=\kappa\left[t^{\theta}v^{1-\theta}+(n-t)^{\theta}(n-v)^{1-\theta}-n\right]$ then

$$\psi'(t)=\kappa\left[\theta\left(\frac{t}{v}\right)^{\theta-1}-\theta\left(\frac{n-t}{n-v}\right)^{\theta-1}\right]$$

and

$$\psi''(t)=\kappa\theta(\theta-1)\left[\left(\frac{t}{v}\right)^{\theta-2}\left(\frac{1}{v}\right)+\left(\frac{n-t}{n-v}\right)^{\theta-2}\left(\frac{1}{n-v}\right)\right]>0.$$

From above it is obvious that $\psi(t)$ has least value when $\frac{t}{v}\left(=\frac{n-t}{n-v}\right)=1$, therefore, $\psi(t)$ is a convex function of u. This shows $\psi(t)\ge 0$ and equality holds for $t=\nu$.

Case II: $R>\delta$

We can write expression (9) in the following manner

$$\left(\sum_{i=1}^{n}\left\{\upsilon_{\mathrm{M}}^{\theta}(z_i)\upsilon_{\mathrm{N}}^{1-\theta}(x_i)+\left(1-\upsilon_{\mathrm{M}}(z_i)\right)^{\theta}\left(1-\upsilon_{\mathrm{N}}(z_i)\right)^{1-\theta}\right\}-1\right)^{\frac{1}{\theta}}$$
$$\ge\left(t^{\theta}v^{1-\theta}+(n-t)^{\theta}(n-v)^{1-\theta}-n\right)^{\frac{1}{\theta}} \quad (10)$$

Further,

$$\sum_{i=1}^{n}\left\{\left[\upsilon_{\mathrm{M}}^{\theta}(z_i)\upsilon_{\mathrm{N}}^{1-\theta}(x_i)+\left(1-\upsilon_{\mathrm{M}}(z_i)\right)^{\theta}\left(1-\upsilon_{\mathrm{N}}(z_i)\right)^{1-\theta}\right]^{\frac{1}{\theta}}-1\right\}$$
$$\ge\left(\sum_{i=1}^{n}\upsilon_{\mathrm{M}}^{\theta}(z_i)\upsilon_{\mathrm{N}}^{1-\theta}(x_i)+\left(1-\upsilon_{\mathrm{M}}(z_i)\right)^{\theta}\left(1-\upsilon_{\mathrm{N}}(z_i)\right)^{1-\theta}-1\right)^{\frac{1}{\theta}} \quad (11)$$

By linking (10) & (11) we develop the following result

$$I_R^{\gamma,\delta}(\mathrm{M},\mathrm{N})\ge\kappa\left(t^{\theta}v^{1-\theta}+(n-u)^{\theta}(n-v)^{1-\theta}-n\right)^{\frac{1}{\theta}}$$

We consider $\psi(t)=\kappa\left[t^{\theta}v^{1-\theta}+(n-t)^{\theta}(n-v)^{1-\theta}-n\right]^{\frac{1}{\theta}}$, then

$$\psi'(t)=\kappa\left[u^{\theta}v^{1-\theta}+(n-t)^{\theta}(n-v)^{1-\theta}-n\right]^{\frac{1}{\theta}-1}\left[\left(\frac{t}{v}\right)^{\theta-1}-\left(\frac{n-t}{n-v}\right)^{\theta-1}\right] \text{ and } \psi''(t)>0 .$$

It is clear that $\psi(t)$ is convex function and gives minimum value only when $\frac{t}{v}$ $\left(=\frac{n-t}{n-v}\right)=1$. Therefore, $\psi(t)>0$ and disappears for $t=v$. This follows that $\psi(t)\geq 0$.

For property 4 & 5

Let $\mathrm{M}=(0.65,0.23,0.82,0.44,0.97)$, $\mathrm{N}=(0.42,0.31,0.05,0.99,0.73)$, $R=0.56$, $\gamma=0.91$ and $\delta=0.31$.

$$I_R^{\gamma,\delta}(\mathrm{M},\mathrm{N})=6.8633 \text{ and } I_R^{\gamma,\delta}(\mathrm{N},\mathrm{M})=3.4373 .$$

This clearly shows that $I_R^{\gamma,\delta}(\mathrm{M},\mathrm{N})\neq I_R^{\gamma,\delta}(\mathrm{N},\mathrm{M})$.

Also, $I_R^{\alpha,\beta}(1-\mathrm{M},1-\mathrm{N})=6.8633$ this result is same as $I_R^{\gamma,\delta}(\mathrm{M},\mathrm{N})$.

The above results for property 4 & 5 are proved with the help of R-software.

Thus, we have proved that $I_R^{\gamma,\delta}(\mathrm{M},\mathrm{N})$ fulfills all the properties of a F.D.D.M. of set M along with set N. Accordingly, the related measure of fuzzy symmetric divergence i.e., $J_R^{\gamma,\delta}(\mathrm{M},\mathrm{N})$ is also a valid measure.

5. MONOTONIC BEHAVIOUR OF R-NORM FUZZY DIRECTED DIVERGENCE MEASURE

In this section, we check the monotonic nature of the R.F.D.D.M. defined in (5) by considering different values of the parameters R, γ and δ given the value of membership functions. We consider the following fuzzy sets $\mathrm{M}_1,\mathrm{M}_2,\mathrm{M}_3$ and $\mathrm{N}_1,\mathrm{N}_2,\mathrm{N}_3$ where,

$$\upsilon_{\mathrm{M}_1}(z)=\{0.65,0.23,0.82,0.44,0.97\},\ \upsilon_{\mathrm{N}_1}(z)=\{0.42,0.53,0.65,0.22,0.76\},$$
$$\upsilon_{\mathrm{M}_2}(z)=\{0.41,0.64,0.91,0.13,0.22\},\ \upsilon_{\mathrm{N}_2}(z)=\{0.82,0.23,0.55,0.60,0.03\},$$
$$\upsilon_{\mathrm{M}_3}(z)=\{0.56,0.11,0.32,0.63,0.84\},\ \upsilon_{\mathrm{N}_3}(z)=\{0.65,0.20,0.43,0.71,0.73\}$$

Table 5.1

Behaviour of $I_R^{\gamma,\delta}(\mathrm{M},\mathrm{N})$ with $\upsilon_{M_j}(z_i)$, $\upsilon_{N_j}(z_i)$ $\forall i=1,2,\ldots 3,\ j=1,2,3$

and Different Value of $\gamma \& \delta$

R	$\gamma=0.81, \delta=0.71$			$\gamma=0.91, \delta=0.80$		
	$I_R^{\gamma,\delta}(\mathrm{M}_1,\mathrm{N}_1)$	$I_R^{\gamma,\delta}(\mathrm{M}_2,\mathrm{N}_2)$	$I_R^{\gamma,\delta}(\mathrm{M}_3,\mathrm{N}_3)$	$I_R^{\gamma,\delta}(\mathrm{M}_1,\mathrm{N}_1)$	$I_R^{\gamma,\delta}(\mathrm{M}_2,\mathrm{N}_2)$	$I_R^{\gamma,\delta}(\mathrm{M}_3,\mathrm{N}_3)$
0.89	0.7829	2.3176	0.1454	0.7155	2.0606	0.1318
6	2.2136	9.1755	0.5845	2.1505	8.8544	0.5486
20	2.5709	11.2119	0.8584	2.5521	11.092	0.8421
34	2.6338	11.6166	0.9126	2.6227	11.5440	0.9031
57	2.6701	11.8573	0.9436	2.6635	11.8130	0.9380
79	2.6850	11.9580	0.9563	2.6802	11.9257	0.9522
108	2.6954	12.0286	0.9651	2.6919	12.0048	0.9622
130	2.7002	12.0613	0.9692	2.6973	12.0414	0.9667
150	2.7033	12.0827	0.9719	2.7008	12.0655	0.9697
160	2.7046	12.0915	0.9729	2.7022	12.0655	0.9709

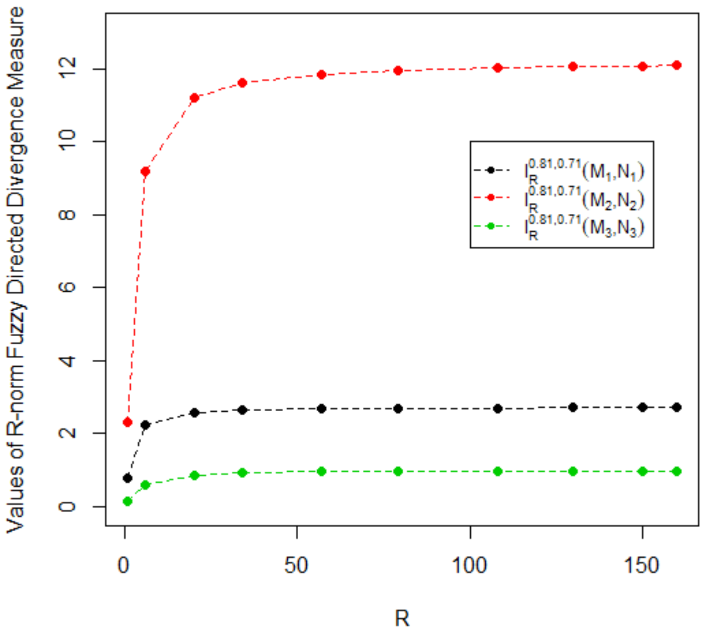

Figure 5.1

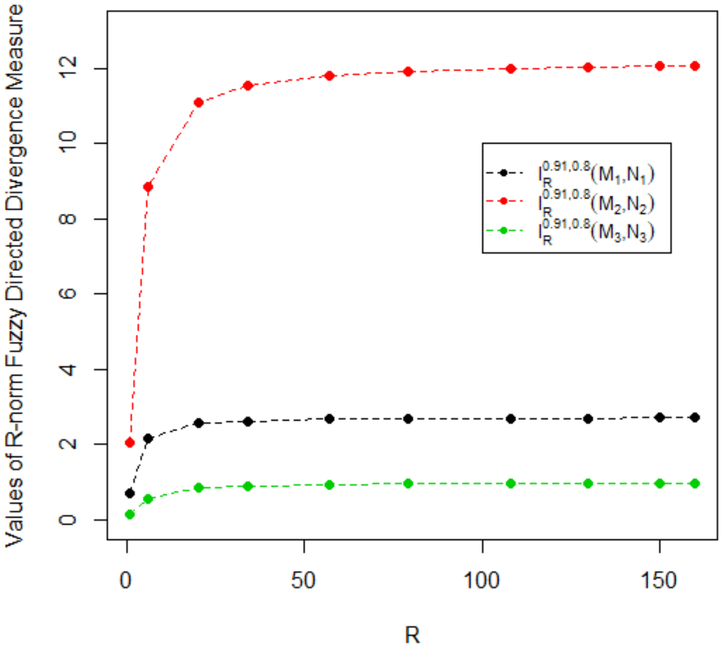

Figure 5.3(a)

The above Table 5.1 show there is direct relation between *R* and divergence measure defined in (5) for different membership functions. This relation is also conveyed in Figures 5.1 and 5.1(a) respectively which clearly shows that proposed measure follows same trend with respect to parameter *R* even if the value of other two parameters i.e., $\gamma \& \delta$ are increased within their range Thus, it is obvious that fuzzy directed divergence measure has monotonic increasing nature with respect to *R*.

Table 5.2

Behaviour of $I_R^{\gamma,\delta}(M,N)$ when γ is Varying, where R and δ have Fixed Values

γ	$I_{0.76}^{\gamma,0.45}(M,N)$			$I_{10.5}^{\gamma,0.91}(M,N)$		
	$I_R^{\gamma,\delta}(M_1,N_1)$	$I_R^{\gamma,\delta}(M_2,N_2)$	$I_R^{\gamma,\delta}(M_3,N_3)$	$I_R^{\gamma,\delta}(M_1,N_1)$	$I_R^{\gamma,\delta}(M_2,N_2)$	$I_R^{\gamma,\delta}(M_3,N_3)$
0.1	1.7971	7.1343	0.3985	2.6847	11.9559	0.9561
0.2	1.3712	5.0166	0.2763	2.6464	11.6997	0.9234
0.3	1.1706	4.0193	0.2286	2.6088	11.4539	0.8911
0.4	1.0572	3.4837	0.2032	2.5719	11.2178	0.8593
0.5	0.9847	3.1569	0.1875	2.5356	10.9908	0.8278
0.6	0.9346	2.9383	0.1768	2.5000	10.7726	0.7971
0.7	0.8978	2.7822	0.1690	2.4649	10.5624	0.7673
0.8	0.8696	2.6652	0.1632	2.4304	10.3600	0.7387
0.9	0.8475	2.5744	0.1586	2.3965	10.1648	0.7114
1	0.8295	2.5019	0.1549	2.3630	9.9765	0.6856

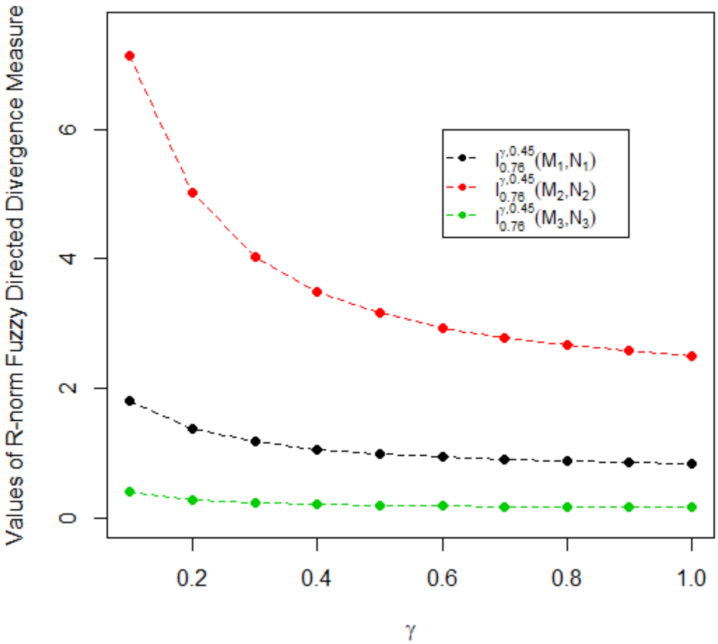

Figure 5.2

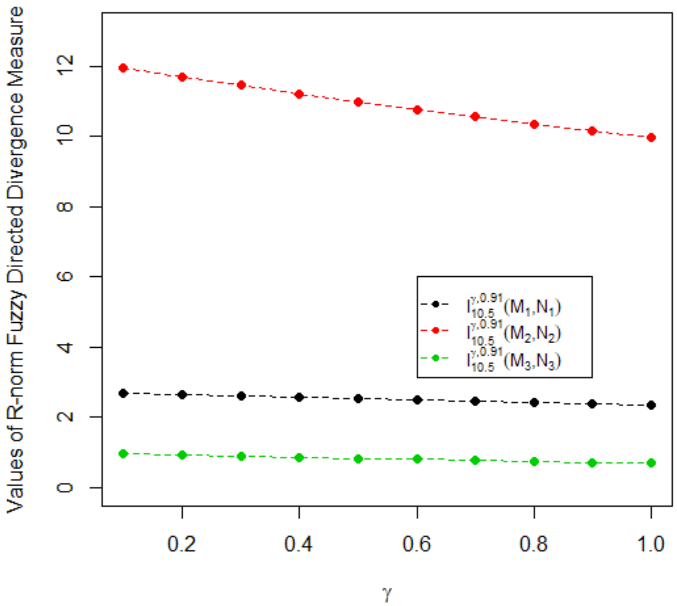

Figure 5.2(a)

The results of Table 5.2 indicate that as we increase the value of γ, for different membership functions, the value of R.F.D.M. decreases. This relation between the parameter γ and measure (5) is displayed in the Figures 5.2 and 5.2 (a).

Table 5.3

Behaviour of $I_R^{\gamma,\delta}(\mathrm{M},\mathrm{N})$ for $\gamma = 0.43, R = 0.69$ and $\gamma = 0.71, R = 12$ with changing δ

δ	$I_{0.69}^{0.43,\delta}(\mathrm{M},\mathrm{N})$			$I_{12}^{0.71,\delta}(\mathrm{M},\mathrm{N})$		
	$I_R^{\gamma,\delta}(\mathrm{M}_1,\mathrm{N}_1)$	$I_R^{\gamma,\delta}(\mathrm{M}_2,\mathrm{N}_2)$	$I_R^{\gamma,\delta}(\mathrm{M}_3,\mathrm{N}_3)$	$I_R^{\gamma,\delta}(\mathrm{M}_1,\mathrm{N}_1)$	$I_R^{\gamma,\delta}(\mathrm{M}_2,\mathrm{N}_2)$	$I_R^{\gamma,\delta}(\mathrm{M}_3,\mathrm{N}_3)$
0.1	1.3059	4.6878	0.2603	2.5098	10.8325	0.8056
0.2	1.2146	4.2340	0.2387	2.5081	10.8220	0.8041
0.3	1.1162	3.7593	0.2163	2.5064	10.8113	0.8026
0.4	1.0106	3.2719	0.1930	2.5046	10.8005	0.8011
0.5	0.8973	2.7801	0.1689	2.5028	10.7895	0.7995
0.6	0.7760	2.2906	0.1440	2.5009	10.7784	0.7979
0.7	0.6463	1.8086	0.1181	2.4991	10.7671	0.7963
0.8	0.5076	1.3388	0.0914	2.4972	10.7556	0.7947
0.9	0.3595	0.8874	0.0638	2.4952	10.7439	0.7931
1	0.2015	0.4625	0.0353	2.4933	10.7320	0.7914

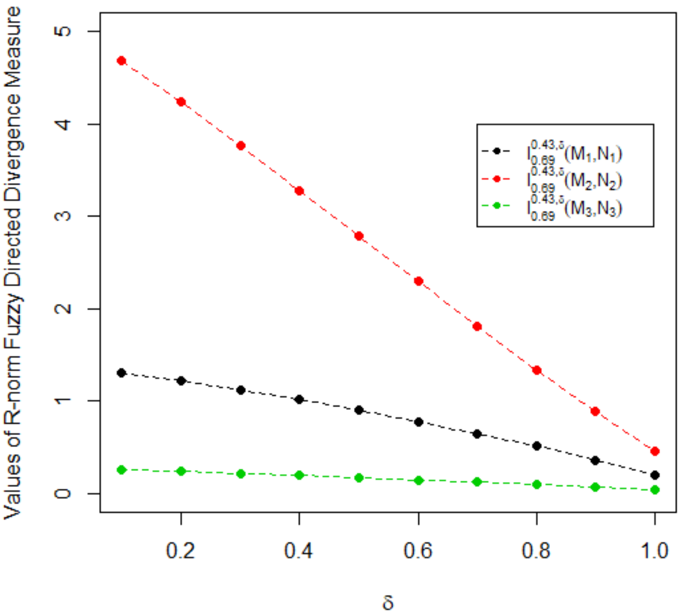

Figure 5.3

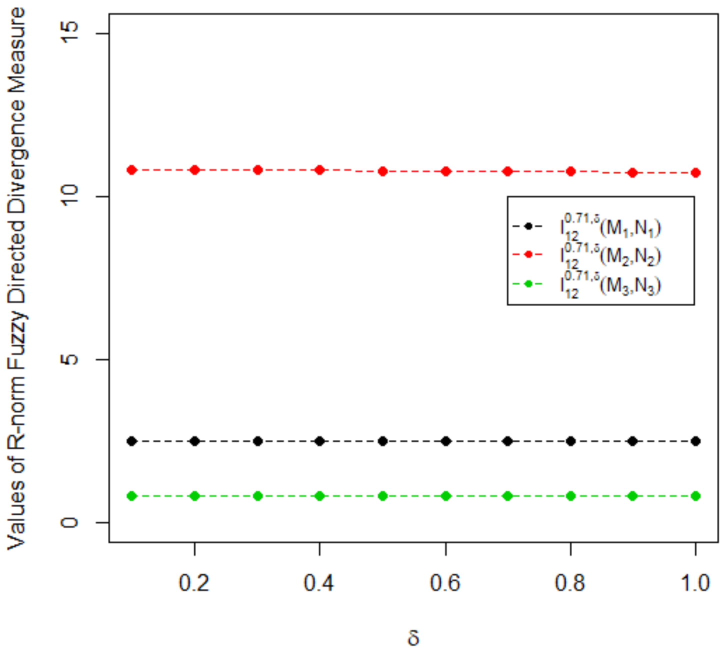

Figure 5.3(a)

The results of Table 5.3 indicate that as we increase the value of δ, the value of R.F.D.D.M. decreases. This table also shows that for small value of parameters R and γ the given measure in (5) shows visible decreasing trend, and for large value of R the measure shows constant decreasing trend. This relation between the parameter δ and the measure (5) is displayed in the Figures 5.3 & 5.3(a).

6. APPLICATIONS OF THE R-NORM FUZZY DIVERGENCE MEASURE

The suitable method to measure vague information is fuzzy sets. Here, in this section application part of R.F.D.D.M. in the perspective of P.R. and M.C.D.M. is given

6.1 Pattern Recognition

New techniques in P.R. have come into existence by the growth of fuzzy sets that helps to outdo the problems concerning uncertainties. Now, we determine the application of the (5) in terms of P.R. as:

Let us consider that we have p known patterns $A_1, A_2, ..., A_p$ that are classified as $Q_1, Q_2, ..., Q_p$ respectively. The following fuzzy set gives the representation of these pattern in the universal set $X = x_1, x_2, ..., x_n$: $A_i = \left\{\left(y_j, \nu_{A_i}\left(y_j\right)\right) / y_j \in Y\right\}$, where

$i = 1, 2, ..., p$ $j = 1, 2, ..., m$. Also, we have given an unknown pattern *B*, i.e., denoted by fuzzy set: $B_i = \left\{ \left(y_j, \nu_{B_i}\left(y_j \right) \right) / y_j \in Y \right\}$.

Our purpose is to categorize *B* to one of the classes $Q_1, Q_2, ..., Q_p$. Following the principle of minimum divergence information among fuzzy sets, the method of allocating B to Q_{l^*} is given by $l^* = \arg\min_l \left\{ I\left(A_l, B \right) \right\}$. This method gives the identification of the pattern which helps in selecting the best class. In the following example, we numerically prove the problem of pattern recognition.

Example 6.1.1:

Let us assume we have five known parameters $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ and Λ_5 that are classified as Q_1, Q_2, Q_3, Q_4 and Q_5 respectively. Following fuzzy set gives the representation of these patterns in the universal set $Y = \{ y_1, y_2, y_3, y_4, y_5 \}$:

$$\Lambda_1 = \left\{ (y_1, 0.6), (y_2, 0.9), (y_3, 0.2), (y_4, 0.5), (y_5, 0.7) \right\}$$
$$\Lambda_2 = \left\{ (y_1, 0.4), (y_2, 0.3), (y_3, 0.7), (y_4, 0.9), (y_5, 0.2) \right\}$$
$$\Lambda_3 = \left\{ (y_1, 0.8), (y_2, 0.5), (y_3, 0.1), (y_4, 0.7), (y_5, 0.4) \right\}$$
$$\Lambda_4 = \left\{ (y_1, 0.7), (y_2, 0.2), (y_3, 0.5), (y_4, 0.3), (y_5, 0.8) \right\}$$
$$\Lambda_5 = \left\{ (y_1, 0.3), (y_2, 0.7), (y_3, 0.6), (y_4, 0.1), (y_5, 0.9) \right\}$$

Also, we have an unknown pattern B that is, denoted by fuzzy set

$$B = \left\{ (y_1, 0.9), (y_2, 0.8), (y_3, 0.4), (y_4, 0.2), (y_5, 0.5) \right\}$$

The purpose is to categorize B to one of the classes Q_1, Q_2, Q_3, Q_4 and Q_5. Using these values we calculate R.F.D.D.M. given in equation (5). The values are represented in the following table:

Table 6.1

Calculation of R-Norm F.D.M $I_R^{\gamma,\delta}\left(\Lambda_i, B \right)$ for different values of R, γ and δ

	$R = 0.66, \gamma = 0.45$ and $\delta = 0.86$	$R = 13, \gamma = 0.75$ and $\delta = 0.36$
$I_R^{\gamma,\delta}\left(\Lambda_1, B \right)$	**0.3766**	**5.0828**
$I_R^{\gamma,\delta}\left(\Lambda_2, B \right)$	1.3975	11.8713
$I_R^{\gamma,\delta}\left(\Lambda_3, B \right)$	0.5720	5.4126
$I_R^{\gamma,\delta}\left(\Lambda_4, B \right)$	0.6197	6.0203
$I_R^{\gamma,\delta}\left(\Lambda_5, B \right)$	0.7465	7.5465

Hence, it is seen that $I_R^{\alpha,\beta}\left(\Lambda_1, B\right)$ has the minimum of all values. Therefore, B is classified to Λ_1 suitably.

6.2 Multi-Criteria Decision Making (M.C.D.M.)

M.C.D.M. is considered as branch of decision/choice making that helps in making decisions based on different criteria. It is usually used in the circumstances where choice/decision makers have inexact information about the criterion. It also helps in obtaining the result that is considered as the best among all alternatives. M.C.D.M. procedures are significant and common mathematical methods that are widely used in various activities. We illustrate the application of the R.F.D.D.M. in the perspective of M.C.D.M.

Assume for a set of alternatives $T=\{T_1,T_2,...,T_m\}$ we have a set of criteria $Q=\{Q_1,Q_2,...,Q_n\}$. We represent the features of the decision T_i in the form of criteria Q as

$$T_i=\left\{\left(Q_j,\nu_{ij}\right),Q_j\in Q\right\},\ i=1,2,..,m\quad j=1,2,...n$$

where ν_{ij} denotes the amount that the decision T_i fulfills criteria Q_j.

Now, we present the method that solves the given M.C.D.M. problem by using equation (5). Following is the way that includes the computational steps:

Step I: The first step is to find the positive ideal solution and negative ideal solution from the fuzzy sets i.e., T_i^+ and T_i^-

where $T_i^+=\left\{\left(\nu_1^+\right),\left(\nu_2^+\right),...,\left(\nu_n^+\right)\right\}$ and $T_i^-=\left\{\left(\nu_1^-\right),\left(\nu_2^-\right),...,\left(\nu_n^-\right)\right\}$

$\forall\ j=1,2,...n$, $\left(\nu_j^+\right)=\left(\max_i \nu_{ij}\right),\&\left(\nu_j^-\right)=\left(\min_i \nu_{ij}\right)$

Step II: Determine $I_R^{\gamma,\delta}\left(T_i,T^+\right)$ and $I_R^{\gamma,\delta}\left(T_i,T^-\right)\forall i$

Step III: In this step, the relative fuzzy R.D.M. is given by the formula as:

$$I_R^{\gamma,\delta}\left(T_i\right)=\frac{I_R^{\gamma,\delta}\left(T_i,T^+\right)}{I_R^{\alpha,\beta}\left(T_i,T^+\right)+I_R^{\alpha,\beta}\left(T_i,T^-\right)}\quad\forall i=1,2,...,m.$$

Step IV: **Select** the option that has least $I_R^{\gamma,\delta}\left(T_i\right)$.

We now consider the following real life example and exhibit the application of (5) with respect to M.C.D.M. problem as:

Example 6.2.1:

Suppose a consumer wishes to buy mobile phone. Consider five types of mobile phones are available in the market $T=\{T_1,T_2,T_3,T_4,T_5\}$. The consumer takes in to consideration five criteria for choosing best option:

(i) Price (c_1)

(ii) Battery efficiency (c_2)

(iii) RAM (c_3)

(iv) Size of Screen (c_4)

(v) Camera (c_5)

On the basis of the above five criteria, the best option i.e., $T_i \; \forall i = 1, 2, ..., 5$ is to be chosen by the decision maker by considering the following:

$$T_1 = \{(c_1, 0.9), (c_2, 0.4), (c_3, 0.1), (c_4, 0.4), (c_5, 0.3)\}$$
$$T_2 = \{(c_1, 0.3), (c_2, 0.6), (c_3, 0.8), (c_4, 0.5), (c_5, 0.2)\}$$
$$T_3 = \{(c_1, 0.5), (c_2, 0.2), (c_3, 0.4), (c_4, 0.6), (c_5, 0.5)\}$$
$$T_4 = \{(c_1, 0.6), (c_2, 0.7), (c_3, 0.5), (c_4, 0.9), (c_5, 0.4)\}$$
$$T_5 = \{(c_1, 0.3), (c_2, 0.5), (c_3, 0.2), (c_4, 0.8), (c_5, 0.5)\}$$

Following are the steps to solve above M.C.D.M. fuzzy problem:

Step I: We find the positive ideal solution and negative ideal solution from the above problem as:

$$T^+ = \{(c_1, 0.9), (c_2, 0.7), (c_3, 0.8), (c_4, 0.9), (c_5, 0.5)\}$$
$$T^- = \{(c_1, 0.3), (c_2, 0.2), (c_3, 0.1), (c_4, 0.4), (c_5, 0.2)\}$$

Step II: Compute the value for $I_R^{\alpha,\beta}(T_i, T^+)$ and $I_R^{\alpha,\beta}(T_i, T^-) \forall i = 1, 2, 3, 4, 5$

Table 6.2

Values of $I_R^{\alpha,\beta}(T_i, T^+) \; \forall i = 1, 2, 3, 4, 5$

	$R = 0.71, \gamma = 0.49$ and $\delta = 0.17$	$R = 7.2, \gamma = 0.65$ and $\delta = 0.47$	$R = 15, \gamma = 0.85$ and $\delta = 0.95$
$I_R^{\gamma,\delta}(T_1, T^+)$	2.3467	5.4702	4.4082
$I_R^{\gamma,\delta}(T_2, T^+)$	2.0915	4.5429	4.8739
$I_R^{\gamma,\delta}(T_3, T^+)$	3.4646	8.3743	8.9274
$I_R^{\gamma,\delta}(T_4, T^+)$	1.4766	4.0991	4.4082
$I_R^{\gamma,\delta}(T_5, T^+)$	2.6474	6.6391	7.1050

Table 6.2(a)

Values of $I_R^{\gamma,\delta}\left(T_i,T^-\right)\ \forall i=1,2,3,4,5$

	$R=0.71,\gamma=0.49$ and $\delta=0.17$	$R=7.2,\gamma=0.65$ and $\delta=0.47$	$R=15,\gamma=0.85$ and $\delta=0.95$
$I_R^{\gamma,\delta}\left(T_1,T^-\right)$	1.0324	2.1331	2.2684
$I_R^{\gamma,\delta}\left(T_2,T^+\right)$	1.2026	1.8283	1.8836
$I_R^{\gamma,\delta}\left(T_3,T^-\right)$	1.7035	2.8369	2.9506
$I_R^{\gamma,\delta}\left(T_4,T^-\right)$	1.3742	2.8281	2.9714
$I_R^{\gamma,\delta}\left(T_5,T^-\right)$	0.8024	1.8315	1.9555

Step III: This step gives the relative fuzzy R-norm divergence measure by the formula as:

$$I_R^{\alpha,\beta}\left(T_i\right)=\frac{I_R^{\gamma,\delta}\left(T_i,T^+\right)}{I_R^{\gamma,\delta}\left(T_i,T^+\right)+I_R^{\gamma,\delta}\left(T_i,T^-\right)}\quad \forall i=1,2,3,4,5$$

for different values of parameters.

Table 6.3

Values of $I_R^{\gamma,\delta}\left(T_i\right)\ \forall i=1,2,3,4,5$

	$I_R^{\gamma,\delta}\left(T_1\right)$	$I_R^{\gamma,\delta}\left(T_2\right)$	$I_R^{\gamma,\delta}\left(T_3\right)$	$I_R^{\gamma,\delta}\left(T_4\right)$	$I_R^{\gamma,\delta}\left(T_5\right)$
$R=0.71,\gamma=0.49$ and $\delta=0.17$	0.6944	0.6349	0.6703	0.5148	0.7674
$R=7.2,\gamma=0.65$ and $\delta=0.47$	0.7914	0.7130	0.7469	0.5917	0.7837
$R=15,\gamma=0.85$ and $\delta=0.95$	0.6602	0.7212	0.7515	0.5973	0.7841

Step IV: Table 6.3 gives the ranking of relative fuzzy R.D.M. as:

$T_4<T_2<T_3<T_1<T_5$ for $R=0.71,\gamma=0.49$ and $\delta=0.17$.

$T_4<T_2<T_3<T_5<T_1$ for $R=7.2,\gamma=0.65$ and $\delta=0.47$.

$T_4<T_1<T_2<T_3<T_5$ for $R=15,\gamma=0.85$ and $\delta=0.95$.

Since, T_4 comes minimum of all the choices in the variations of parameters, the result gives T_4 as the most suitable option.

7. CONCLUSION

In this article, we have presented parametric R.F.I.M. and its corresponding directed divergence measure. We have verified with the help of numerical data that the defined measures are correct. The important properties and the specific cases of the proposed measures are discussed which gives the generality of previously confirmed R-norm measures. Also, the nature of the proposed measures is analyzed by substituting numerous values to the parameters. Further, the application of the R-norm F.D.M. is discussed in terms of P.R. and M.C.D.M. that is exemplified with the help of illustrations.

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