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BETA GENERATED KUMARASWAMY-G FAMILY OF DISTRIBUTIONS

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ABSTRACT

A new generalization of the family of Kumaraswamy-*G* distribution that includes three recently proposed families namely the Garhy generated family, Beta-Dagum and Beta-Singh-Maddala distributions is proposed by constructing beta generated Kumaraswamy-*G* distribution. Useful expansions of the probability density function and the cumulative distribution function of the proposed family are derived and seen as infinite mixtures of the Kumaraswamy-*G* distribution. Order statistics, probability weighted moments, moment generating function, Rényi entropy, quantile power series, random sample generation, asymptotes and shapes are also investigated. Two methods of parameter estimation are presented. Suitability of the proposed model in comparison to its sub models is carried out by considering two real life data sets modeling.

KEYWORDS

Beta generated family; Kumaraswamy-*G* family; Exponentiated family; Akaike Information Criterion; Power weighted moment.

1. INTRODUCTION

Here we briefly introduce the beta-*G* (Eugene *et al.*, 2002 and Jones, 2004) and Kumaraswamy-*G* ($Kw-G$) (Cordeiro and de Castro, 2011) family of distributions**.**

1.1 Beta-*G* family of Distribution

For a given distribution with probability density function (pdf) $f(t)$ and cumulative distribution function (cdf) $F(t)$, the cdf of beta-*G* family of distribution is given respectively by

$$F^{BG}(t;m,n) = \left(1/B(m,n)\right) \int_o^{F(t)} v^{m-1}(1-v)^{n-1}\,dv$$

$$= B_{F(t)}(m,n)\big/B(m,n) = I_{F(t)}(m,n) \tag{1}$$

and pdf

$$f^{BG}(t;m,n) = \left(1/B(m,n)\right) f(t)\,F(t)^{m-1}\,\bar{F}(t)^{n-1}, \tag{2}$$

where $I_t(m,n)=B(m,n)^{-1}\int_0^t x^{m-1}(1-x)^{n-1}dx$ is the incomplete beta function ratio. The corresponding survival function (sf), hazard rate function (hrf), reverse hazard rate function (rhrf) and cumulative hazard rate function (chrf) are given respectively by

$$\bar{F}^{BG}(t;m,n)=P[T>t]=1-I_{F(t)}(m,n)=\left\{B(m,n)-B_{F(t)}(m,n)\right\}\Big/B(m,n),$$

$$h^{BG}(t;m,n)=\left\{f(t)F(t)^{m-1}\bar{F}(t)^{n-1}\right\}\Big/\left\{B(m,n)-B_{F(t)}(m,n)\right\},$$

$$r^{BG}(t;m,n)=\left\{f(t)F(t)^{m-1}\bar{F}(t)^{n-1}\right\}\Big/B_{F(t)}(m,n)$$

and

$$H^{BG}(t;m,n)=-\log\left[\left\{B(m,n)-B_{F(t)}(m,n)\right\}\Big/B(m,n)\right].$$

Some of the well-known beta-generated (beta-*G*) families are the beta Gumbel distribution (Nadarajah and Kotz, 2004), beta generalized exponential distribution (Barreto-Souza *et al.*, 2010), beta generalized Weibull distribution (Singla *et al.*, 2012), beta generalized Rayleigh distribution (Cordeiro *et al.*, 2013), beta extended half normal distribution (Cordeiro *et al.*, 2014a), beta log-logistic distribution (Lemonte, 2014), beta Marshall-Olkin family of distribution (Alizadeh *et al.*, 2015b), beta exponential Frechet distribution (Mead *et al.*, 2016), beta-Dagum and beta-Singh-Maddala distribution (Domma and Condino, 2016) among others.

1.2 Kumaraswamy-*G* ($Kw-G$) Family of Distribution

Given the baseline cdf $G(t)$ with pdf $g(t)$, Cordeiro and de Castro (2011) defined $Kw-G$ distribution with respectively cdf and pdf

$$F^{KwG}(t;a,b)=1-\left[1-G(t)^a\right]^b \tag{3}$$

and

$$f^{KwG}(t;a,b)=a\,b\,g(t)G(t)^{a-1}\left[1-G(t)^a\right]^{b-1}, \tag{4}$$

where $t>0,\ g(t)=G'(t)$ and $a>0,\ b>0$ are the shape parameters in addition to those in the baseline distribution. Corresponding sf, hrf, rhrf and chrf are respectively given by

$$\bar{F}^{KwG}(t;a,b)=1-F^{KwG}(t)=\left[1-G(t)^a\right]^b \tag{5}$$

$$h^{KwG}(t;a,b)=a\,b\,g(t)G(t)^{a-1}\left[1-G(t)^a\right]^{-1}$$

$$r^{KwG}(t;a,b)=a\,b\,g(t)G(t)^{a-1}\left[1-G(t)^a\right]^{b-1}\left\{1-\left[1-G(t)^a\right]^b\right\}^{-1}$$

and $H^{KwG}(t;a,b)=-b\log\left[1-G(t)^a\right]$ respectively.

Some of the notable distributions derived under the scheme of $Kw-G$ family are Kumaraswamy Normal distribution (Correa *et al.*, 2012), Kumaraswamy generalized Pareto distribution (Nadarajah and Eljbri, 2013), Kumaraswamy linear exponential distribution (Elbatal, 2013), Kumaraswamy modified Weibull distribution (Cordeiro *et al.*, 2014b), Kumaraswamy Marshall-Olkin family of distribution (Alizadeh *et al.*, 2015a), Marshall-Olkin Kumaraswamy-*G* family of distribution (Handique *et al.*, 2017) and generalized Marshall-Olkin Kumaraswamy-*G* family of distribution (Chakraborty and Handique, 2017) among others. For detail see Tahir and Nadarajah (2015).

In this article we propose a new family of beta generated $Kw-G$ distribution by considering the $Kw-G$ family as the base line distribution *G* in the beta-*G* family and investigate some of its general properties. The rest of this article is organized in five more Sections. In Section 2 the new family is defined along with its physical basis and list of some important sub models. In Section 3 we discuss some general results of the family. Different methods of estimation of parameters are presented in Section 4. Application of the proposed family is considered in Section 5. The paper ends with a conclusion in Section 6.

2. A NEW GENERALIZATION: BETA KUMARSWAMY-*G* ($BKw-G$) FAMILY OF DISTRIBUTIONS

Here we propose a new beta generated family by considering the cdf, pdf and sf of $Kw-G$ distribution in (3), (4) and (5) as the $F(t)$, $f(t)$ and $\bar{F}(t)$ respectively in the beta-*G* formulation in (2) and call it $BKw-G$ distribution. The pdf and cdf of $BKw-G$ are given respectively by

$$f^{BKwG}(t;a,b,m,n) = \{1/B(m,n)\}\left\{a\,b\,g(t)\,G(t)^{a-1}\left[1-G(t)^{a}\right]^{bn-1}\left[1-\left[1-G(t)^{a}\right]^{b}\right]^{m-1}\right\}, \quad 0<t<\infty,\ 0<a,b<\infty,\ m,n>0 \tag{6}$$

and

$$F^{BKwG}(t;a,b,m,n) = I_{1-[1-G(t)^{a}]^{b}}(m,n). \tag{7}$$

The sf, hrf, rhrf and chrf of $BKw-G$ distribution are respectively obtained as

$$\bar{F}^{BKwG}(t;a,b,m,n) = 1 - I_{1-[1-G(t)^{a}]^{b}}(m,n)$$

$$h^{BKwG}(t;a,b,m,n) = \frac{a\,b\,g(t)\,G(t)^{a-1}\left[1-G(t)^{a}\right]^{bn-1}\left[1-\left[1-G(t)^{a}\right]^{b}\right]^{m-1}}{B(m,n)\left[1-I_{1-[1-G(t)^{a}]^{b}}(m,n)\right]} \tag{8}$$

$$r^{BKwG}(t;a,b,m,n)=\frac{a\,b\,g(t)\,G(t)^{a-1}\left[1-G(t)^{a}\right]^{bn-1}\left[1-\left[1-G(t)^{a}\right]^{b}\right]^{m-1}}{B(m,n)\ I_{1-[1-G(t)^{a}]^{b}}(m,n)} \tag{9}$$

$$H^{BKwG}(t;a,b,m,n)=-\log\left[1-I_{1-[1-G(t)^{a}]^{b}}(m,n)\right].$$

This new family can also be referred to as the new generalized $Kw-G$ family of distribution. For

i) $n=1$, it reduces to the recently proposed Garhy generated family that is $BKw-G(a,b,m,1)\equiv GH-G(a,b,m)$ with cdf and pdf respectively given by

$F^{GHG}(t;a,b,m)=[1-\{1-G(t)^{a}\}^{b}]^{m}$ and

$f^{GHG}(t;a,b,m)=a\,b\,m\,g(t)\,G(t)^{a-1}[1-G(t)^{a}]^{b-1}[1-\{1-G(t)^{a}\}^{b}]^{m-1}$, $t>0$,

where $a,b,m>0$ are three shape parameters.

ii) $m=n=1$, $BKw-G(a,b,1,1)=Kw-G(a,b)$ and

iii) $a=b=1$, then $BKw-G(1,1,m,n)=B(m,n)$.

2.1 Genesis of the Family

Theorem 1:

If m and n are both integers, then the probability distribution of $BKw-G(a,b,m,n)$ arises as distribution of the m^{th} order statistics of a random sample of size $m+n-1$ from $Kw\text{-}G(a,b)$ distribution.

Proof:

Let $T_1, T_2, ..., T_{m+n-1}$ be a random sample of size $m+n-1$ from $Kw\text{-}G(a,b)$ distribution with cdf $1-[1-G(t)^{a}]^{b}$. Then the pdf of the m^{th} order statistics $T_{(m)}$ is given by

$$=(m+n-1)!\Big/\left\{(m-1)!\ \left[(m+n-1)-m\right]!\right\}$$

$$\times\left[1-\left[1-G(t)^{a}\right]^{b}\right]^{m-1}\left[\left\{1-G(t)^{a}\right\}^{b}\right]^{(m+n-1)-m} a\,b\,g(t)\,G\left(t\right)^{a-1}\left[1-G(t)^{a}\right]^{b-1}$$

$$=\Gamma(m+n)/\{\Gamma(m)\ \Gamma(n)\}\left[1-\left[1-G\left(t\right)^{a}\right]^{b}\right]^{m-1}$$

$$\left[\left\{1-G(t)^{a}\right\}^{b}\right]^{n-1} a\,b\,g(t)\,G\left(t\right)^{a-1}\left[1-G(t)^{a}\right]^{b-1}$$

$$=(1/B(m,n))\ a\,b\,g(t)G(t)^{a-1}[1-G(t)^{a}]^{bn-1}[1-[1-G(t)^{a}]^{b}]^{m-1}.$$

2.2 Plots of the PDF and HRF

In this section we have plotted the pdf and hrf of the $BKw-G(a,b,m,n)$ taking G to be exponential (E), Weibull (W), Lomax (L) and Frechet (F) for some chosen values of the parameters to show the variety of shapes assumed by the family.

The pdf and hrf of these distributions are obtained from $BKw-G(a,b,m,n)$ as follows:

- The $BKw-$Weibull ($BKw-W$) distribution
  Considering the Weibull distribution (Weibull, 1951) with parameters $\lambda>0$ and $\beta>0$ having pdf and cdf $g(t)=\lambda\beta\, t^{\beta-1}e^{-\lambda t^{\beta}}$ and $G(t)=1-e^{-\lambda t^{\beta}}$ respectively we get the pdf and hrf of $BKw-W(a,b,m,n,\lambda,\beta)$ distribution as

$$f^{BKwW}(t)=\frac{ab\lambda\beta t^{\beta-1}e^{-\lambda t^{\beta}}\left\{1-e^{-\lambda t^{\beta}}\right\}^{a-1}\left[1-\left\{1-e^{-\lambda t^{\beta}}\right\}^{a}\right]^{bn-1}\left[1-\left[1-\left\{1-e^{-\lambda t^{\beta}}\right\}^{a}\right]^{b}\right]^{m-1}}{B(m,n)}$$

$$h^{BKwW}(t)=\frac{ab\lambda\beta t^{\beta-1}e^{-\lambda t^{\beta}}\left\{1-e^{-\lambda t^{\beta}}\right\}^{a-1}\left[1-\left\{1-e^{-\lambda t^{\beta}}\right\}^{a}\right]^{bn-1}\left[1-\left[1-\left\{1-e^{-\lambda t^{\beta}}\right\}^{a}\right]^{b}\right]^{m-1}}{B(m,n)\left[1-I_{1-\left[1-\left\{1-e^{-\lambda t^{\beta}}\right\}^{a}\right]^{b}}(m,n)\right]}.$$

Taking $\beta=1$ in $BKw-W(a,b,m,n,\lambda,\beta)$ we get the $BKw-E(a,b,m,n,\lambda)$ with pdf

$$f^{BKwE}(t)=\frac{1}{B(m,n)}ab\lambda e^{-\lambda t}\left\{1-e^{-\lambda t}\right\}^{a-1}\left[1-\left\{1-e^{-\lambda t}\right\}^{a}\right]^{bn-1}\left[1-\left[1-\left\{1-e^{-\lambda t}\right\}^{a}\right]^{b}\right]^{m-1}$$

$$h^{BKwE}(t)=\frac{ab\lambda e^{-\lambda t}\left\{1-e^{-\lambda t}\right\}^{a-1}\left[1-\left\{1-e^{-\lambda t}\right\}^{a}\right]^{bn-1}\left[1-\left[1-\left\{1-e^{-\lambda t}\right\}^{a}\right]^{b}\right]^{m-1}}{B(m,n)\left[1-I_{1-\left[1-\left\{1-e^{-\lambda t}\right\}^{a}\right]^{b}}(m,n)\right]}$$

- The $BKw-$Lomax ($BKw-L$) distribution
  Considering the Lomax distribution (Lomax, 1954) with pdf and cdf given by $g(t:\beta,\delta)=(\beta/\delta)\left[1+(t/\delta)\right]^{-(\beta+1)}$, $t>0$, and $G(t:\beta,\delta)=1-\left[1+(t/\delta)\right]^{-\beta}$, $\beta>0$ and $\delta>0$ the pdf and hrf of the $BKw-L(a,b,m,n,\beta,\delta)$ distribution are given respectively by

$$f^{BKwL}(t)=\frac{ab(\beta/\delta)\left[1+(t/\delta)\right]^{-(\beta+1)}\left\{1-\left[1+(t/\delta)\right]^{-\beta}\right\}^{a-1}\left[1-\left\{1-\left[1+(t/\delta)\right]^{-\beta}\right\}^{a}\right]^{bn-1}}{B(m,n)}$$

$$\times[1-[1-\{1-[1+(t/\delta)]^{-\beta}\}^{a}]^{b}]^{m-1}$$

$$h^{BKwL}(t)=\frac{ab(\beta/\delta)\left[1+(t/\delta)\right]^{-(\beta+1)}\left\{1-\left[1+(t/\delta)\right]^{-\beta}\right\}^{a-1}\left[1-\left\{1-\left[1+(t/\delta)\right]^{-\beta}\right\}^{a}\right]^{bn-1}}{B(m,n)\left[1-I_{1-\left[1-\left\{1-\left[1+(t/\delta)\right]^{-\beta}\right\}^{a}\right]^{b}}(m,n)\right]}$$

$$\times[1-[1-\{1-[1+(t/\delta)]^{-\beta}\}^{a}]^{b}]^{m-1}.$$

- The $BKw-$Frechet ($BKw-F$) distribution

  Suppose the base line distribution is the Frechet distribution (Krishna *et al.*, 2013) with pdf and cdf given by $g(t)=\lambda\delta^{\lambda}t^{-(\lambda+1)}e^{-(\delta/t)^{\lambda}}$ and $G(t)=e^{-(\delta/t)^{\lambda}}$, $t>0$ respectively, then the corresponding pdf and hrf of $BKw-F(a,b,m,n,\lambda,\delta)$ distribution becomes

$$f^{BKwF}(t)=\frac{ab\lambda\delta^{\lambda}t^{-(\lambda+1)}e^{-(\delta/t)^{\lambda}}\left\{e^{-(\delta/t)^{\lambda}}\right\}^{a-1}\left[1-\left\{e^{-(\delta/t)^{\lambda}}\right\}^{a}\right]^{bn-1}\left[1-\left[1-\left\{e^{-(\delta/t)^{\lambda}}\right\}^{a}\right]^{b}\right]^{m-1}}{B(m,n)}$$

$$f^{BKwF}(t)=\frac{ab\lambda\delta^{\lambda}t^{-(\lambda+1)}e^{-(\delta/t)^{\lambda}}\left\{e^{-(\delta/t)^{\lambda}}\right\}^{a-1}\left[1-\left\{e^{-(\delta/t)^{\lambda}}\right\}^{a}\right]^{bn-1}\left[1-\left[1-\left\{e^{-(\delta/t)^{\lambda}}\right\}^{a}\right]^{b}\right]^{m-1}}{B(m,n)\left[1-I_{1-\left[1-\left\{e^{-(\delta/t)^{\lambda}}\right\}^{a}\right]^{b}}(m,n)\right]}.$$

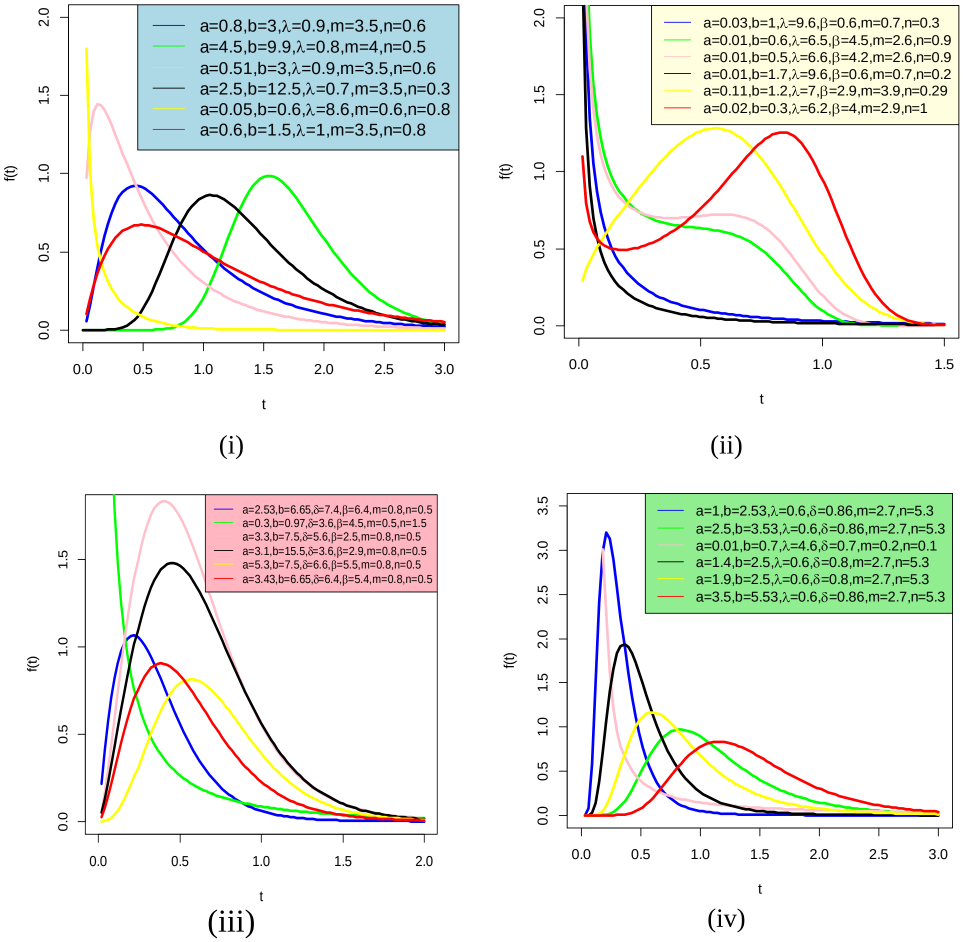


Fig 1: Density plots of (i) $BKw-E$ **, (ii)** $BKw-W$ **,**
(iii) $BKw-L$ **and (iv)** $BKw-F$ **distributions.**

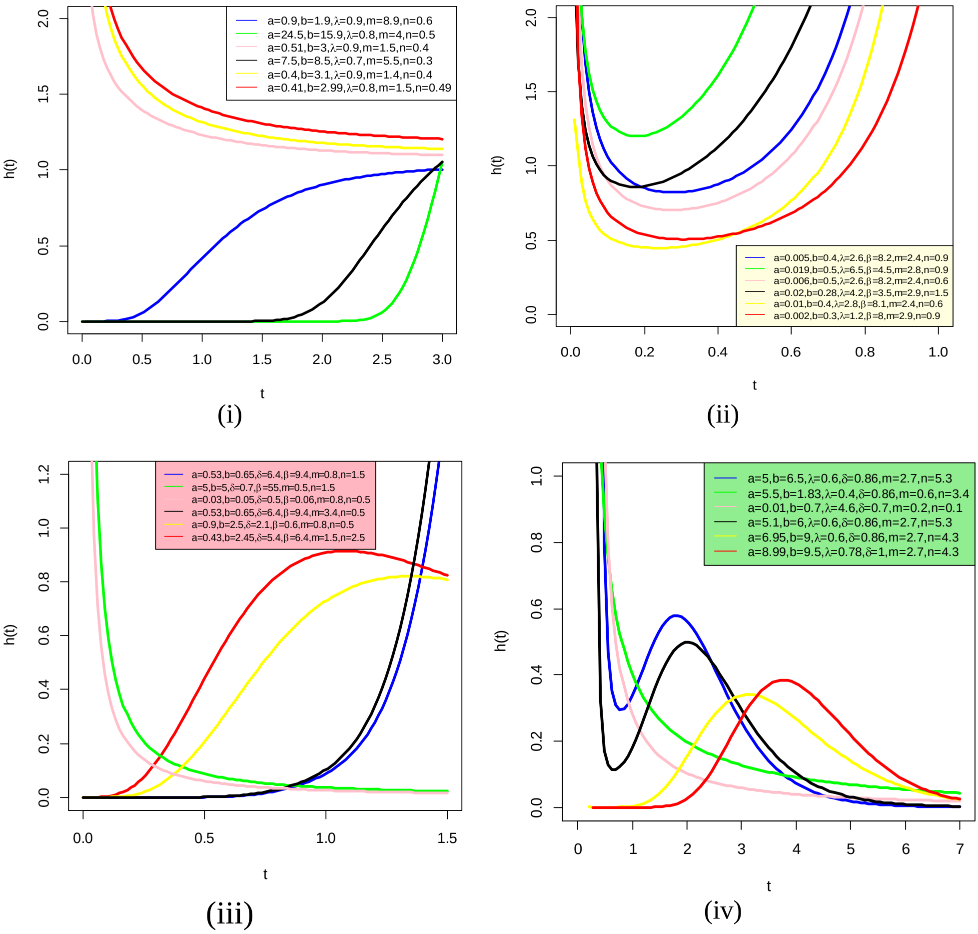


Fig 2: Hazard plots of (i) $BKw-E$, (ii) $BKw-W$, (iii) $BKw-L$ and (iv) $BKw-F$ distributions

From the plots in Figure 1 and 2 it can be seen that the family is very flexible and can offer many different types of shapes. It offers IFR, DFR and even bath tub shaped hazard rate functions.

3. GENERAL RESULTS

In this Section we derive some general results for the proposed $BKw-G(a,b,m,n)$ family.

3.1 Expansions of pdf and cdf

By using binomial expansion in (6), we obtain

$$f^{BKwG}(t;a,b,m,n)=\left(1/B(m,n)\right)ab\,g(t)G\left(t\right)^{a-1}\left[1-G\left(t\right)^{a}\right]^{bn-1}\left[1-\left[1-G\left(t\right)^{a}\right]^{b}\right]^{(m-1)}$$

$$=(1/B(m,n))\;ab\,g(t)G\left(t\right)^{a-1}\left[1-G\left(t\right)^{a}\right]^{bn-1}\sum_{j=0}^{m-1}\binom{m-1}{j}\left(-1\right)^{j}\left[1-G(t)^{a}\right]^{bj} \quad (10)$$

$$=\sum_{j=0}^{m-1}\beta'_j\; f^{KwG}\left(t;a,b(j+n)\right) \quad (11)$$

where

$$f^{KwG}\left(t;a,b(j+n)\right)=ab(j+n)\,g(t)G(t)^{a-1}\left[1-G(t)^{a}\right]^{b(j+n)-1}$$

and

$$\beta'_j=\frac{(-1)^{j+1}}{B(m,n)(j+n)}\binom{m-1}{j}.$$

Adjusting further we get from (10)

$$=\sum_{j=0}^{m-1}\beta'_j\,\frac{d}{dt}\left[\bar{F}^{KwG}(t;a,b)\right]^{j+n} \quad (12)$$

$$=f^{KwG}(t;a,b)\sum_{j=0}^{m-1}\left\{\beta'_j(j+n)\right\}\left[\bar{F}^{KwG}(t;a,b)\right]^{j+n-1}$$

$$=f^{KwG}(t;a,b)\sum_{j=0}^{m-1}\beta_j\left[\bar{F}^{KwG}(t;a,b)\right]^{j+n-1}, \quad (13)$$

where $\beta_j=\beta'_j(j+n)$.

Alternatively, we can expand the pdf as

$$f^{BKwG}(t;a,b,m,n)\;=f^{KwG}\left(t;a,b\right)\sum_{l=0}^{j+n-1}\eta_l\left[F^{KwG}(t;a,b)\right]^{l}\;, \quad (14)$$

where $\eta_l=\sum_{j=0}^{m-1}(-1)^{l}\beta_j\binom{j+n-1}{l}$.

Now to expand the cdf $F^{BKwG}(t;a,b,m,n)=I_{1-[1-G(t)^a]^b}(m,n)$ we use the result to get

$$I_z(a,b)=\frac{B_z(a,b)}{B(a,b)}=\frac{z^a}{B(a,b)}\sum_{i=0}^{\infty}\binom{b-1}{i}\frac{\left(-1\right)^{i}}{\left(a+i\right)}z^{i} \quad (15)$$

Remembering that $B(z;a,b)=B_z(a,b)=z^a\sum_{i=0}^{\infty}\frac{(1-b)_i}{i!\,(a+i)}z^i$, where $(x)_i$ is a Pochhammer symbol.

$$=z^a\sum_{i=0}^{\infty}\frac{(-1)^i\,(b-1)!}{i!\,(b-i-1)!\,(a+i)}z^i=z^a\sum_{i=0}^{\infty}\binom{b-1}{i}\frac{(-1)^i}{(a+i)}z^i$$

(See "Incomplete Beta Function" *From Math World*-A Wolfram Web Resource. http://mathworld. Wolfram.com/Incomplete Beta Function. html).

$$F^{BKwG}(t;a,b,m,n)=\frac{1}{B(m,n)}\left(1-\left[1-G(t)^a\right]^b\right)^m\sum_{i=0}^{\infty}\binom{n-1}{i}\frac{(-1)^i}{(m+i)}\left(1-\left[1-G(t)^a\right]^b\right)^i$$

$$=\sum_{i,j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^{i+j+k}}{B(m,n)(m+i)}\binom{n-1}{i}\binom{m+i}{j}\binom{j}{k}\left[F^{KwG}(t;a,b)\right]^k.$$

Now in the summation exchanging the indices j and k in the sum symbol, we get

$$F^{BKwG}(t;a,b,m,n)=\sum_{i,k=0}^{\infty}\sum_{j=k}^{\infty}\frac{(-1)^{i+j+k}}{B(m,n)(m+i)}\binom{n-1}{i}\binom{m+i}{j}\binom{j}{k}\left[F^{KwG}(t;a,b)\right]^k$$

and then

$$F^{BKwG}(t;a,b,m,n)=\sum_{k=0}^{\infty}\mu_k\,F^{KwG}(t;a,b)^k\quad,\tag{16}$$

where

$$\mu_k=\sum_{i=0}^{\infty}\sum_{j=k}^{\infty}\frac{(-1)^{i+j+k}}{B(m,n)(m+i)}\binom{n-1}{i}\binom{m+i}{j}\binom{j}{k}.$$

Similarly, an expansion for the cdf of $BKw-G(a,b,m,n)$ can be derived as (Osborn and Madey, 1968).

$$F^{BKwG}(t;a,b,m,n)=I_{1-\left[1-G(t)^a\right]^b}(m,n)=I_{F^{KwG}(t;a,b)}(m,n)$$

On using $I_x(a,\,b)=\sum_{k=a}^{a+b-1}\binom{a+b-1}{k}x^k\,(1-x)^{a+b-1-k}$ we get

$$=\sum_{p=m}^{m+n-1}\binom{m+n-1}{p}\left[F^{KwG}(t;a,b)\right]^p\left[\bar{F}^{KwG}(t;a,b)\right]^{m+n-1-p}$$

$$=\sum_{p=m}^{m+n-1}\sum_{q=0}^{m+n-1-p}\lambda_{p,q}\left[F^{KwG}(t;a,b)\right]^{p+q}\quad,$$

where $\lambda_{p,q}=\binom{m+n-1}{p}\binom{m+n-1-p}{q}(-1)^q$.

3.2 Order Statistics

Consider a random sample $T_1, T_2, .., T_\vartheta$ from any $BKw-G(a,b,m,n)$ distribution. Let $T_{r:\vartheta}$ denote the r^{th} order statistic. The pdf of $T_{r:\vartheta}$ can be expressed as

$$f_{r:\vartheta}(t;a,b,m,n) = \{\vartheta!/(r-1)!(\vartheta-r)!\}\, f^{BKwG}(t)\, F^{BKwG}(t)^{r-1}\left\{1-F^{BKwG}(t)\right\}^{\vartheta-r}$$
$$= \{\vartheta!/(r-1)!\,(\vartheta-r)!\}\sum_{j=0}^{\vartheta-r}(-1)^j\binom{\vartheta-r}{j} f^{BKwG}(t)\, F^{BKwG}(t)^{j+r-1}\ .$$

Now using the general expansion of the pdf and cdf of the $BKw-G(a,b,m,n)$ distribution we get the pdf of the r^{th} order statistic for of the $BKw-G(a,b,m,n)$ as

$$f_{r:\vartheta}(t;a,b,m,n) = f^{KwG}(t;a,b)\sum_{z=0}^{k+l}\chi_z\left[\bar{F}^{KwG}(t;a,b)\right]^z \qquad (17)$$
$$= \sum_{z=0}^{k+l}\frac{\chi_z}{z+1}\frac{d}{dt}\left[\bar{F}^{KwG}(t;a,b)\right]^{z+1} = \sum_{z=0}^{k+l}\chi'_z\, f^{KwG}\left(t;a,b(z+1)\right)\ ,$$

where $\chi'_z = \sum_{l=0}^{j+n-1}\sum_{k=0}^{\infty}\frac{(-1)^{z+1}\gamma_{l,k}}{z+1}\binom{k+l}{z}$, $\chi_z = \chi'_z(z+1)$

$$\gamma_{l,k} = \frac{\vartheta!}{(r-1)!\,(\vartheta-r)!}\sum_{j=0}^{\vartheta-r}(-1)^j\binom{\vartheta-r}{j}\eta_l\, d_{j+r-1,k} \text{ and}$$

$$d_{j+r-1,k} = \frac{1}{k\mu_0}\sum_{c=1}^{k}\left[c(j+r)-k\right]\mu_c\, d_{j+r-1,\,k-c}$$ and η_l, μ_c as defined above.

3.3 Probability Weighted Moments

The probability weighted moments (PWM), first proposed by Greenwood *et al*. (1979), are expectations of certain functions of a random variable whose mean exists. The $(p,q,r)^{th}$ PWM of *T* is defined by $\Gamma_{p,q,r} = \int_{-\infty}^{\infty} t^p F(t)^q\left[1-F(t)\right]^r f(t)\,dt$.

From equations (13) and (14) the s^{th} moment of T can be written either as

$$E(T^s) = \int_{-\infty}^{\infty} t^s\, f^{BKwG}(t;a,b,m,n)\,dt$$
$$= \sum_{j=0}^{m-1}\beta_j\int_{-\infty}^{\infty} t^s\left[\left\{1-G(t)^a\right\}^b\right]^{j+n-1} a\,b\,g(t)\,G(t)^{a-1}\left[1-G(t)^a\right]^{b-1}dt$$
$$= \sum_{j=0}^{m-1}\beta_j\,\Gamma_{s,0,\,j+n-1}$$

and $$E(T^s) = \sum_{l=0}^{j+n-1}\eta_l\,\Gamma_{s,\,l,\,0}$$

where $\Gamma_{p,q,r}=\int_{-\infty}^{\infty} t^{p}\left[1-\left\{1-G(t)^{a}\right\}^{b}\right]^{q}\left[\left\{1-G(t)^{a}\right\}^{b}\right]^{r} a\, b\, g(t)\, G(t)^{a-1}\left[1-G(t)^{a}\right]^{b-1} dt$

is the PWM of $Kw\text{-}G(a,b)$ distribution.

Therefore the moments of the $BKw-G(a,b,m,n)$ may be expressed in terms of the PWMs of $Kw-G(a,b)$. The PWM method can generally be used for estimating parameters quantiles of generalized distributions. These moments have low variance and no severe biases, and compare favourably with estimators obtained by maximum likelihood.

Proceeding as above we can derive s^{th} moment of the r^{th} order statistic $T_{r:n}$, in a random sample of size n from $BKw-G(a,b,m,n)$ on using equation (17) as

$$E\left(T^{s}{}_{r;n}\right)=\sum_{z=0}^{k+l}\chi_{z}\,\Gamma_{s,0,z}\,,\quad \text{where } \beta_{j}\,,\eta_{l} \text{ and } \chi_{z} \text{ defined in above.}$$

3.4 Moment Generating Function

The moment generating function of $BKw-G(a,b,m,n)$ family can be easily expressed in terms of those of the exponentiated $Kw-G$ distribution using the results of Section 3.1. For example using equation (12) it can be seen that

$$M_{T}(s)=E\left[e^{sT}\right]=\int_{-\infty}^{\infty} e^{st} f(t;m,n,a,b)\,dt=\int_{-\infty}^{\infty} e^{st}\sum_{j=0}^{m-1}\beta'_{j}\frac{d}{dt}\left[\bar{F}^{KwG}(t;a,b)\right]^{j+n} dt$$
$$=\sum_{j=0}^{m-1}\beta'_{j}\int_{-\infty}^{\infty} e^{st}\frac{d}{dt}\left\{\bar{F}^{KwG}(t;a,b)\right\}^{j+n} dt=\sum_{j=0}^{m-1}\beta_{j}\, M_{X}(s),$$

where $M_{X}(s)$ is the mgf of a $Kw-G$ distribution.

3.5 Rényi Entropy

The entropy of a random variable is a measure of uncertainty and has been used in various situations in science and engineering. The Rényi entropy of a random variable having pdf $f(t)$ is given by

$$I_{R}(\delta)=(1-\delta)^{-1}\log\left(\int_{-\infty}^{\infty}\left[f(t)\right]^{\delta} dt\right),$$

where $\delta>0$ and $\delta\neq 1$. For further details, see Song (2001). Applying binomial expansion in (6) we can write the pdf of $BKw-G(a,b,m,n)$ in equation (6) by expanding the last expression as

$$f^{BKwG}(t;a,b,m,n)^{\delta}=\frac{1}{B(m,n)^{\delta}}\left[a\,b\,g(t)\,G(t)^{a-1}\left[1-G(t)^{a}\right]^{bn-1}\right]^{\delta}$$
$$\sum_{\alpha=0}^{(m-1)\delta}\binom{(m-1)\delta}{\alpha}(-1)^{\alpha}\left[1-G(t)^{a}\right]^{b\alpha}.$$

Thus the Rényi entropy of T can be obtained as

$$I_R(\delta)=(1-\delta)^{-1}\log\left(\sum_{\alpha=0}^{(m-1)\delta} R_\alpha \int_{-\infty}^{\infty}\left[abg(t)\,G(t)^{a-1}\left[1-G(t)^a\right]^{bn-1}\right]^{\delta}\left[1-G(t)^a\right]^{b\alpha} dt\right)$$
$$=(1-\delta)^{-1}\log\left(\sum_{\alpha=0}^{(m-1)\delta} R_\alpha \int_{-\infty}^{\infty}\left[f^{\,KwG}(t;a,bn)\right]^{\delta}\left[\bar{F}^{\,KwG}(t;a,b)\right]^{\alpha} dt\right)$$

where $R_\alpha=\dfrac{1}{B(m,n)^{\delta}}\dbinom{(m-1)\delta}{\alpha}(-1)^{\alpha}$.

3.6 Quantile Power Series and Random Sample Generation

The quantile function of T, $t=Q(u)=F^{-1}(u)$, can be obtained by inverting (7). Let $z=Q_{m,n}(u)$ be the beta quantile function. Then,

$$F^{BKwG}(t;a,b,m,n)=I_{1-\left[1-G(t)^a\right]^b}(m,n)=u$$
$$\Rightarrow 1-\left\{1-G(t)^a\right\}^b=Q_{m,n}(u)\ \Rightarrow t=Q_G\left[\left[1-\left\{1-Q_{m,n}(u)\right\}^{1/b}\right]^{1/a}\right].$$

It is possible to obtain an expansion for $Q_{m,n}(u)$ as

$$z=Q_{m,n}(u)=\sum_{i=0}^{\infty} e_i\,u^{\,i/m}\quad \text{(Bornemann and Weisstein, 2004)},$$

where $e_i=\left[\,m\,B(m,n)\,\right]^{1/m} d_i$ and $d_0=0,\ d_1=1,\ d_2=(n-1)/(m+1)$,

$$d_3=\frac{(n-1)(m^2+3mn-m+5n-4)}{2(m+1)^2(m+2)}$$

$$d_4=(n-1)\Big[m^4+(6n-1)m^3+(n+2)(8n-5)m^2$$
$$+(33n^2-30n+4)m+n(31n-47)+18\Big]\Big/\Big[3(m+1)^3(m+2)(m+3)\Big]$$

and so on.

The Bowley skewness (Kenney and Keeping, 1962) and Moors kurtosis (Moors, 1988) measures are known to be robust and less sensitive to outliers and exist even for distributions without moments. For $BKw-G(a,b,m,n)$ family these measures can be obtained

$$B=\frac{Q(3/4)+Q(1/4)-2Q(1/2)}{Q(3/4)-Q(1/4)}\quad\text{and}\quad M=\frac{Q(3/8)-Q(1/8)+Q(7/8)-Q(5/8)}{Q(6/8)-Q(2/8)}$$

For example, let the G be exponential distribution with parameter $\lambda > 0$, having pdf and cdf as $g(t:\lambda) = \lambda e^{-\lambda t}, t > 0$ and $G(t:\lambda) = 1 - e^{-\lambda t}$, respectively. Then the p^{th} quantile is obtained as $-(1/\lambda)\log[1-p]$. Therefore, the p^{th} quantile t_p, of $BKw-E$ is given by

$$t_p = -(1/\lambda)\log\left[1-\left[\left\{1-\left\{1-Q_{m,n}(p)\right\}^{1/b}\right\}^{1/a}\right]\right].$$

3.7 Asymptotes and Shapes

Here we investigate the asymptotic shapes of the proposed family using the method followed in Alizadeh *et al*., (2015a).

Proposition 1: The asymptotes of equations in (6), (7) and (8) as $t \to 0$ are given by

$$f(t;a,b,m,n) \sim \left\{ a\,b\,g(t)G(t)^{a-1}\left[1-\left[1-G(t)^a\right]^b\right]^{m-1}\right\} \Big/ B(m,n) \text{ as } t \to 0$$

$$F(t;a,b,m,n) \sim \left\{1-\left[1-G(t)^a\right]^b\right\}^m \Big/ B(m,n)\,m \text{ as } t \to 0$$

$$h(t;a,b,m,n) \sim \left\{ a\,b\,g(t)G(t)^{a-1}\left[1-\left[1-G(t)^a\right]^b\right]^{m-1}\right\} \Big/ B(m,n) \text{ as } t \to 0$$

Proposition 2: The asymptotes of equations in (6), (7) and (8) as $t \to \infty$ are given by

$$f(t;a,b,m,n) \sim \left\{ a\,b\,g(t)\left[1-G(t)^a\right]^{bn-1}\right\} \Big/ B(m,n) \text{ as } t \to \infty$$

$$1-F(t;a,b,m,n) \sim \left[1-G(t)^a\right]^{bn} \Big/ n\,B(m,n) \text{ as } t \to \infty$$

$$h(t;a,b,m,n) \sim a\,b\,n\,g(t)\left[1-G(t)^a\right]^{-1} \text{ as } t \to \infty$$

The shapes of the density and hazard rate functions can be described analytically. The critical points of the $BKw-G(a,b,m,n)$ density function are the roots of the equation

$$d/dt\left\{\log\left[f(t;a,b,m,n)\right]\right\} = 0$$

$$\Rightarrow \frac{g'(t)}{g(t)} + (a-1)\frac{g(t)}{G(t)} + a(1-bn)\frac{g(t)G(t)^{a-1}}{1-G(t)^a}$$

$$+(m-1)\frac{a\,b\,g(t)G(t)^{a-1}\left[1-G(t)^a\right]^{b-1}}{1-\left[1-G(t)^a\right]^b} = 0 \qquad (18)$$

There may be more than one root to (18). If $t = t_0$ is a root of (18), then it corresponds to a local maximum, a local minimum or a point of inflexion depending on whether

$$\kappa(t_0) < 0,\ \kappa(t_0) > 0 \ \ or \ \ \kappa(t_0) = 0,$$

where

$$\kappa(t) = \left\{d^2/dt^2\right\}\log\left[f(t;a,b,m,n)\right]$$

$$= \left\{g(t)\,g''(t) - \left[g'(t)\right]^2\right\}\Big/ g(t)^2 + (a-1)\left\{G(t)\,g'(t) - g(t)^2\right\}\Big/ G(t)^2$$

$$+ a(1-bn)\left[\frac{g'(t)\,G(t)^{a-1}}{1-G(t)^a} + \frac{(a-1)\,g(t)^2\,G(t)^{a-2}}{1-G(t)^a} + \frac{a\,g(t)^2\,G(t)^{2a-2}}{\left[1-G(t)^a\right]^2}\right]$$

$$+ \frac{(m-1)\,a\,b\left[g'(t)\,G(t)^{a-1}\left[1-G(t)^a\right]^{b-1}\right]}{1-\left[1-G(t)^a\right]^b}$$

$$+ \frac{(m-1)\,a\,b\left[(a-1)\,g(t)^2\,G(t)^{a-2}\left[1-G(t)^a\right]^{b-1}\right]}{1-\left[1-G(t)^a\right]^b}$$

$$- \left\{(m-1)\,ab\left[a(b-1)\,g(t)^2\,G(t)^{2a-2}\left[1-G(t)^a\right]^{b-2}\right]\right\}\Big/\left\{1-\left[1-G(t)^a\right]^b\right\}$$

$$-(m-1)\left[\left\{a\,b\,g(t)\,G(t)^{a-1}\left[1-G(t)^a\right]^{b-1}\right\}\Big/ 1-\left[1-G(t)^a\right]^b\right]^2$$

The critical points of $h(t)$ are the roots of the equation

$$(d/dt)\log\left[h(t;a,b,m,n)\right] = 0$$

$$\Rightarrow \frac{g'(t)}{g(t)} + (a-1)\frac{g(t)}{G(t)} + a(1-bn)\frac{g(t)\,G(t)^{a-1}}{1-G(t)^a}$$

$$+(m-1)\frac{a\,b\,g(t)\,G(t)^{a-1}\left[1-G(t)^a\right]^{b-1}}{1-\left[1-G(t)^a\right]^b}$$

$$+ \frac{a\,b\,g(t)\,G(t)^{a-1}\left[1-G(t)^a\right]^{bn-1}\left[1-[1-G(t)^a]^b\right]^{m-1}}{B(m,n)\left[1-I_{1-[1-G(t)^a]^b}(m,n)\right]} = 0 \qquad (19)$$

There may be more than one root to (19). If $t=t_0$ is a root of (19), then it corresponds to a local maximum, a local minimum or a point of inflexion depending on whether

$\omega(t_0)<0$, $\omega(t_0)>0$ or $\omega(t_0)=0$, where $\omega(t)=\left\{d^2/dt^2\right\}\log\left[h(t;a,b,m,n)\right]$

$$
\begin{aligned}
\omega(t) =&\left\{g(t)\,g''(t)-\left[\,g'(t)\right]^2\right\}\Big/ g(t)^2+(a-1)\,\left\{G(t)\,g'(t)-g(t)^2\right\}\Big/ G(t)^2 \\
&+a\,(1-b\,n)\left[\frac{g'(t)\,G(t)^{a-1}}{1-G(t)^a}+\frac{(a-1)\,g(t)^2\,G(t)^{a-2}}{1-G(t)^a}+\frac{a\,g(t)^2\,G(t)^{2a-2}}{\left[1-G(t)^a\right]^2}\right] \\
&+\frac{(m-1)\,a\,b\left[\,g'(t)\,G(t)^{a-1}\left[1-G(t)^a\right]^{b-1}\right]}{1-\left[1-G(t)^a\right]^b} \\
&+\frac{(m-1)\;a\,b\left[(a-1)\,g(t)^2\,G(t)^{a-2}\left[1-G(t)^a\right]^{b-1}\right]}{1-\left[1-G(t)^a\right]^b} \\
&-\frac{(m-1)\;a\,b\left[\,a\,(b-1)\,g(t)^2\,G(t)^{2a-2}\left[1-G(t)^a\right]^{b-2}\right]}{1-\left[1-G(t)^a\right]^b} \\
&-(m-1)\left[\frac{a\,b\,g(t)\,G(t)^{a-1}\left[1-G(t)^a\right]^{b-1}}{1-\left[1-G(t)^a\right]^b}\right]^2 \\
&+\left\{a\,b\,g'(t)\,G(t)^{a-1}\left[1-G(t)^a\right]^{b\,n-1}\left[1-\left[1-G(t)^a\right]^b\right]^{m-1}\right\}\Big/ \\
&\qquad\qquad\left\{B(m,n)\;\left[1-I_{1-\left[1-G(t)^a\right]^b}\,(m,n)\right]\right\} \\
&+\left\{a\,b\,(a-1)\,g(t)^2\,G(t)^{a-2}\left[1-G(t)^a\right]^{b\,n-1}\left[1-\left[1-G(t)^a\right]^b\right]^{m-1}\right\}\Big/ \\
&\qquad\qquad\left\{B\,(m,n)\left[[1-I_{1-\left[1-G(t)^a\right]^b}\,(m,n)\right]\right\}
\end{aligned}
$$

$$-\left\{a^2\, b\,(bn-1)\, g\left(t\right)^2\, G\left(t\right)^{2a-2}\left[1-G(t)^a\right]^{b\,n-2}\left[1-\left[1-G\left(t\right)^a\right]^b\right]^{m-1}\right\}\Big/$$

$$\left\{B\,(m,n)\left[[1-I_{1-\left[1-G\left(t\right)^a\right]^b}\,(m,n)\right]\right\}$$

$$+\left\{a^2b^2\,(m-1)\, g(t)^2\, G(t)^{2a-2}\left[1-G(t)^a\right]^{b\,(n+1)-2}\times\left[1-\left[1-G(t)^a\right]^b\right]^{m-2}\right\}\Big/$$

$$\left\{B\,(m,n)\left[1-I_{1-[1-G(t)^a]^b}\,(m,n)\right]\right\}$$

$$+\left[\left\{a\, b\, g(t)\, G(t)^{\,a-1}\left[1-G(t)^a\right]^{b\,n-1}\left[1-\left[1-G(t)^{\,a}\right]^b\right]^{m-1}\right\}\Big/\right.$$

$$\left.\left\{B\,(m,n)\left[1-I_{1-\left[1-G(t)^a\right]^b}\,(m,n)\right]\right\}\right]^2$$

4. ESTIMATION

4.1 Maximum Likelihood Estimation (MLE) for $BKw-G$

The model parameters of the $BKw-G(a,b,m,n)$ distribution can be estimated by maximum likelihood. Let $t=\left(t_1,t_2,...,t_n\right)^T$ be a random sample of size n from $BKw-G(a,b,m,n)$ with parameter vector $\theta=(a,b,m,n,\beta^T)^T$, where $\beta=\left(\beta_1,\beta_2,...,\beta_q\right)^T$ corresponds to the parameter vector of the baseline distribution *G*. Then the log-likelihood function for θ is given by

$$\ell=\ell(\theta)\ =r\log(a\,b)+\sum_{i=0}^{r}\log\left[g\,(t_i,\beta)\right]-r\log\left[B(m,n)\right]+(a-1)\sum_{i=0}^{r}\log\left[G\left(t_i,\beta\right)\right]$$
$$+(b\,n-1)\sum_{i=0}^{r}\log\left[1-G\left(t_i,\beta\right)^a\right]+(m-1)\sum_{i=1}^{r}\log\left[1-\left[1-G\,(t_i,\beta)^{\,a}\right]^b\right] \quad (20)$$

This log-likelihood function cannot be solved analytically because of its complex form but it can be maximized numerically by employing global optimization methods available with mathematical software's or by directly solving the nonlinear likelihood function in equation in (20) by differentiating.

By taking the partial derivatives of the log-likelihood function with respect to a,b,m,n and β components of the score vector $U_\theta=(U_a,U_b,U_m,U_n,U_{\beta^T})^T$ can be obtained as follows:

$$U_a = \frac{\partial \ell}{\partial a} = \frac{r}{a} + \sum_{i=0}^{r} \log\left[G(t_i,\beta)\right] + (1-bn)\sum_{i=0}^{r} \frac{G(t_i,\beta)^a \log\left[G(t_i,\beta)\right]}{1-G(t_i,\beta)^a}$$

$$+(m-1)\sum_{i=1}^{r} \frac{b\left[1-G(t_i,\beta)^a\right]^{b-1} G(t_i,\beta)^a \log\left[G(t_i,\beta)\right]}{1-\left[1-G(t_i,\beta)^a\right]^b}$$

$$U_b = \frac{\partial \ell}{\partial b} = \frac{r}{b} + n\sum_{i=0}^{r} \log[1-G(t_i,\beta)^a]$$

$$+(1-m)\sum_{i=0}^{r} \frac{\left[1-G(t_i,\beta)^a\right]^b \log\left[1-G(t_i,\beta)^a\right]}{1-\left[1-G(t_i,\beta)^a\right]^b}$$

$$U_m = \partial \ell / \partial m = -r\psi(m) + r\psi(m+n) + \sum_{i=1}^{r} \log\left[1-\left[1-G\left(t_i,\beta\right)^a\right]^b\right]$$

$$U_n = \partial \ell / \partial n = -r\psi(n) + r\psi(m+n) + b\sum_{i=0}^{r} \log\left[1-G(t_i,\beta)^a\right] \text{ and}$$

$$U_\beta = \frac{\partial \ell}{\partial \beta} = \sum_{i=0}^{r} \frac{g^{(\beta)}(t_i,\beta)}{g(t_i,\beta)} + (a-1)\sum_{i=0}^{r} \frac{G^{(\beta)}(t_i,\beta)}{G(t_i,\beta)}$$

$$+(1-bn)\sum_{i=0}^{r} \frac{a\,G(t_i,\beta)^{a-1} G^{(\beta)}(t_i,\beta)}{1-G(t_i,\beta)^a}$$

$$+(m-1)\sum_{i=0}^{r} \frac{b\left[1-G(t_i,\beta)^a\right]^{b-1} a\,G(t_i,\beta)^{a-1} G^{(\beta)}(t_i,\beta)}{1-\left[1-G(t_i,\beta)^a\right]^b},$$

where $\psi(.)$ is the digamma function.

4.2 Asymptotic Standard Error for the MLE

The asymptotic variance-covariance matrix of the MLEs of parameters can obtained by inverting the Fisher information matrix $\mathrm{I}(\theta)$ which can be derived using the second partial derivatives of the log-likelihood function with respect to each parameter. The $i\,j^{th}$ elements of $\mathrm{I}_n(\theta)$ are given by

$$\mathrm{I}_{ij} = -E\left[\partial^2 l(\theta) / \partial\theta_i\, \partial\theta_j\right], \quad i, j = 1, 2, \cdots, 3+q.$$

The exact evaluation of the above expectations may be cumbersome. In practice one can estimate $\mathrm{I}_n(\theta)$ by the observed Fisher's information matrix $\hat{\mathrm{I}}_n(\hat{\theta}) = \left(\hat{\mathrm{I}}_{ij}\right)$ defined as:

$$\hat{\mathrm{I}}_{ij} \approx \left(-\partial^2 l(\theta) / \partial\theta_i\, \partial\theta_j\right)_{\theta=\hat{\theta}}, \quad i,\ j = 1, 2, \cdots, 3+q.$$

Using the general theory of MLEs under some regularity conditions on the parameters as $n \to \infty$ the asymptotic distribution of $\sqrt{n}\,(\hat{\theta}-\theta)$ is $N_k(0,V_n)$ where $V_n = (v_{jj}) = \mathrm{I}_n^{-1}(\theta)$. The asymptotic behaviour remains valid if V_n is replaced by $\hat{V}_n = \hat{\mathrm{I}}^{-1}(\hat{\theta})$. This result can be used to provide large sample standard errors for the model parameters. Thus an approximate standard error for the MLE of j^{th} parameter θ_j is given by $\sqrt{\hat{v}_{jj}}$.

4.3 Estimation by Method of Moments

Here an alternative method of estimation of the parameters is discussed. Since the moments are not in closed form, the estimation by the usual method of moments may not be tractable. Therefore following (Barreto-Souza *et al*., 2013) we get

$$E\left\{\left[1-\left[1-G(t)^a\right]^b\right]\right\}$$

$$=\int_{-\infty}^{\infty}\left\{\left[1-\left[1-G(t)^a\right]^b\right]\right\}\frac{a\,b\,g(t)G(t)^{a-1}\left[1-G(t)^a\right]^{bn-1}}{B(m,n)\left[1-\left[1-G(t)^a\right]^b\right]^{1-m}}dt$$

$$=\frac{1}{B(m,n)}\int_{-\infty}^{\infty}\frac{a\,b\,g(t)G(t)^{a-1}\left[1-G(t)^a\right]^{bn-1}}{\left[1-\left[1-G(t)^a\right]^b\right]^{-m}}dt$$

$$=\frac{1}{B(m,n)}\int_0^1 u^m\,(1-u)^{n-1}\,du$$

$$=\frac{1}{B(m,n)}\int_0^1 u^{(m+1)-1}\,(1-u)^{n-1}\,du=\frac{B(m+1,n)}{B(m,n)}$$

with $u=1-\left[1-G(t)^a\right]^b$ and $du=a\,b\,g(t)G(t)^{a-1}\left[1-G(t)^a\right]^{b-1}dt$.

$$E\left\{\left[1-\left[1-G(t)^a\right]^b\right]^v\right\}$$

$$=\int_{-\infty}^{\infty}\left\{\left[1-\left[1-G(t)^a\right]^b\right]^v\right\}\frac{a\,b\,g(t)G(t)^{a-1}\left[1-G(t)^a\right]^{bn-1}}{B(m,n)\left[1-\left[1-G(t)^a\right]^b\right]^{1-m}}dt$$

$$=\frac{1}{B(m,n)}\int_{-\infty}^{\infty}\frac{a\,b\,g(t)G(t)^{a-1}\left[1-G(t)^{a}\right]^{bn-1}}{\left[1-\left[1-G(t)^{a}\right]^{b}\right]^{1-m-v}}dt$$

$$=\frac{1}{B(m,n)}\int_{0}^{1}\frac{(1-u)^{n-1}}{u^{1-m-v}}du=\frac{1}{B(m,n)}\int_{0}^{1}u^{v+m-1}(1-u)^{n-1}du=\frac{B(v+m,n)}{B(m,n)}$$

with $u=1-\left[1-G(t)^{a}\right]^{b}$ and $du=a\,b\,g(t)G(t)^{a-1}\left[1-G(t)^{a}\right]^{b-1}dt$.

Therefore we have,

$$E\left[\left[1-\left[1-G(t)^{a}\right]^{b}\right]^{v}\right]=B(m+v,n)/B(m,n)\ ,\ \text{for } m,n>0,\ v=1,2,3,... \quad (21)$$

For a random sample $t_1,t_2,..,t_r$ from a population with $BKw-G(a,b,m,n)$ distribution, the parameters may be estimated by numerically solving the equations

$$\frac{1}{r}\sum_{i=1}^{r}\left[\left[1-\left[1-G(t_i)^{a}\right]^{b}\right]^{v}\right]=\frac{B(m+v,n)}{B(m,n)}\ ,\ \text{for } m,n>0,\ v=1,2,3,\cdots.$$

5. MODELLING APPLICATIONS

In this subsection, we consider two real data sets to establish the advantage of using the $BKw-G(a,b,m,n)$ distributions over some of the distributions nested with in it by choosing the Weibull distribution as the base line *G*. Namely we consider here B-*W*, *Kw*-*W* and *GH*-*W* for comparative fitting against $BKw-W$. We have used MLE to estimates the parameters and provided standard errors of the estimators. Models are compared employing the Akaike Information Criterion (AIC). $AIC=2k-2l$, where k is the number of parameters in the distribution and l is the maximized value of the log-likelihood function under the considered model. The Kolmogorov-Smirnov (KS) test is also used to compare the fitted models.

Example I

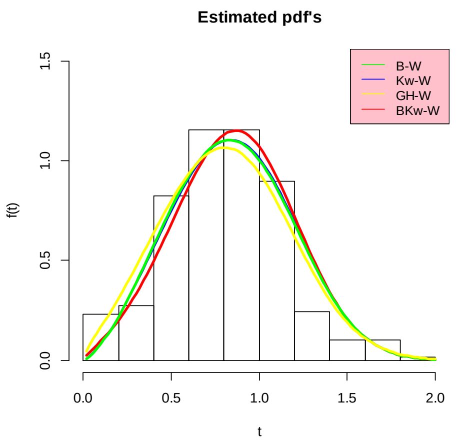
The data set I is about 346 nicotine measurements made from several brands of cigarettes in 1998. This data was originally collected by the Federal Trade Commission an independent agency of the US government, whose main mission is the promotion of consumer protection [http://www.ftc.gov/reports/tobacco or http://pw1.netcom.com/rdavis2/smoke.html].

Example II

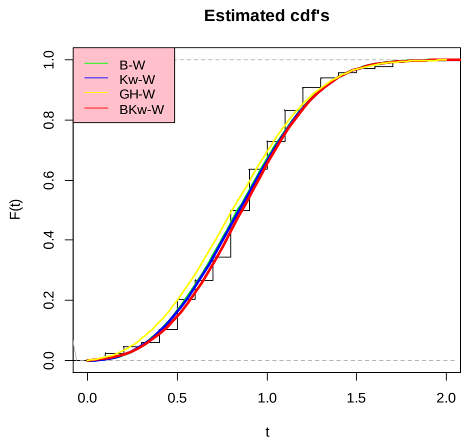
The data set II for the second application is a subset of the data reported by Bekker *et al*. (2000) which corresponds to the survival times (in years) of a group of patients given chemotherapy treatment alone. This data set consists of survival times (in years) for 46 patients.

Table 1
MLEs, Standard Errors (in parentheses) AIC and KS (*p*-value) values for the data set I

Models	$\hat{a}$	$\hat{b}$	$\hat{m}$	$\hat{n}$	$\hat{\lambda}$	$\hat{\beta}$	$(-l_{\max})$	**AIC**	**KS test (*p*-value)**
$B-W$ (m,n,λ,β)	---	---	0.777 (0.155)	2.027 (1.996)	0.431 (0.461)	3.158 (0.403)	113.08	234.16	0.26 (0.09)
$Kw-W$ (a,b,λ,β)	0.792 (0.149)	0.430 (0.188)	---	---	2.326 (1.060)	3.001 (0.259)	112.58	233.16	0.27 (0.08)
$GH-W$ (a,b,m,λ,β)	0.240 (0.012)	0.525 (0.102)	2.750 (1.135)	---	2.588 (0.646)	2.486 (0.211)	112.19	234.38	0.28 (0.09)
$BKw-W$ (a,b,m,n,λ,β)	0.207 (0.282)	0.776 (0.191)	2.647 (2.816)	0.298 (0.090)	4.636 (0.534)	2.912 (0.201)	**110.06**	**232.12**	0.25 (**0.10**)



(i)

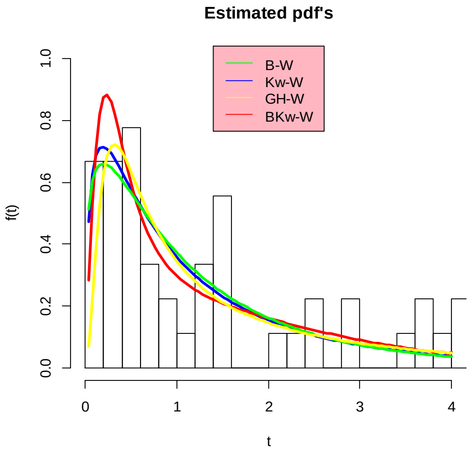


(ii)

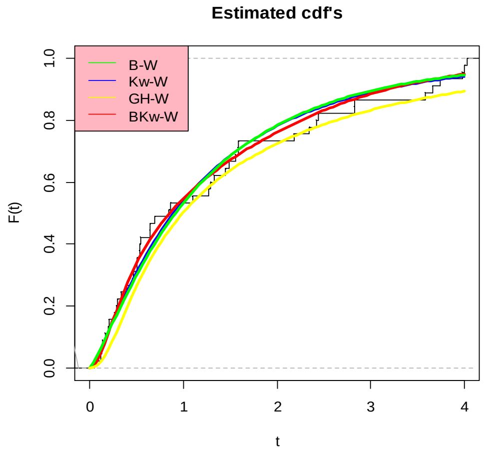
Fig. 3: Plots of the (i) Observed histogram and estimated pdf's and (ii) Estimated cdf's for the $B-W$, $Kw-W$, $GH-W$ and $BKw-W$ for the data set I

Table 2
MLEs, Standard Errors (in parentheses) AIC and KS (*p*-value) values for the data set II

Models	$\hat{a}$	$\hat{b}$	$\hat{m}$	$\hat{n}$	$\hat{\lambda}$	$\hat{\beta}$	$(-l_{\max})$	AIC	KS test (*p*-value)
$B-W$ (m, n, λ, β)	---	---	2.024 (2.856)	3.418 (6.538)	0.467 (0.807)	0.695 (0.555)	58.07	124.14	0.09 (0.73)
$Kw-W$ (a, b, λ, β)	2.160 (1.584)	0.208 (0204)	---	---	4.521 (3.900)	0.836 (0.196)	57.72	123.44	0.08 (0.86)
$GH-W$ $(a, b, m, \lambda, \beta)$	7.717 (1.428)	0.132 (0.004)	0.420 (0.201)	---	4.108 (3.773)	0.993 (0.413)	57.77	125.54	0.09 (0.74)
$BKw-W$ $(a, b, m, n, \lambda, \beta)$	6.525 (0.079)	0.317 (0.323)	0.223 (0.054)	0.224 (0.189)	5.183 (0.004)	1.388 (0.002)	55.46	122.92	0.07 (0.96)



(i)



(ii)

Fig. 4: Plots of the (i) Observed histogram and estimated pdf's and (ii) Estimated cdf's for the $B-W$**,** $Kw-W$**,** $GH-W$ **and** $BKw-W$ **for the data set II.**

In Table 1 and 2, the estimates, MLEs, standard errors (SEs) (in parentheses) of the parameters for all the fitted models along with their AIC and KS with *p*-values for the examples I and II are presented. It is evident that in all the examples the $BKw-W$ distribution turned out to be the best model than of its sub models with lowest values of the AIC and highest *p*-value of KS statistic. These findings are further validated by providing plots of fitted densities with histogram of the observed data in Figures 3(i) and 4(i) and fitted cdfs with observed cdf of data in Figures 3(ii) and 4(ii) for the examples I and II respectively. These plots also indicate that the proposed distributions provide closest fit to the data sets considered here.

6. CONCLUSION

A new beta extended Kumaraswamy generalized family of distributions which includes some well-known distribution is introduced and some of its important properties are studied. The maximum likelihood and moment method for estimating the parameters are also discussed. Comparative data modeling application of the proposed model with some of its sub-models is carried out considering two data sets reveal its superiority.

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REFERENCES

1. Alizadeh, M., Tahir, M.H., Cordeiro, G.M., Zubair, M. and Hamedani, G.G. (2015a). The Kumaraswamy Marshall-Olkin family of distributions. *Journal of the Egyptian Mathematical Society,* 23(3), 546-557.
2. Alizadeh, M., Tahir, M.H., Cordeiro, G.M., Zubair, M. and Hamedani, G.G. (2015b). The beta Marshall-Olkin family of distributions. *Journal of Statistical Distributions and Applications,* 2(4), 1-18. DOI: 10.1186/s40488-015-0027-7.
3. Barreto-Souza, W., Santos, A. and Cordeiro, G.M. (2010). The beta generalized exponential distribution. *J. Stat. Comput. Simul.*, 80(2), 159-17.
4. Barreto-Souza, W., Lemonte, A.J. and Cordeiro, G.M. (2013). General results for the Marshall and Olkin's family of distributions. *Anais da Academia Brasileira de Ciências*, 85(1), 3-21.
5. Bekker, A., Roux, J. and Mostert, P. (2000). A generalization of the compound Rayleigh distribution: Using Bayesian methods on cancer survival times. *Commun. Statist. Theor. Meth.*, 29(7), 1419-1433.
6. Bornemann, F. and Weisstein, E.W. (2004). *Power series*, From MathWorld-A Wolfram Web Resource. http://mathworld.wolfram.com/PowerSeries.html.
7. Chakraborty, S. and Handique, L. (2017). The generalized Marshall-Olkin-Kumaraswamy-*G* family of distributions. *Journal of Data Science,* forthcoming.
8. Cordeiro, G.M. and de Castro, M. (2011). A new family of generalized distributions. *J. Stat. Comput. Simul.*, 81(7), 883-893.
9. Cordeiro, G.M., Cristino, C.T., Hashimoto, E.M. and Ortega, E.M.M. (2013). The beta generalized Rayleigh distribution with applications to lifetime data. *Stat. Pap.*, 54(1), 133-161.
10. Cordeiro, G.M., Silva, G.O., Pescim, R.R. and Ortega, E.M.M. (2014a). General properties for the beta extended half-normal distribution. *J. Stat. Comput. Simul.*, 84(4), 881-901.
11. Cordeiro, G.M., Silva, G.O., Pescim, R.R. and Ortega, E.M.M. (2014b). The Kumaraswamy modified Weibull distribution: theory and application. *J. Stat. Comput. Simul.*, 84(7), 881-901.
12. Correa, M.A., Nogueira, D.A. and Ferreira, E.B. (2012). Kumaraswamy normal and Azzalini's skew normal modelling asymmetry. *Sigmae Alfenas*, 1(1), 65-83.
13. David, O. and Richard, M. (1968). The incomplete beta function and its ratio to the complete beta function. *Mathematics of Computation*, 22(101), 159-162.
14. Domma, F. and Condino, F. (2016). Use of the Beta-Dagum and Beta-Singh-Maddala distributions for modeling hydrologic data. *Stoch Environ Res Risk Assess*, published on line 20 February 2016. DOI 10.1007/s00477-016-1226-6.

15. Elbatal, L. (2013). Kumaraswamy linear exponential distribution. *Pioneer J. Theor. Appl. Statist*., 5(2), 59-73.
16. Eugene, N., Lee, C. and Famoye, F. (2002). Beta-normal distribution and its applications. *Commun. Statist. Theor. Meth.*, 31(4), 497-512.
17. Greenwood, J.A., Landwehr, J.M., Matalas, N.C. and Wallis, J.R. (1979). Probability weighted moments: definition and relation to parameters of several distributions expressible in inverse form. *Water Resour. Res.*, 15(5), 1049-1054.
18. Handique, L., Chakraborty, S. and Hamedani, G.G. (2017). The Marshall-Olkin-Kumaraswamy-*G* family of distributions. *Journal of Statistical Theory and Applications*, forthcoming.
19. Jones, M.C. (2004). Families of distributions arising from the distributions of order statistics. *Test*, 13(1), 1-43.
20. Kenney, J.F. and Keeping, E.S. (1962). *Mathematics of Statistics*, Part 1. Van Nostrand, New Jersey, 3rd edition.
21. Krishna, E., Jose, K.K. and Ristić, M. (2013). Applications of Marshall-Olkin Fréchet distribution. *Comm. Stat. Simulat. Comput*, 42 (1), 76-89.
22. Lemonte, A.J. (2014). The beta log-logistic distribution. *Braz. J. Probab. Statist.*, 28(3), 313-332.
23. Lomax, K.S. (1954). Business Failures; Another example of the analysis of failure data. *Journal of the American Statistical Association*, 49, 847-852.
24. Mead, M.E., Afify, A.Z., Hamedani, G.G. and Ghosh, I. (2016). The Beta exponential Frechet distribution with applications. *Austrian Journal of Statistics*, 46, 41-63 doi: 10.17713/ajs.v46i1.144.
25. Moors, J.J.A. (1988). A quantile alternative for kurtosis. *The Statistician*, 37(1), 25-32.
26. Mohamed E., Amal S.H. and Mohamed R. (2016). Garhy-generated family of distributions with application. *Mathematical Theory and Modeling*, 6(2), 1-15.
27. Nadarajah, S. and Kotz, S. (2004). The beta Gumbel distribution, *Mathematical Problems in Engineering*, 10(4), 323-332.
28. Nadarajah, S. and Eljabri, S. (2013). The Kumaraswamy generalized Pareto distribution. *J. Data Sci.*, 11, 739-766.
29. Ramos, M.W.A., Marinho, P.R.D., da Silva, R.V. and Cordeiro, G.M. (2013). The exponentiated Lomax Poisson distribution with an application to lifetime data. *Advances and Applications in Statistics*, 34, 107-135.
30. Song, K.S. (2001). Rényi information, log likelihood and an intrinsic distribution measure. *J. Stat. Plan. Infer.*, 93(1-2), 51-69.
31. Singla N., Jain, K. and Sharma, S.K. (2012). The beta generalized Weibull distribution: Properties and applications. *Reliab. Eng. Syst Safety*, 102, 5-15.
32. Tahir, M.H. and Nadarajah, S. (2015). Parameter induction in continuous univariate distributions: Well-established *G* families. *Anais da Academia Brasileira de Ciências* (*Annals of the Brazilian Academy of Sciences*), 87(2), 539-568.
33. Weibull, W. (1951). A statistical distribution function of wide applicability. *J. Appl. Mech.-Trans. ASME*, 18(3), 293-297.
34. Weisstein, Eric W. "Incomplete Beta Function" From *MathWorld* – A Wolfram Web Resource. http://mathworld.Wolfram.com/IncompleteBetaFunction.html.