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PERFORMANCE OF SOME BIASED ESTIMATORS IN MISSPECIFIED MODEL UNDER MAHALANOBIS LOSS FUNCTION

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ABSTRACT

In econometric research, misspecification due to omission of relevant variables is a very common phenomena, which leads to biased and inconsistent estimation of parameters. It is needless to mention that the explanatory variables in the misspecified model may be multicollinear and have adverse effects on the least squares estimator. To combat the problem of multicollinearity, several biased estimators have been introduced and widely analyzed under mean squared error criterion when there is no misspecification. Furthermore, Mahalanobis loss function, which is a weighted quadratic loss function has gained quite an attention as a comparison criterion in recent years. Although, not much literature is available on the comparison of the estimators in the presence of multicollinearity under the Mahalanobis loss function when there is no misspecification and is almost negligible in case of misspecification due to omission of relevant variables. This article evaluates the performance of some biased estimators in the presence of multicollinearity under the Mahalanobis loss function using average loss criterion in the misspecified model. Necessary and sufficient conditions for the dominance of one estimator over the others have been derived. Further, the findings are supported by a Monte Carlo simulation and numerical example.

KEYWORDS

Omission of relevant variables, multicollinearity, biased estimators, Mahalanobis loss function, average loss criterion.

1. INTRODUCTION

Multicollinearity among the regressors is a common problem in regression analysis, its one of the major consequences is that it produces larger sampling variance of the ordinary least squares (OLS) estimator and makes it unsteady. A solution to this problem is found in making use of more consistent biased estimators such as, the ordinary ridge regression (ORR) estimator, the principal component regression (PCR) estimator, the $r-k$ class estimator, the Liu estimator, the $r-d$ class estimator and the two-parameter class estimator, introduced by Hoerl and Kennard (1970), Massy (1965), Baye and Parker (1984), Liu (1993), Kaçiranlar and Sakallioğlu (2001) and Özkale and Kaçiranlar (2007), respectively. Recently, Özkale (2012) merged the approaches of the two-parameter class estimator and the PCR estimator and put forward the $r-(k,d)$ class estimator.

In applied work misspecification may occur due to one reason or another. There are mainly two reasons for misspecification, inclusion of superfluous variables or omission of relevant variables. The misspecification in linear regression model has many undesirable effects on the properties of the estimators (see Johnston DiNardo (1997)). Several authors have studied the effect of misspecification on estimation of regression coefficients, for instance, Kadiyala (1986); Sarkar (1989); Wijekoon and Trenkler (1989), Dube et al. (1991), Groß et al. (1998), Dube (1999), Hubert and Wijekoon (2004), Uemukai (2010), Jianwen and Hu (2012), Şiray (2015) and Wu (2016) among others.

The criterion of average loss under squared error loss function, famously known as mean squared error (MSE) criterion is one of the most popular criterion to evaluate the performance of the estimators. Baye and Parker (1984) revealed the MSE superiority of the $r-k$ class estimator over the PCR estimator. Nomura and Ohkubo (1985) derived the dominance conditions of the $r-k$ class estimator over the OLS and ORR estimators under MSE criterion and Sarkar (1996) obtained conditions for the superiority of the $r-k$ class estimator over the OLS, PCR and the ORR estimators by the MSE matrix criterion. Özkale and Kaçiranlar (2007) have also compared the $r-d$ class estimator with the PCR estimator, $r-k$ class estimator and Liu estimator by the MSE matrix criterion. Sarkar (1989) compared the performances of the $r-k$ class estimator with the OLS, ORR and PCR in misspecified linear regression model. Şiray (2015) made a comparison between the performance of the $r-d$ class estimator with others when some relevant variables get omitted. Jianwen and Hu (2012) compared the stochastic restricted Liu estimator with others when there is inclusion of superfluous variables.

Furthermore, Peddada et al. (1989) pointed out that the Malahanobis loss function (Mahalanobis (1936)), can be used to compare two estimators if at least one of them is biased, and compared the generalized ridge estimator and the OLS estimator in a linear regression model. Şiray et al. (2012) examined the performance of the $r-k$ class estimator over the OLS, ORR and PCR estimators in terms of the Mahalanobis loss function using average loss criterion, further referred as average Mahalanobis loss criterion. Sarkar and Chandra (2015) obtained conditions of superiority of the $r-(k,d)$ class estimator over the OLS, PCR and the two-parameter class estimators by means of the average Mahalanobis loss criterion.

Sarkar (1989) discussed the effect of omission of relevant variables on the dominance conditions of the $r-k$ class estimator over the other competing estimators. Şiray (2015) studied the performance of the $r-d$ class estimator and others in case of omission of relevant variables. Wu (2016) compared the $r-k$ class estimator with the OLS, ORR and the PCR estimators under the average Mahalanobis loss criterion when there is misspecification due to omission of relevant variables. Chandra and Tyagi (2017) studied consequences of omission of relevant variables on the superiority conditions of the $r-(k,d)$ class estimator over the others in mean squared error sense.

This paper discusses the effect of misspecification due to omission of relevant variables on the performance of the $r-(k,d)$ class estimator with other competing estimators such as the OLS, ORR, PCR, Liu estimator, $r-d$ class estimator, $r-k$ class estimator and the two-parameter class estimator under the Mahalanobis loss function using average loss criterion. The rest of the paper is designed in the following manner:

Section 2 discusses the model and the estimators under consideration, comparison of the estimators under the Mahalanobis loss has been done in Section 3 and the conditions of dominance of several estimators over the other estimators have also been derived. The estimators are compared by Monte Carlo simulation study in Section 4 and Section 5 illustrates theoretical findings with a numerical example. The paper is then concluded in Section 6.

2. MODEL AND ESTIMATORS

Let us consider the classical linear regression model which is assumed to be a true model

$$y = X\beta + Z\gamma + \varepsilon, \tag{2.1}$$

where y is an $n \times 1$ vector of dependent variable, X is an $n \times p$ matrix of explanatory variables of rank p and Z is a matrix of some other relevant explanatory variables of rank q, $p + q < n$, β and γ are the corresponding $p \times 1$ and $q \times 1$ vectors of parameters associated with X and Z, respectively. ε is an $n \times 1$ vector of disturbance term and assume that $\varepsilon \sim N(0, \sigma^2 I_n)$.

In regression analysis, omission of relevant variables is a very common problem. It increases the bias in the estimator and also affects the validity of conventional inference procedures. Now, suppose that investigator has unknowingly excluded variables of Z matrix, thus the misspecified model is given by

$$y = X\beta + u, \tag{2.2}$$

where, $u = Z\gamma + \varepsilon$. Misspecification occurs when investigator assumes the disturbance vector u to be normally distributed with mean vector 0 and variance $\sigma^2 I_n$.

In order to define the estimators, let us assume that $T = (t_1, t_2, \dots, t_p)$ is an orthogonal matrix of order $p \times p$ with $T'X'XT = \Lambda = diag\ (\lambda_1, \lambda_2, \dots, \lambda_p)$ is a $p \times p$ diagonal matrix of eigen values of $X'X$ matrix such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$. Now, let $T_r = (t_1, t_2, \dots, t_r)$ be $p \times r$ orthogonal matrix after deleting last $p - r$ columns from T matrix, where $r \leq p$. Thus, $T_r'X'XT_r = \Lambda_r$ where $\Lambda_r = diag(\lambda_1, \lambda_2, \dots, \lambda_r)$ and $T_{p-r}'X'XT_{p-r} = \Lambda_{p-r}$,$\Lambda_{p-r} = diag\ (\lambda_{r+1}, \lambda_{r+2}, \dots, \lambda_p)$. Also, $T'T = T_r'T_r + T_{p-r}'T_{p-r}$.

Özkale (2012) introduced an estimator known as $r - (k, d)$ class estimator to deal with multicollinearity. This estimator for the misspecified model in (2.2) can be written as

$$\hat{\beta}_r^*\ (k, d) = T_r(T_r'X'XT_r + kI)^{-1}(T_r'X'y + kdT_r'\hat{\beta}_r^*)\ k \geq 0, 0 < d < 1. \tag{2.3}$$

This is a general class of estimators which includes the OLS, ORR, Liu, PCR, $r - k$ class estimator, $r -$ d class estimator and the two-parameter class estimator as its special cases.

(i) $\hat{\beta}_p^*\ (0{,}0) = \hat{\beta}^* = (X'X)^{-1}X'y$, is the OLS estimator,
(ii) $\hat{\beta}_p^*\ (k, 0) = \hat{\beta}^*\ (k) = (X'X + kI)^{-1}X'y$, is the ORR estimator,
(iii) $\hat{\beta}_r^*\ (0{,}0) = \hat{\beta}_r^* = T_r\ (T_r'X'XT_r)^{-1}T_r'X'y$, is the PCR estimator,

(iv) $\hat{\beta}_p^*\,(1,d) = \hat{\beta}^*\,(d) = (X'X + I)^{-1}\,(X'y + d\hat{\beta}^*)$, is the Liu estimator.

(v) $\hat{\beta}_r^*\,(k,0) = \hat{\beta}_r^*\,(k) = T_r\,(T_r{}'X'XT_r + kI)^{-1}T_r{}'X'y$, is the $r-k$ class estimator,

(vi) $\hat{\beta}_r^*\,(1,d) = \hat{\beta}_r^*\,(d) = T_r\,(T_r{}'X'XT_r + I)^{-1}\,(T_r{}'X'y + dT_r{}'\hat{\beta}_r^*)$, is the $r-d$ class estimator.

(vii) $\hat{\beta}_p^*\,(k,d) = \hat{\beta}^*\,(k,d) = (X'X + kI)^{-1}\,(X'y + kd\hat{\beta}^*)$, is the two-parameter class estimator.

After some straightforward calculations, expectation and covariance of the $r-\ (k,d)$ class estimator in misspecified model can be obtained as

$$E\,(\hat{\beta}_r^*(k,d)) = T_rS_r(k)^{-1}S_r\,(kd)T_r{}'\beta + T_rS_r\,(k)^{-1}\Lambda_r^{-1}S_r\,(kd)T_r{}'X'Z\gamma.$$

Further,

$$Cov\,(\hat{\beta}_r^*(k,d)) = \sigma^2T_rS_r\,(k)^{-2}\Lambda_r^{-1}S_r(kd)^2T_r{}' \qquad (2.4)$$

where $S_r\,(k) = (\Lambda_r + kI)$ and $S_r\,(kd) = (\Lambda_r + kdI)$.

3. AVERAGE LOSS COMPARISONS UNDER MAHALANOBIS LOSS

For an estimator $\tilde{\beta}$ of β Mahalanobis loss function is defined as

$$L\,(\tilde{\beta}) = \ (\tilde{\beta} - \beta)'\,(Cov\,(\tilde{\beta}))^{-1}\,(\tilde{\beta} - \beta).$$

The above form of the loss function can not be applied for the estimators such as the PCR, $r-k$ class estimator, $r-d$ class estimator and the $r-\ (k,d)$ class estimator as the covariance matrix for these estimators is singular. So, a more general form of the loss function can be given as

$$L\,(\tilde{\beta}) = (\tilde{\beta} - \beta)'\,(Cov\,(\tilde{\beta}))^{+}\,(\tilde{\beta} - \beta),$$

where '+' denotes Moore-Penrose inverse of the matrix.

Therefore, Mahalanobis loss function for the $r-(k,d)$ class estimator in misspecified model can be written as

$$L\,(\hat{\beta}_r^*\,(k,d)) = (\hat{\beta}_r^*\,(k,d) - \beta)'\,(Cov\,(\hat{\beta}_r^*\,(k,d)))^{+}(\hat{\beta}_r^*\,(k,d) - \beta). \qquad (3.1)$$

The Moore-Penrose inverse of covariance matrix $Cov\,(\hat{\beta}_r^*\,(k,d))$ in (2.4) is given as

$$\left(Cov\left(\hat{\beta}_r^*(k,d)\right)\right)^{+} = \sigma^{-2}T_rS_r\,(k)^2\Lambda_r\,S_r\,(kd)^{-2}T_r'.$$

On substituting the expressions for $\hat{\beta}_r^*\,(k,d)$ from (2.3) and $\left(Cov\left(\hat{\beta}_r^*(k,d)\right)\right)^{+}$ in (3.1), and on further simplification, we get

$$L\,(\hat{\beta}_r^*\,(k,d)) = \frac{1}{\sigma^2}\,(y'XT_r\Lambda_r^{-1}T_r{}'X'y - 2\beta'T_rS_r\,(k)\Lambda_rS_r\,(kd)^{-1}\Lambda_r^{-1}T_r{}'X'y + \beta'T_rS_r\,(k)^2\Lambda_rS_r\,(kd)^{-2}T_r{}'\beta).$$

Further, risk under the Mahalanobis loss function or average Mahalanobis loss can be obtained as follows

$$R\ (\hat{\beta}_r^*(k,d)) = E(L(\hat{\beta}_r^*\ (k,d)))$$
$$= \frac{1}{\sigma^2} E\ (y'XT_r\Lambda_r^{-1}T_r{}'X'y - 2\beta' T_r S_r\ (k)\Lambda_r S_r\ (kd)^{-1}\Lambda_r^{-1}T_r{}'X'y$$
$$+\beta' T_r S_r\ (k)^2 \Lambda_r S_r\ (kd)^{-2} T_r{}'\beta)$$
$$= r + \sigma^{-2}k^2\ (d-1)^2\beta' T_r\Lambda_r S_r\ (kd)^{-2}T_r{}'\beta$$
$$-2k(1-d)\beta' T_r S_r (kd)^{-1} T_r{}'\delta + \sigma^{-2}\delta' T_r\Lambda_r^{-1}T_r{}'\delta. \tag{3.2}$$

On further simplification,we get

$$R\left(\hat{\beta}_r^*(k,d)\right) = r + \sigma^{-2}$$
$$\left[k^2(1-d)^2 \sum_{i=1}^r \frac{\lambda_i\alpha_i^2}{(\lambda_i+kd)^2} - 2k(1-d)\sum_{i=1}^r \frac{\alpha_i\eta_i}{(\lambda_i+kd)} + \sum_{i=1}^r \frac{\eta_i^2}{\lambda_i}\right], \tag{3.3}$$

where, $\delta = X'Z\gamma$, $\alpha_i = t_i{}'\beta$ and $\eta_i = t_i{}'\delta$. From (3.3) it is easy to notice that the last two terms are due to misspecification of omission of Z variables. The risk expressions for the other estimators under the Mahalanobis loss function for the misspecified model can be easily obtained by substituting suitable values of r, k and d in (3.3).

3.1 Comparison of $\hat{\boldsymbol{\beta}}_r^*\ (\boldsymbol{k},\boldsymbol{d})$ and $\hat{\boldsymbol{\beta}}^*$

The risk expression for the OLS estimator can be obtained by putting $r = p, k = 0$ and $d = 0$ in (3.3) and is given by

$$R(\hat{\beta}^*) = p + \sigma^{-2}\sum_{i=1}^p \frac{\eta_i^2}{\lambda_i}. \tag{3.4}$$

Using (3.3) and (3.4), we have

$$R\ (\hat{\beta}^*) - R\ (\hat{\beta}_r^*\ (k,d)) = p - r - \sigma^{-2}$$
$$\left[\sum_{i=1}^r \frac{\lambda_i\alpha_i^2}{(\lambda_i+kd)^2} - 2k\ (1-d)\sum_{i=1}^r \frac{\alpha_i\eta_i}{(\lambda_i+kd)} + \sum_{i=1}^r \frac{\eta_i^2}{\lambda_i}\right]. \tag{3.5}$$

Since $p \geq r$ and η_i^2/λ_i is positive for all $i = r+1, r+2, \ldots, p$, a sufficient condition for $R\ (\hat{\beta}) - R(\hat{\beta}_r\ (k,d))$ to be positive when $\alpha_i\eta_i > 0$ for $i = 1,2,\ldots,r$, is that

$$-k^2\ (1-d)^2 \frac{\lambda_i\alpha_i^2}{(\lambda_i+kd)^2} + 2k\ (1-d)\frac{\alpha_i\eta_i}{\lambda_i+kd} \geq 0.$$

which is equivalent to $k\ (1-d) \leq 2\ (\lambda_i + kd)\alpha_i\eta_i/\lambda_i\alpha_i^2$ for all $i = 1,2 \ldots, r$.

However, when $\alpha_i\eta_i < 0$, $i = 1,2,\ldots,r$, a necessary and sufficient condition for $R(\hat{\beta}) - R(\hat{\beta}_r\ (k,d))$ to be positive, is that the expression in (3.5) should be positive. Hence the conditions can be given in the form of the following theorem.

Theorem 3.1

The $r-(k,d)$ class estimator is superior to the OLS estimator in misspecified model in terms of average Mahalanobis loss

(i) *when $\alpha_i\eta_i > 0$ for all $i = 1,2,\ldots,r$, $r-(k,d)$ estimator dominates the OLS estimator if*

$$k(1-d) < \frac{2(\lambda_i+kd)\alpha_i\eta_i}{\lambda_i\alpha_i^2}, for\ all i = 1,2,\ldots,r. \tag{3.6}$$

(ii) *When* $\alpha_i\eta_i < 0$ *for all* $i = 1,2, \dots, r$, $R\ (\hat{\beta}^*) - R\ (\hat{\beta}_r^*\ (k, d)) \geq 0$ *if*

$$k(1-d) \leq \frac{\sigma^2\ (p-r) + \sum_{i=1}^{r} \frac{\eta_i^2}{\lambda_i}}{\sum_{i=1}^{r} \frac{k\ (1-d)\lambda_i\alpha_i^2}{(\lambda_i+kd)^2} - 2\sum_{i=1}^{r} \frac{\alpha_i\eta_i}{(\lambda_i+kd)}}. \quad (3.7)$$

Remark 3.1.1

For $d = 0$ the condition (3.6) reduces to the superiority conditions of the $r - k$ class estimator over the OLS estimator in misspecified model obtained by Wu (2016).

Remark 3.1.2

Condition (3.7) gives the comparison between the $r - (k, d)$ class estimator and the OLS estimator in case of no misspecification ($\eta_i = 0, i = 1,2, \dots, r$) obtained by Sarkar and Chandra (2015).

3.2 Comparison of $\hat{\boldsymbol{\beta}}_r^*\ (\boldsymbol{k}, \boldsymbol{d})$ and $\hat{\boldsymbol{\beta}}_r^*$

The expression of risk of the PCR estimator, $\hat{\beta}_r^*$, in misspecified model under the Mahalanobis loss function can be obtained by substituting $k = 0$ and $d = 0$ in (3.3). Thus, we have

$$R\ (\hat{\beta}_r^*) = r + \sigma^{-2} \sum_{i=1}^{r} \frac{\eta_i^2}{\lambda_i}. \quad (3.8)$$

Using (3.3) and (3.8), we have

$$R\ (\hat{\beta}_r^*) - R\ (\hat{\beta}_r^*\ (k, d)) = -\sigma^{-2}k^2\ (1-d)^2 \sum_{i=1}^{r} \frac{\lambda_i\alpha_i^2}{(\lambda_i+kd)^2} + \sigma^{-2}2k\ (1-d) \sum_{i=1}^{r} \frac{\alpha_i\eta_i}{(\lambda_i+kd)}. \quad (3.9)$$

From (3.9), it is clear that when $\alpha_i\eta_i < 0$, $i = 1,2, \dots, r$, $R\ (\hat{\beta}_r) - R\ (\hat{\beta}_r\ (k, d))$ is always negative and thus the PCR estimator is superior to the $r - (k, d)$ class estimator for all $k > 0$ and $0 < d < 1$.

However, when $\alpha_i\eta_i > 0$, $i = 1,2, \dots, r$, the difference of risks in (3.9) is positive if

$$k(1-d) \frac{\lambda_i\alpha_i^2}{(\lambda_i+kd)^2} < 2 \frac{\alpha_i\eta_i}{(\lambda_i+kd)} \text{ for all } i = 1,2, \dots, r.$$

Thus we have the dominance conditions of the estimators in the theorem below:

Theorem 3.2

(i) *If* $\alpha_i\eta_i > 0$ *for all* $i = 1,2, \dots, r$, *a necessary and sufficient condition for superiority of the* $r - (k, d)$ *class estimator over the PCR estimator is that* $k(1-d) < \frac{2(\lambda_i+kd)\alpha_i\eta_i}{\lambda_i\alpha_i^2}$ *for all* $i = 1,2, \dots, r$.

(ii) *If* $\alpha_i\eta_i < 0$ *for all* $i = 1,2, \dots, r$, *the* $r - (k, d)$ *class estimator is always superior to the PCR estimator under average Mahalanobis loss criterion.*

Remark 3.2.1

For $d = 0$, the condition reduces to comparison between the $r - k$ class estimator and the PCR in misspecified model obtained by Wu (2016).

Remark 3.2.2

It may be noted that in case of no misspecification Sarkar and Chandra (2015) showed that the $r - (k, d)$ class estimator is always superior to the PCR estimator, however in the misspecified model there may be some situations where the $r - (k, d)$ class estimator dominates the PCR estimator.

3.3 Comparison of $\widehat{\boldsymbol{\beta}}_r^* (\boldsymbol{k}, \boldsymbol{d})$ and $\widehat{\boldsymbol{\beta}}^*(\boldsymbol{k})$

The risk expression for the ORR estimator obtained by substituting $r = p$ and $d = 0$ in (3.3) is given as

$$R(\hat{\beta}^* (k)) = p + \sigma^{-2} \left[k^2 \sum_{i=1}^{p} \frac{\alpha_i^2}{\lambda_i} - 2k \sum_{i=1}^{p} \frac{\alpha_i \eta_i}{\lambda_i} + \sum_{i=1}^{p} \frac{\eta_i^2}{\lambda_i}\right] \tag{3.10}$$

Using (3.3) and (3.10), we get

$$R\left(\hat{\beta}^*(k)\right) - R\left(\hat{\beta}_r^*(k,d)\right) = p - r + \sigma^{-2} k^2 d \left[\sum_{i=1}^{r} \frac{(\lambda_i+k)\left(\lambda_i+kd+\lambda_i(1-d)\right)\alpha_i^2}{\lambda_i\,(\lambda_i+kd)^2} - 2k \sum_{i=1}^{r} d\, \frac{(\lambda_i+k)\alpha_i\eta_i}{\lambda_i\,(\lambda_i+kd)}\right] + \sigma^{-2} \sum_{i=r+1}^{p} \frac{(k\alpha_i-\eta_i)^2}{\lambda_i} \tag{3.11}$$

When $\alpha_i \eta_i > 0$ for $i = 1,2, \dots, r$, $R(\hat{\beta}\ (k) - R(\hat{\beta}_r\ (k, d))$ is positive when the quantity in the brackets is positive. Thus, equivalently we have

$$\frac{2d\ (\lambda_i+kd)\alpha_i\eta_i}{(\lambda_i+kd+\ (1-d)\lambda_i)\alpha_i^2} < 1, \text{ for all } i = 1,2, \dots, r.$$

Clearly, when $\alpha_i \eta_i < 0$, $i = 1,2, \dots, r$, the $r -\ (k, d)$ class estimator performs better than the ORR estimator under the average Mahalanobis loss function criterion in misspecified model.

Theorem 3.3

(i) *When $\alpha_i \eta_i > 0, i = 1,2, \dots, r$ a sufficient condition for the $r - (k, d)$ class estimator to be superior to the ORR estimator under the average Mahalanobis loss criterion in the misspecified model is $\frac{2d\ (\lambda_i+kd)\alpha_i\eta_i}{(\lambda_i+kd+(1-d)\lambda_i)\alpha_i^2} < 1$, for all $i = 1,2, \dots, r$.*

(ii) *For $\alpha_i \eta_i < 0$ for all $i = 1,2, \dots, r$, the $r -\ (k, d)$ class estimator outperforms the ORR estimator under the average Mahalanobis loss criterion in the misspecified model.*

Remark 3.3.1

When $\alpha_i \eta_i > 0$ for $i = 1,2, \dots, r$, the condition stated in above theorem always holds good for $d = 0$, that gives the superiority of the $r - k$ class estimator over the ORR estimator in misspecified model under the average Mahalanobis loss criterion. This result is also obtained by Wu (2016).

Remark 3.3.2

It is evident from (3.11) that, in case of no misspecification (i.e. $\eta_i = 0, i = 1,2, \dots p$) the $r-$ (k,d) class estimator is always superior to the ORR estimator.

3.4 Comparison of $\hat{\boldsymbol{\beta}}_r^*(\boldsymbol{k},\boldsymbol{d})$ and $\hat{\boldsymbol{\beta}}^*(\boldsymbol{d})$

The Liu estimator is a special case of the $r-$ (k,d) class estimator for $r = p$ and $k = 1$. Thus, the risk of the Liu estimator can be obtained by substituting these values in (3.3), given as

$$R(\hat{\beta}^*(d)) = p + \sigma^{-2}\left[(1-d)^2 \sum_{i=1}^{p} \frac{\lambda_i \alpha_i^2}{(\lambda_i + d)^2} - 2(1-d)\sum_{i=1}^{p} \frac{\alpha_i \eta_i}{(\lambda_i + d)} + \sum_{i=1}^{p} \frac{\eta_i^2}{\lambda_i}\right]. \tag{3.12}$$

Further, the difference of the risks of the $r-$ (k,d) class estimator and the Liu estimator is given as

$$\begin{aligned} R\left(\hat{\beta}^*(d)\right) - R\left(\hat{\beta}_r^*(k,d)\right) &= p - r + \sigma^{-2}(1-d)(1-k) \\ &\left[(1-d)\sum_{i=1}^{r} \frac{(\lambda_i(1+k)+2kd)(\lambda_i^2\alpha_i^2)}{(\lambda_i+d)^2\,(\lambda_i+kd)^2} - 2\sum_{i=1}^{r} \frac{\lambda_i\alpha_i\eta_i}{(\lambda_i+d)(\lambda_i+kd)}\right] \\ &+ \sigma^{-2}\sum_{i=r+1}^{p} \frac{(\,(1-d)\lambda_i\alpha_i - (\lambda_i+d)\eta_i)^2}{\lambda_i\,(\lambda_i+d)^2}. \end{aligned} \tag{3.13}$$

Since $p > r$, from the above expression in (3.13), it is clear that when $k < 1$ the difference of risks is positive if

$$(1-d)\sum_{i=1}^{r} \frac{(\lambda_i\,(1+k)+2kd)\lambda_i^2\alpha_i^2}{(\lambda_i+d)^2\,(\lambda_i+kd)^2} - 2\sum_{i=1}^{r} \frac{\lambda_i\alpha_i\eta_i}{(\lambda_i+d)\,(\lambda_i+kd)} \geq 0.$$

Evidently, if $\alpha_i\eta_i < 0$, $i = 1,2, \dots, r$, the above expression holds for all values of $d(0 < d < 1)$, whereas, if $\alpha_i\eta_i > 0$, $i = 1,2, \dots, r$, the following condition will assure to hold the expression.

$$\frac{\sum_{i=1}^{r} \frac{(\lambda_i\,(1+k)+2kd)\lambda_i^2\alpha_i^2}{(\lambda_i+d)^2\,(\lambda_i+kd)^2}}{\sum_{i=1}^{r} \frac{\lambda_i\alpha_i\eta_i}{(\lambda_i+d)\,(\lambda_i+kd)}} \geq \frac{2}{(1-d)}.$$

On the contrary, when $k > 1$ and $\alpha_i\eta_i > 0, i = 1,2, \dots, r$ the difference of risks in (3.13) will be positive if

$$(1-d)\sum_{i=1}^{r} \frac{(\lambda_i\,(1+k)+2kd)\lambda_i^2\alpha_i^2}{(\lambda_i+d)^2\,(\lambda_i+kd)^2} - 2\sum_{i=1}^{r} \frac{\lambda_i\alpha_i\eta_i}{(\lambda_i+d)\,(\lambda_i+kd)} \leq 0.$$

Alternatively, when

$$\frac{\sum_{i=1}^{r} \frac{(\lambda_i\,(1+k)+2kd)\lambda_i^2\alpha_i^2}{(\lambda_i+d)^2\,(\lambda_i+kd)^2}}{\sum_{i=1}^{r} \frac{\lambda_i\alpha_i\eta_i}{(\lambda_i+d)\,(\lambda_i+kd)}} \leq \frac{2}{(1-d)}.$$

Moreover, when $k > 1$ and $\alpha_i\eta_i < 0, i = 1,2, \dots, r$, it will be sufficient for $R(\hat{\beta}(d)) - R(\hat{\beta}_r(k,d))$ to be positive that

$$p - r + \sigma^{-2} (1-d)(1-k) \left[(1-d)\sum_{i=1}^{r} \frac{(\lambda_i (1+k)+2kd)\lambda_i^2 \alpha_i^2}{(\lambda_i+d)^2 (\lambda_i+kd)^2} - 2\sum_{i=1}^{r} \frac{\lambda_i \alpha_i \eta_i}{(\lambda_i+d)(\lambda_i+kd)}\right] \geq 0.$$

Thus, the dominance conditions can be stated as given in the following theorem.

Theorem 3.4

The sufficient conditions for $r-(k,d)$ class estimator to be superior to the Liu estimator under the average Mahalanobis loss criterion are:

(i) *When $\alpha_i \eta_i > 0$ for all $i = 1,2,\ldots,r$*

a) $k > 1$ *and* $\sum_{i=1}^{r} \frac{(\lambda_i (1+k)+2kd)\lambda_i^2 \alpha_i^2}{(\lambda_i+d)^2 (\lambda_i+kd)^2} / \sum_{i=1}^{r} \frac{\lambda_i \alpha_i \eta_i}{(\lambda_i+d)(\lambda_i+kd)} < \frac{2}{1-d}$.

b) $0 < k < 1$ *and* $\sum_{i=1}^{r} \frac{(\lambda_i (1+k)+2kd)\lambda_i^2 \alpha_i^2}{(\lambda_i+d)^2 (\lambda_i+kd)^2} / \sum_{i=1}^{r} \frac{\lambda_i \alpha_i \eta_i}{(\lambda_i+d)(\lambda_i+kd)} > \frac{2}{1-d}$.

(ii) *When $\alpha_i \eta_i < 0$ for all $i = 1,2,\ldots,r$ and*

a) $k > 1$ *and* $\frac{p-r}{(1-d)(1-k)} > \sum_{i=1}^{r} \left[\frac{(1-d)(\lambda_i (1+k)+2kd)\lambda_i^2 \alpha_i^2}{(\lambda_i+d)^2 (\lambda_i+kd)^2} - 2\frac{\lambda_i \alpha_i \eta_i}{(\lambda_i+d)(\lambda_i+kd)}\right]$.

b) $0 < k < 1$ *and* $0 < d < 1$.

Remark 3.4.1

On substitution of $k = 1$ in expression (3.13), we obtain a straightforward superiority of the $r-d$ class estimator over the Liu estimator in misspecified model under the average Mahalanobis loss criterion.

Remark 3.4.2

When there is no omission of relevant variables, that is ($\eta_i = 0, i = 1,2,\ldots p$), the expression in (3.13) suggests that the $r-(k,d)$ class estimator performs better than the $r-d$ class estimator for all values of $k < 1$, however for $k > 1$ the range where the $r-(k,d)$ class estimator dominates the *r-d* class estimator may be obtained from condition (ii) (a) in Theorem 3.4, as $\frac{p-r}{(1-d)(1-k)} > \sum_{i=1}^{r} \left[\frac{(1-d)(\lambda_i (1+k)+2kd)\lambda_i^2 \alpha_i^2}{(\lambda_i+d)^2 (\lambda_i+kd)^2}\right]$.

3.5 Comparison of $\widehat{\boldsymbol{\beta}}_r^*(\boldsymbol{k},\boldsymbol{d})$ and $\widehat{\boldsymbol{\beta}}^*(\boldsymbol{k},\boldsymbol{d})$

Risk under the Mahalanobis loss function of the two-parameter class estimator can be obtained by substituting $r = p$ in (3.3). The expression can be written as

$$R(\hat{\beta}^*(k,d)) = p + \sigma^{-2} \left[k^2(1-d)^2 \sum_{i=1}^{p} \frac{\lambda_i \alpha_i^2}{(\lambda_i+kd)^2} - 2k(1-d)\sum_{i=1}^{p} \frac{\alpha_i \eta_i}{(\lambda_i+kd)} + \sum_{i=1}^{p} \frac{\eta_i^2}{\lambda_i}\right]. \quad (3.14)$$

Further, we have

$$R(\hat{\beta}^*(k,d)) - R(\hat{\beta}_r^*(k,d)) = p - r + \sigma^{-2} \left[k^2 (1-d)^2 \sum_{i=r+1}^{p} \frac{\lambda_i \alpha_i^2}{(\lambda_i+kd)^2} - 2k(1-d)\sum_{i=r+1}^{p} \frac{\alpha_i \eta_i}{(\lambda_i+kd)} + \sum_{i=r+1}^{p} \frac{\eta_i^2}{\lambda_i}\right]$$

$$= p - r + \sigma^{-2} \sum_{i=r+1}^{p} \frac{(k(1-d)\lambda_i \alpha_i - (\lambda_i+kd)\eta_i)^2}{\lambda_i (\lambda_i+kd)^2}. \quad (3.15)$$

From the above expression, it is clear that the $r-(k,d)$ class estimator always dominates the two-parameter class estimator in misspecified model.

Theorem 3.5

The $r-(k,d)$ class estimator dominates the two-parameter class estimator under the average Mahalanobis loss criterionin the misspecified model.

Remark 3.5.1

When $\eta_i=0$ for all $i=r+1,\dots,p$, that is when there is no misspecification due to omission, the result is same as obtained by Sarkar and Chandra (2015) when there is no misspecification.

Remark 3.5.2

For $d=0$, it gives us the comparison between the $r-k$ class estimator and the ORR estimator which was obtained by Wu (2016).

Remark 3.5.3

For $k=1$ it gives the comparison of the $r-d$ class estimator with the Liu estimator and for $k=0$ it compares the PCR and the OLS estimators under average Mahalanobis loss criterionin misspecified model.

3.6 Comparison of $\hat{\boldsymbol{\beta}}_r^*(\boldsymbol{k},\boldsymbol{d})$ and $\hat{\boldsymbol{\beta}}_r^*(\boldsymbol{k})$

Risk expression for the $r-k$ class estimator for misspecified model can be obtained by substituting $d=0$ in (3.3), given as

$$R\,(\hat{\beta}_r^*\,(k))=r+\sigma^{-2}\left[k^2\sum_{i=1}^{r}\frac{\alpha_i^2}{\lambda_i}-2k\sum_{i=1}^{r}\frac{\alpha_i\eta_i}{\lambda_i}+\sum_{i=1}^{r}\frac{\eta_i^2}{\lambda_i}\right]. \tag{3.16}$$

Further, we have

$$R\,(\hat{\beta}_r^*\,(k))-R\,(\hat{\beta}_r^*\,(k,d))=\sigma^{-2}k^2d\left[\sum_{i=1}^{r}\frac{(\lambda_i+k)\,(\lambda_i+kd+(1-d)\lambda_i)\alpha_i^2}{\lambda_i\,(\lambda_i+kd)^2}-2\sum_{i=1}^{r}\frac{d\alpha_i\eta_i}{\lambda_i\,(\lambda_i+kd)}\right]. \tag{3.17}$$

From the above expression, the conditions of superiority of the $r-(k,d)$ class estimator over the $r-k$ class estimator are very straightforward. When $\alpha_i\eta_i<0$, $i=1,2,\dots,r$, the right hand side of (3.17) is always positive and so is the difference of risks. However, when $\alpha_i\eta_i>0$, $i=1,2,\dots,r$, the $r-(k,d)$ class estimator will dominate the $r-k$ class estimator if

$$\frac{(\lambda_i+k)\,(\lambda_i+kd+(1-d)\lambda_i)\alpha_i^2}{\lambda_i\,(\lambda_i+kd)^2}-2\frac{d\alpha_i\eta_i}{\lambda_i\,(\lambda_i+kd)}>0 \text{ for all } i=1,2,\dots,r$$

holds.

Hence, we have the conditions of dominance stated in the form of the following theorem.

Theorem 3.6

(i) *When* $\alpha_i\eta_i > 0,\ i = 1,2,\dots,r,$ *a necessary and sufficient condition for the superiority of the* $r-\ (k,d)$ *class estimator over the* $r-k$ *class estimator under the average Mahalanobis loss criterionin the misspecified model is* $\frac{2d\ (\lambda_i+kd)\alpha_i\eta_i}{(\lambda_i+kd+\ (1-d)\lambda_i)\alpha_i^2} < 1, for\ all\ i = 1,2,\dots,r.$

(ii) *When* $\alpha_i\eta_i < 0,\ i = 1,2,\dots,r$ *the* $r-\ (k,d)$ *class estimator is superior to the* $r-k$ *class estimator under the average Mahalanobis loss criterion in the misspecified model for all values of* $k > 0$ *and* $0 < d < 1$.

Remark 3.6.1

When $\eta_i = 0$, that is in case of no misspecification the condition in the above theorem always holds. Hence the $r-\ (k,d)$ class estimator always performs better than the $r-k$ class estimator in the model with no misspecification under the Mahalanobis loss function using average loss criterion.

3.7 Superiority of the $r-\ (k,d)$ Class Estimator Over the $r-d$ Class Estimator

The $r-d$ class estimator is a special case of the $r-\ (k,d)$ class estimator for $k = 1$. Thus, the risk of the $r-d$ class estimator can be obtained by substituting these values in (3.3), given as

$$R\ (\hat{\beta}_r^*\ (d)) = r + \sigma^{-2}\left[\ (1-d)^2\sum_{i=1}^{r}\frac{\lambda_i\alpha_i^2}{(\lambda_i+d)^2} - 2\ (1-d)\sum_{i=1}^{r}\frac{\alpha_i\eta_i}{(\lambda_i+d)} + \sum_{i=1}^{r}\frac{\eta_i^2}{\lambda_i}\right] \tag{3.18}$$

Further, from (3.3) and (3.18), we have

$$R\ (\hat{\beta}_r^*\ (d)) - R\ (\hat{\beta}_r^*\ (k,d)) = \sigma^{-2}\ (1-d)\ (1-k) \left[\ (1-d)\sum_{i=1}^{r}\frac{(\lambda_i\ (1+k)+2kd)\lambda_i^2\alpha_i^2}{(\lambda_i+d)^2\ (\lambda_i+kd)^2} - 2\sum_{i=1}^{r}\frac{\lambda_i\alpha_i\eta_i}{(\lambda_i+d)\ (\lambda_i+kd)}\right] \tag{3.19}$$

Following the steps similar to the comparison of the $r-\ (k,d)$ class estimator with the Liu estimator in section 3.4, the conditions for superiority of the $r-\ (k,d)$ class estimator over the $r-d$ class estimator can be given in the following theorem.

Theorem 3.7

i) *When* $\alpha_i\eta_i > 0$ *for all* $i = 1,2,\dots,r,$ *the necessary and sufficient conditions for superiority of the* $r-\ (k,d)$ *class estimator over the* $r-d$ *class estimator in the misspecified model under the average Mahalanobis loss criterion are*

a) $k > 1,\ \alpha_i\eta_i > 0$ *for all* $i = 1,2,\dots,r$ *and* $\sum_{i=1}^{r}\frac{(\lambda_i\ (1+k)+2kd)\lambda_i^2\alpha_i^2}{(\lambda_i+d)^2\ (\lambda_i+kd)^2} / \sum_{i=1}^{r}\frac{\lambda_i\alpha_i\eta_i}{(\lambda_i+d)\ (\lambda_i+kd)} < \frac{2}{1-d}.$

b) $k < 1$ *and* $\sum_{i=1}^{r}\frac{(\lambda_i\ (1+k)+2kd)\lambda_i^2\alpha_i^2}{(\lambda_i+d)^2\ (\lambda_i+kd)^2} / \sum_{i=1}^{r}\frac{\lambda_i\alpha_i\eta_i}{(\lambda_i+d)\ (\lambda_i+kd)} > \frac{2}{1-d}.$

ii) *When* $\alpha_i\eta_i < 0$ *for all* $i = 1,2,\dots,r,$

a) *for* $k < 1$, *the* $r-\ (k,d)$ *class estimator dominates the* $r-d$ *class estimator for all values of* $0 < d < 1$.

b) *For $k > 1$, the $r-d$ class estimator dominates the $r-(k,d)$ class estimator in misspecified model under the average Mahalanobis loss criterion.*

Remark 3.7.1

When there is no ommission of relevant variables in the model (When $\eta_i = 0$ for all $i = 1,2,\dots,r$), the $r-(k,d)$ class estimator performs better than the $r-d$ class estimator for $k < 1$, and for $k > 1$ the $r-d$ class estimator outperforms the $r-(k,d)$ class estimator under the average Mahalanobis loss critetion.

Now, we further examine the performance of the estimators in misspecified model under the average Mahalanobis loss criterion by performing a simulation study.

4. MONTE CARLO SIMULATION

In this section a simulation study has been carried out to illustrate theoretical results obtained in previous section. To compare the dominance of the estimators in misspecified model, we take $p = 5$, $q = 3$ and design matrix X and Z have been generated by the method given in [McDonald Galarneau (1975)] and [Gibbons (1981)], that is

$$x_{ij} = (1-\rho^2)^{1/2}\bar{z}_{ij} + \rho\bar{z}_{ip+1}, i = 1,2,\dots,n, j = 1,2,\dots,p$$

$$z_{il} = (1-\rho^2)^{1/2}\bar{z}_{il} + \rho\bar{z}_{iq+1}, i = 1,2,\dots,n, l = 1,2,\dots,q.$$

where $\bar{z}_{ij}$ and $\bar{z}_{il}$ are independent standard normal pseudo-random numbers and ρ^2 gives the correlation between any two explanatory variables. The dependent variable y has been generated as follows:

$$y = X\beta + Z\gamma + u;\ u \sim N\ (0,\sigma^2) \tag{4.1}$$

X matrix has been standardized such that $X'X$ forms correlation matrix of explanatory variables. u is a vector of normal pseudo random numbers with standard deviation σ. β and γ vectors have been taken such that $\beta'\beta = 1$ and $\gamma'\gamma = 1$. The estimator of β for misspecified model has been estimated using X and y only and information on Z has not been considered (i.e., Z are omitted from the model).

The simulation has been conducted for n =50, $\rho =$ 0.50,0.70,0.80,0.90,0.95,0.99, $\sigma^2 =$ 0.1,1,10, $k =$ 0.01,0.1,0.3,0.5,0.7,0.9,2 and $d =$ 0.1,0.3,0.5,0.7,0.9,0.99. The value of r has been taken as the number of eigen values of $X'X$ matrix greater than 1, here it is obtained to be 1. For each parametric combination, the simulation process has been repeated 2000 times and average Mahalanobis loss is calculated by the following formula

$$R\ (\tilde{\beta}) = \frac{1}{2000}\sum_{i=1}^{2000}\ (\tilde{\beta}_{(i)} - \beta)'\ (\text{cov}\ (\tilde{\beta}_{(i)}))^{+}\ (\tilde{\beta}_{(i)} - \beta) \tag{4.2}$$

where $\tilde{\beta}_{(i)}$ is an estimator of β in i^{th} iteration.

The simulation results are summarized in Tables 1-4. The following observations were made:

The average Mahalanobis loss decreases as the value of k increases, and it increases as the value of d increases for the values considered in this study. As it is evident from the expression of average Mahalanobis loss, it decreases with the increase in the value of

σ^2. From Tables 1-3, we see that when $\rho = 0.95$, the $r- \ (k, d)$ class estimator performs better than all the other estimators except the $r-k$ and $r-d$ class estimators for $k < 1$, however for $k = 2$, only $r-k$ class estimator has lesser average loss than the $r- \ (k, d)$ class estimator. When $\rho = 0.99$, for $k < 1$ the $r- \ (k, d)$ class estimator is superior to all the estimators except the PCR estimator, nevertheless the PCR and $r-d$ class estimators perform better than the $r- \ (k, d)$ class estimator for $k > 1$.

The average Mahalanobis loss in misspecified model for different values of ρ for $\sigma^2 = 1$ and some arbitrarity selected values of k and d is presented in Table 4. The results in the Table 4 exhibit that when $\rho = 0.50, 0.99$, the PCR estimator performs better than all the other estimators. When $\rho = 0.70, 0.80, 0.90, 0.95$, for smaller values of k the $r-d$ class estimator outperforms the others and for larger values of k the $r-k$ class estimator starts dominating the other estimators. Moreover, if we examine the superiority of the $r-(k, d)$ class estimator, we observe that the $r-(k, d)$ class estimator outperforms all the other estimators except the PCR estimator for $\rho = 0.50, 0.99$, while for $\rho = 0.70 - 0.95$, the $r-(k, d)$ class estimator dominates the OLS, ORR, Liu, two-parameter class and the PCR estimators. Hence from these results, we can say that for low and strong collinearity the PCR estimator is a good choice and when the multicollinearity is moderate, the $r-d$ class estimator for small value of d and *r*-*k* class estimator for large values of d may be suggested for practical purposes.

Furthermore, the results show that the superiority of the estimators changes with the value of ρ, which is evident from the theoretical results that the dominance depends on the sign of $\alpha_i \eta_i, i = 1, 2, \dots, r$. In this simulation it is positive when $\rho = 0.50, 0.99$ and is negative for $\rho = 0.70, 0.80, 0.90, 0.95$. Hence, same dominance can be seen for $\rho = 0.50$ and 0.99; and for other values of ρ. Additionally it can also be verified that the conditions of the dominance given in the theorems are also satisfied. For example in one iteration of simulation when $\rho = 0.50, k = 0.1, d = 0.1$, the values of α_1 and η_1 are -0.6776841 and -5.90645, resulting in positive values of $\alpha_i \eta_i, \ i = 1, 2, \dots, r$ (*r* is obtained to be 1). Further, the eigen values of *X’X* matrix are 138.30927, 52.87401, 39.35769, 29.83332, 21.02597, hence the value of $2d(\lambda_i + kd)\alpha_i \eta_i / (\lambda_i + kd + (1-d)\lambda_i)\alpha_i^2$ for $i = 1$ is obtained to be 0.1935084 which is less than 1. Thus the condition in Theorem 3.6 (i) for the superiority of the $r-(k, d)$ class estimator over the $r-k$ class estimator is satisfied which gives an evidence to the result obtained in Table 5 that the $r-(k, d)$ class estimator outperforms the $r-k$ class estimator for when $\rho = 0.50, k = 0.1$, $d = 0.1$.

Table 1
Simulated Average Mahalanobis Loss in Misspecified Model when $\sigma^2 = 0.1$

k		$\rho = 0.95$					
		***d*=0.1**	***d*=0.3**	***d*=0.5**	***d*=0.7**	***d*=0.9**	***d*=0.99**
		$\widehat{\beta}$=45.199135, $\widehat{\beta}_r$=1.090234					
0.01	$\hat{\beta}\ (k)$	45.14713	45.14713	45.14713	45.14713	45.14713	45.14713
	$\hat{\beta}\ (d)$	40.96507	42.04656	43.03864	43.95293	44.79905	45.15978
	$\hat{\beta}\ (k,d)$	45.15234	45.16276	45.17316	45.18356	45.19395	45.19862
	$\hat{\beta}_r\ (k)$	1.089961	1.089961	1.089961	1.089961	1.089961	1.089961
	$\hat{\beta}_r\ (d)$	1.069107	1.073211	1.077653	1.082433	1.08755	1.089962
	$\hat{\beta}_r\ (k,d)$	1.089988	1.090043	1.090097	1.090152	1.090207	1.090231
0.1	$\hat{\beta}\ (k)$	44.6826	44.6826	44.6826	44.6826	44.6826	44.6826
	$\hat{\beta}\ (d)$	40.96507	42.04656	43.03864	43.95293	44.79905	45.15978
	$\hat{\beta}\ (k,d)$	44.73541	44.84024	44.94404	45.04682	45.14861	45.19409
	$\hat{\beta}_r\ (k)$	1.087542	1.087542	1.087542	1.087542	1.087542	1.087542
	$\hat{\beta}_r\ (d)$	1.069107	1.073211	1.077653	1.082433	1.08755	1.089962
	$\hat{\beta}_r\ (k,d)$	1.087807	1.088341	1.088877	1.089417	1.089961	1.090207
0.3	$\hat{\beta}\ (k)$	43.67305	43.67305	43.67305	43.67305	43.67305	43.67305
	$\hat{\beta}\ (d)$	40.96507	42.04656	43.03864	43.95293	44.79905	45.15978
	$\hat{\beta}\ (k,d)$	43.83567	44.15387	44.46304	44.76359	45.05593	45.1849
	$\hat{\beta}_r\ (k)$	1.082416	1.082416	1.082416	1.082416	1.082416	1.082416
	$\hat{\beta}_r\ (d)$	1.069107	1.073211	1.077653	1.082433	1.08755	1.089962
	$\hat{\beta}_r\ (k,d)$	1.083163	1.084682	1.08623	1.087809	1.089418	1.090152
0.5	$\hat{\beta}\ (k)$	42.69484	42.69484	42.69484	42.69484	42.69484	42.69484
	$\hat{\beta}\ (d)$	40.96507	42.04656	43.03864	43.95293	44.79905	45.15978
	$\hat{\beta}\ (k,d)$	42.97208	43.50706	44.0177	44.50579	44.97295	45.17673
	$\hat{\beta}_r\ (k)$	1.077634	1.077634	1.077634	1.077634	1.077634	1.077634
	$\hat{\beta}_r\ (d)$	1.069107	1.073211	1.077653	1.082433	1.08755	1.089962
	$\hat{\beta}_r\ (k,d)$	1.078799	1.081192	1.08367	1.086233	1.088879	1.090098
0.7	$\hat{\beta}\ (k)$	41.74799	41.74799	41.74799	41.74799	41.74799	41.74799
	$\hat{\beta}\ (d)$	40.96507	42.04656	43.03864	43.95293	44.79905	45.15978
	$\hat{\beta}\ (k,d)$	42.14368	42.89721	43.60456	44.27025	44.89821	45.16942
	$\hat{\beta}_r\ (k)$	1.073195	1.073195	1.073195	1.073195	1.073195	1.073195
	$\hat{\beta}_r\ (d)$	1.069107	1.073211	1.077653	1.082433	1.08755	1.089962
	$\hat{\beta}_r\ (k,d)$	1.074713	1.077873	1.081198	1.084689	1.088344	1.090043
0.9	$\hat{\beta}\ (k)$	40.83249	40.83249	40.83249	40.83249	40.83249	40.83249
	$\hat{\beta}\ (d)$	40.96507	42.04656	43.03864	43.95293	44.79905	45.15978
	$\hat{\beta}\ (k,d)$	41.34956	42.32193	43.22057	44.05429	44.83054	45.16283
	$\hat{\beta}_r\ (k)$	1.069101	1.069101	1.069101	1.069101	1.069101	1.069101
	$\hat{\beta}_r\ (d)$	1.069107	1.073211	1.077653	1.082433	1.08755	1.089962
	$\hat{\beta}_r\ (k,d)$	1.070906	1.074722	1.078813	1.083177	1.087814	1.089989
2	$\hat{\beta}\ (k)$	36.35766	36.35766	36.35766	36.35766	36.35766	36.35766
	$\hat{\beta}\ (d)$	40.96507	42.04656	43.03864	43.95293	44.79905	45.15978
	$\hat{\beta}\ (k,d)$	37.55609	39.68568	41.52382	43.13019	44.54914	45.13576
	$\hat{\beta}_r\ (k)$	1.052731	1.052731	1.052731	1.052731	1.052731	1.052731
	$\hat{\beta}_r\ (d)$	1.069107	1.073211	1.077653	1.082433	1.08755	1.089962
	$\hat{\beta}_r\ (k,d)$	1.054948	1.060414	1.067247	1.075435	1.084967	1.089692

Table 1 (Contd….)

k		$\rho = 0.99$					
		***d*=0.1**	***d*=0.1**	***d*=0.1**	***d*=0.1**	***d*=0.1**	***d*=0.1**
		$\widehat{\beta}$=37.30482, $\widehat{\beta}_r$=2.921312					
0.01	$\hat{\beta}\ (k)$	37.3988	37.3988	37.3988	37.3988	37.3988	37.3988
	$\hat{\beta}\ (d)$	45.81878	42.23921	40.09346	38.68299	37.69432	37.34125
	$\hat{\beta}\ (k,d)$	37.38925	37.37024	37.35138	37.33265	37.31406	37.30574
	$\hat{\beta}_r\ (k)$	2.92308	2.92308	2.92308	2.92308	2.92308	2.92308
	$\hat{\beta}_r\ (d)$	3.083709	3.046953	3.010581	2.974589	2.938976	2.923074
	$\hat{\beta}_r\ (k,d)$	2.922903	2.922549	2.922196	2.921842	2.921489	2.92133
0.1	$\hat{\beta}\ (k)$	38.26221	38.26221	38.26221	38.26221	38.26221	38.26221
	$\hat{\beta}\ (d)$	45.81878	42.23921	40.09346	38.68299	37.69432	37.34125
	$\hat{\beta}\ (k,d)$	38.15073	37.93949	37.74262	37.55877	37.38674	37.31289
	$\hat{\beta}_r\ (k)$	2.939029	2.939029	2.939029	2.939029	2.939029	2.939029
	$\hat{\beta}_r\ (d)$	3.083709	3.046953	3.010581	2.974589	2.938976	2.923074
	$\hat{\beta}_r\ (k,d)$	2.937253	2.933704	2.930159	2.926617	2.923079	2.921489
0.3	$\hat{\beta}\ (k)$	40.29422	40.29422	40.29422	40.29422	40.29422	40.29422
	$\hat{\beta}\ (d)$	45.81878	42.23921	40.09346	38.68299	37.69432	37.34125
	$\hat{\beta}\ (k,d)$	39.85099	39.09176	38.46649	37.94383	37.50133	37.32381
	$\hat{\beta}_r\ (k)$	2.974713	2.974713	2.974713	2.974713	2.974713	2.974713
	$\hat{\beta}_r\ (d)$	3.083709	3.046953	3.010581	2.974589	2.938976	2.923074
	$\hat{\beta}_r\ (k,d)$	2.969334	2.958603	2.947906	2.937242	2.926614	2.921842
0.5	$\hat{\beta}\ (k)$	42.48254	42.48254	42.48254	42.48254	42.48254	42.48254
	$\hat{\beta}\ (d)$	45.81878	42.23921	40.09346	38.68299	37.69432	37.34125
	$\hat{\beta}\ (k,d)$	41.55741	40.11351	39.04392	38.22403	37.57825	37.33089
	$\hat{\beta}_r\ (k)$	3.01073	3.01073	3.01073	3.01073	3.01073	3.01073
	$\hat{\beta}_r\ (d)$	3.083709	3.046953	3.010581	2.974589	2.938976	2.923074
	$\hat{\beta}_r\ (k,d)$	3.001681	2.983655	2.965724	2.947888	2.930147	2.922194
0.7	$\hat{\beta}\ (k)$	44.82717	44.82717	44.82717	44.82717	44.82717	44.82717
	$\hat{\beta}\ (d)$	45.81878	42.23921	40.09346	38.68299	37.69432	37.34125
	$\hat{\beta}\ (k,d)$	43.26482	41.02851	39.51816	38.43873	37.63395	37.33591
	$\hat{\beta}_r\ (k)$	3.04708	3.04708	3.04708	3.04708	3.04708	3.04708
	$\hat{\beta}_r\ (d)$	3.083709	3.046953	3.010581	2.974589	2.938976	2.923074
	$\hat{\beta}_r\ (k,d)$	3.034293	3.00886	2.983613	2.958553	2.933679	2.922547
0.9	$\hat{\beta}\ (k)$	47.32811	47.32811	47.32811	47.32811	47.32811	47.32811
	$\hat{\beta}\ (d)$	45.81878	42.23921	40.09346	38.68299	37.69432	37.34125
	$\hat{\beta}\ (k,d)$	44.96901	41.85482	39.91666	38.60966	37.67651	37.33969
	$\hat{\beta}_r\ (k)$	3.083763	3.083763	3.083763	3.083763	3.083763	3.083763
	$\hat{\beta}_r\ (d)$	3.083709	3.046953	3.010581	2.974589	2.938976	2.923074
	$\hat{\beta}_r\ (k,d)$	3.06717	3.034217	3.001574	2.969239	2.937211	2.922898
2	$\hat{\beta}\ (k)$	63.87723	63.87723	63.87723	63.87723	63.87723	63.87723
	$\hat{\beta}\ (d)$	45.81878	42.23921	40.09346	38.68299	37.69432	37.34125
	$\hat{\beta}\ (k,d)$	54.15715	45.31404	41.35079	39.16787	37.80701	37.35104
	$\hat{\beta}_r\ (k)$	3.291475	3.291475	3.291475	3.291475	3.291475	3.291475
	$\hat{\beta}_r\ (d)$	3.083709	3.046953	3.010581	2.974589	2.938976	2.923074
	$\hat{\beta}_r\ (k,d)$	3.252732	3.176406	3.10162	3.028361	2.956618	2.924826

Table 2
Simulated Average Mahalanobis Loss in Misspecified Model when $\sigma^2 = 1$

k		ρ =0.95					
		$d = 0.1$	$d = 0.1$	$d = 0.1$	$d = 0.1$	$d = 0.1$	$d = 0.1$
		$\hat{\beta}$=9.125653, $\hat{\beta}_r$=1.048431					
0.01	$\hat{\beta}\ (k)$	9.12045	9.12045	9.12045	9.12045	9.12045	9.12045
	$\hat{\beta}\ (d)$	8.702021	8.810268	8.909543	9.001012	9.085643	9.121717
	$\hat{\beta}\ (k,d)$	9.120971	9.122013	9.123055	9.124095	9.125134	9.125601
	$\hat{\beta}_r\ (k)$	1.048414	1.048414	1.048414	1.048414	1.048414	1.048414
	$\hat{\beta}_r\ (d)$	1.047244	1.047448	1.047686	1.047959	1.048265	1.048414
	$\hat{\beta}_r\ (k,d)$	1.048415	1.048419	1.048422	1.048425	1.048429	1.04843
0.1	$\hat{\beta}\ (k)$	9.073971	9.073971	9.073971	9.073971	9.073971	9.073971
	$\hat{\beta}\ (d)$	8.702021	8.810268	8.909543	9.001012	9.085643	9.121717
	$\hat{\beta}\ (k,d)$	9.079255	9.089744	9.10013	9.110414	9.120598	9.125149
	$\hat{\beta}_r\ (k)$	1.048264	1.048264	1.048264	1.048264	1.048264	1.048264
	$\hat{\beta}_r\ (d)$	1.047244	1.047448	1.047686	1.047959	1.048265	1.048414
	$\hat{\beta}_r\ (k,d)$	1.048281	1.048313	1.048346	1.04838	1.048414	1.048429
0.3	$\hat{\beta}\ (k)$	8.972958	8.972958	8.972958	8.972958	8.972958	8.972958
	$\hat{\beta}\ (d)$	8.702021	8.810268	8.909543	9.001012	9.085643	9.121717
	$\hat{\beta}\ (k,d)$	8.989232	9.021073	9.052009	9.08208	9.111327	9.124229
	$\hat{\beta}_r\ (k)$	1.047958	1.047958	1.047958	1.047958	1.047958	1.047958
	$\hat{\beta}_r\ (d)$	1.047244	1.047448	1.047686	1.047959	1.048265	1.048414
	$\hat{\beta}_r\ (k,d)$	1.048001	1.048091	1.048184	1.048281	1.04838	1.048425
0.5	$\hat{\beta}\ (k)$	8.87508	8.87508	8.87508	8.87508	8.87508	8.87508
	$\hat{\beta}\ (d)$	8.702021	8.810268	8.909543	9.001012	9.085643	9.121717
	$\hat{\beta}\ (k,d)$	8.902826	8.956364	9.00746	9.056294	9.103028	9.123412
	$\hat{\beta}_r\ (k)$	1.047685	1.047685	1.047685	1.047685	1.047685	1.047685
	$\hat{\beta}_r\ (d)$	1.047244	1.047448	1.047686	1.047959	1.048265	1.048414
	$\hat{\beta}_r\ (k,d)$	1.04775	1.047886	1.048031	1.048185	1.048346	1.048422
0.7	$\hat{\beta}\ (k)$	8.780337	8.780337	8.780337	8.780337	8.780337	8.780337
	$\hat{\beta}\ (d)$	8.702021	8.810268	8.909543	9.001012	9.085643	9.121717
	$\hat{\beta}\ (k,d)$	8.819943	8.895357	8.966137	9.032738	9.095554	9.122681
	$\hat{\beta}_r\ (k)$	1.047447	1.047447	1.047447	1.047447	1.047447	1.047447
	$\hat{\beta}_r\ (d)$	1.047244	1.047448	1.047686	1.047959	1.048265	1.048414
	$\hat{\beta}_r\ (k,d)$	1.047527	1.047698	1.047887	1.048092	1.048313	1.048419
0.9	$\hat{\beta}\ (k)$	8.68873	8.68873	8.68873	8.68873	8.68873	8.68873
	$\hat{\beta}\ (d)$	8.702021	8.810268	8.909543	9.001012	9.085643	9.121717
	$\hat{\beta}\ (k,d)$	8.740489	8.837812	8.927735	9.011145	9.08879	9.122022
	$\hat{\beta}_r\ (k)$	1.047244	1.047244	1.047244	1.047244	1.047244	1.047244
	$\hat{\beta}_r\ (d)$	1.047244	1.047448	1.047686	1.047959	1.048265	1.048414
	$\hat{\beta}_r\ (k,d)$	1.047331	1.047527	1.047751	1.048002	1.048281	1.048415
2	$\hat{\beta}\ (k)$	8.24093	8.24093	8.24093	8.24093	8.24093	8.24093
	$\hat{\beta}\ (d)$	8.702021	8.810268	8.909543	9.001012	9.085643	9.121717
	$\hat{\beta}\ (k,d)$	8.36096	8.574176	8.758126	8.91881	9.060684	9.119319
	$\hat{\beta}_r\ (k)$	1.046739	1.046739	1.046739	1.046739	1.046739	1.046739
	$\hat{\beta}_r\ (d)$	1.047244	1.047448	1.047686	1.047959	1.048265	1.048414
	$\hat{\beta}_r\ (k,d)$	1.046753	1.046887	1.047158	1.047565	1.048108	1.048397

Table 2 (Contd….)

k		ρ =0.99					
		$d=0.1$	$d=0.1$	$d=0.1$	$d=0.1$	$d=0.1$	$d=0.1$
		$\hat{\beta}$=8.318031, $\hat{\beta}_r$=1.253865					
0.01	$\hat{\beta}\ (k)$	8.327405	8.327405	8.327405	8.327405	8.327405	8.327405
	$\hat{\beta}\ (d)$	9.167709	8.810531	8.596413	8.455639	8.356931	8.32167
	$\hat{\beta}\ (k,d)$	8.326452	8.324557	8.322675	8.320807	8.318953	8.318123
	$\hat{\beta}_r\ (k)$	1.254052	1.254052	1.254052	1.254052	1.254052	1.254052
	$\hat{\beta}_r\ (d)$	1.270966	1.267099	1.26327	1.25948	1.255727	1.254051
	$\hat{\beta}_r\ (k,d)$	1.254033	1.253996	1.253959	1.253921	1.253884	1.253867
0.1	$\hat{\beta}\ (k)$	8.413538	8.413538	8.413538	8.413538	8.413538	8.413538
	$\hat{\beta}\ (d)$	9.167709	8.810531	8.596413	8.455639	8.356931	8.32167
	$\hat{\beta}\ (k,d)$	8.402417	8.381345	8.361706	8.343365	8.326203	8.318836
	$\hat{\beta}_r\ (k)$	1.255733	1.255733	1.255733	1.255733	1.255733	1.255733
	$\hat{\beta}_r\ (d)$	1.270966	1.267099	1.26327	1.25948	1.255727	1.254051
	$\hat{\beta}_r\ (k,d)$	1.255546	1.255172	1.254798	1.254425	1.254052	1.253884
0.3	$\hat{\beta}\ (k)$	8.616275	8.616275	8.616275	8.616275	8.616275	8.616275
	$\hat{\beta}\ (d)$	9.167709	8.810531	8.596413	8.455639	8.356931	8.32167
	$\hat{\beta}\ (k,d)$	8.572058	8.496316	8.433937	8.381791	8.337639	8.319925
	$\hat{\beta}_r\ (k)$	1.259493	1.259493	1.259493	1.259493	1.259493	1.259493
	$\hat{\beta}_r\ (d)$	1.270966	1.267099	1.26327	1.25948	1.255727	1.254051
	$\hat{\beta}_r\ (k,d)$	1.258926	1.257795	1.256668	1.255545	1.254424	1.253921
0.5	$\hat{\beta}\ (k)$	8.834643	8.834643	8.834643	8.834643	8.834643	8.834643
	$\hat{\beta}\ (d)$	9.167709	8.810531	8.596413	8.455639	8.356931	8.32167
	$\hat{\beta}\ (k,d)$	8.742344	8.59829	8.491575	8.409767	8.345322	8.320633
	$\hat{\beta}_r\ (k)$	1.263286	1.263286	1.263286	1.263286	1.263286	1.263286
	$\hat{\beta}_r\ (d)$	1.270966	1.267099	1.26327	1.25948	1.255727	1.254051
	$\hat{\beta}_r\ (k,d)$	1.262333	1.260435	1.258546	1.256666	1.254797	1.253958
0.7	$\hat{\beta}\ (k)$	9.068642	9.068642	9.068642	9.068642	9.068642	9.068642
	$\hat{\beta}\ (d)$	9.167709	8.810531	8.596413	8.455639	8.356931	8.32167
	$\hat{\beta}\ (k,d)$	8.912758	8.689633	8.538933	8.431216	8.350889	8.321135
	$\hat{\beta}_r\ (k)$	1.267112	1.267112	1.267112	1.267112	1.267112	1.267112
	$\hat{\beta}_r\ (d)$	1.270966	1.267099	1.26327	1.25948	1.255727	1.254051
	$\hat{\beta}_r\ (k,d)$	1.265766	1.263089	1.26043	1.25779	1.255169	1.253996
0.9	$\hat{\beta}\ (k)$	9.318271	9.318271	9.318271	9.318271	9.318271	9.318271
	$\hat{\beta}\ (d)$	9.167709	8.810531	8.596413	8.455639	8.356931	8.32167
	$\hat{\beta}\ (k,d)$	9.082874	8.772142	8.578744	8.448304	8.355147	8.321513
	$\hat{\beta}_r\ (k)$	1.270972	1.270972	1.270972	1.270972	1.270972	1.270972
	$\hat{\beta}_r\ (d)$	1.270966	1.267099	1.26327	1.25948	1.255727	1.254051
	$\hat{\beta}_r\ (k,d)$	1.269226	1.265758	1.262322	1.258916	1.255541	1.254033
2	$\hat{\beta}\ (k)$	10.97063	10.97063	10.97063	10.97063	10.97063	10.97063
	$\hat{\beta}\ (d)$	9.167709	8.810531	8.596413	8.455639	8.356931	8.32167
	$\hat{\beta}\ (k,d)$	10.0004	9.117806	8.722217	8.504241	8.368253	8.322654
	$\hat{\beta}_r\ (k)$	1.292796	1.292796	1.292796	1.292796	1.292796	1.292796
	$\hat{\beta}_r\ (d)$	1.270966	1.267099	1.26327	1.25948	1.255727	1.254051
	$\hat{\beta}_r\ (k,d)$	1.288729	1.280712	1.27285	1.265142	1.257586	1.254236

Table 3
Simulated Average Mahalanobis Loss in Misspecified Model when $\sigma^2 = 10$

k		ρ =0.95					
		$d = 0.1$	$d = 0.1$	$d = 0.1$	$d = 0.1$	$d = 0.1$	$d = 0.1$
		$\widehat{\beta}$=5.493342, $\widehat{\beta}_r$=1.046878					
0.01	$\hat{\beta}\ (k)$	5.492821	5.492821	5.492821	5.492821	5.492821	5.492821
	$\hat{\beta}\ (d)$	5.450908	5.461763	5.471712	5.480872	5.489341	5.492949
	$\hat{\beta}\ (k,d)$	5.492873	5.492978	5.493082	5.493186	5.49329	5.493337
	$\hat{\beta}_r\ (k)$	1.04688	1.04688	1.04688	1.04688	1.04688	1.04688
	$\hat{\beta}_r\ (d)$	1.047052	1.047008	1.046966	1.046928	1.046894	1.04688
	$\hat{\beta}_r\ (k,d)$	1.04688	1.046879	1.046879	1.046879	1.046878	1.046878
0.1	$\hat{\beta}\ (k)$	5.488165	5.488165	5.488165	5.488165	5.488165	5.488165
	$\hat{\beta}\ (d)$	5.450908	5.461763	5.471712	5.480872	5.489341	5.492949
	$\hat{\beta}\ (k,d)$	5.488694	5.489745	5.490786	5.491816	5.492836	5.493292
	$\hat{\beta}_r\ (k)$	1.046894	1.046894	1.046894	1.046894	1.046894	1.046894
	$\hat{\beta}_r\ (d)$	1.047052	1.047008	1.046966	1.046928	1.046894	1.04688
	$\hat{\beta}_r\ (k,d)$	1.046892	1.046889	1.046886	1.046883	1.04688	1.046878
0.3	$\hat{\beta}\ (k)$	5.478046	5.478046	5.478046	5.478046	5.478046	5.478046
	$\hat{\beta}\ (d)$	5.450908	5.461763	5.471712	5.480872	5.489341	5.492949
	$\hat{\beta}\ (k,d)$	5.479677	5.482867	5.485967	5.488979	5.491908	5.4932
	$\hat{\beta}_r\ (k)$	1.046929	1.046929	1.046929	1.046929	1.046929	1.046929
	$\hat{\beta}_r\ (d)$	1.047052	1.047008	1.046966	1.046928	1.046894	1.04688
	$\hat{\beta}_r\ (k,d)$	1.046923	1.046913	1.046902	1.046892	1.046883	1.046879
0.5	$\hat{\beta}\ (k)$	5.46824	5.46824	5.46824	5.46824	5.46824	5.46824
	$\hat{\beta}\ (d)$	5.450908	5.461763	5.471712	5.480872	5.489341	5.492949
	$\hat{\beta}\ (k,d)$	5.471021	5.476388	5.481507	5.486399	5.491078	5.493118
	$\hat{\beta}_r\ (k)$	1.046966	1.046966	1.046966	1.046966	1.046966	1.046966
	$\hat{\beta}_r\ (d)$	1.047052	1.047008	1.046966	1.046928	1.046894	1.04688
	$\hat{\beta}_r\ (k,d)$	1.046957	1.046938	1.04692	1.046902	1.046886	1.046879
0.7	$\hat{\beta}\ (k)$	5.458747	5.458747	5.458747	5.458747	5.458747	5.458747
	$\hat{\beta}\ (d)$	5.450908	5.461763	5.471712	5.480872	5.489341	5.492949
	$\hat{\beta}\ (k,d)$	5.462719	5.47028	5.477372	5.484043	5.490331	5.493045
	$\hat{\beta}_r\ (k)$	1.047008	1.047008	1.047008	1.047008	1.047008	1.047008
	$\hat{\beta}_r\ (d)$	1.047052	1.047008	1.046966	1.046928	1.046894	1.04688
	$\hat{\beta}_r\ (k,d)$	1.046993	1.046964	1.046938	1.046913	1.046889	1.046879
0.9	$\hat{\beta}\ (k)$	5.449568	5.449568	5.449568	5.449568	5.449568	5.449568
	$\hat{\beta}\ (d)$	5.450908	5.461763	5.471712	5.480872	5.489341	5.492949
	$\hat{\beta}\ (k,d)$	5.454761	5.46452	5.473531	5.481884	5.489655	5.492979
	$\hat{\beta}_r\ (k)$	1.047052	1.047052	1.047052	1.047052	1.047052	1.047052
	$\hat{\beta}_r\ (d)$	1.047052	1.047008	1.046966	1.046928	1.046894	1.04688
	$\hat{\beta}_r\ (k,d)$	1.047032	1.046993	1.046956	1.046923	1.046892	1.04688
2	$\hat{\beta}\ (k)$	5.404688	5.404688	5.404688	5.404688	5.404688	5.404688
	$\hat{\beta}\ (d)$	5.450908	5.461763	5.471712	5.480872	5.489341	5.492949
	$\hat{\beta}\ (k,d)$	5.41675	5.438153	5.456591	5.472675	5.486855	5.49271
	$\hat{\beta}_r\ (k)$	1.04736	1.04736	1.04736	1.04736	1.04736	1.04736
	$\hat{\beta}_r\ (d)$	1.047052	1.047008	1.046966	1.046928	1.046894	1.04688
	$\hat{\beta}_r\ (k,d)$	1.047296	1.047179	1.047075	1.046986	1.046911	1.046881

Table 3 (Contd….)

k		ρ =0.99					
		$d=0.1$	$d=0.1$	$d=0.1$	$d=0.1$	$d=0.1$	$d=0.1$
		$\hat{\beta}$=5.405501, $\hat{\beta}_r$=1.073579					
0.01	$\hat{\beta}\ (k)$	5.406432	5.406432	5.406432	5.406432	5.406432	5.406432
	$\hat{\beta}\ (d)$	5.489926	5.454455	5.433187	5.419196	5.409376	5.405864
	$\hat{\beta}\ (k,d)$	5.406337	5.406149	5.405962	5.405777	5.405593	5.405511
	$\hat{\beta}_r\ (k)$	1.0736	1.0736	1.0736	1.0736	1.0736	1.0736
	$\hat{\beta}_r\ (d)$	1.075561	1.075114	1.07467	1.074231	1.073795	1.0736
	$\hat{\beta}_r\ (k,d)$	1.073598	1.073594	1.073589	1.073585	1.073581	1.073579
0.1	$\hat{\beta}\ (k)$	5.414979	5.414979	5.414979	5.414979	5.414979	5.414979
	$\hat{\beta}\ (d)$	5.489926	5.454455	5.433187	5.419196	5.409376	5.405864
	$\hat{\beta}\ (k,d)$	5.413875	5.411784	5.409836	5.408016	5.406312	5.405581
	$\hat{\beta}_r\ (k)$	1.073796	1.073796	1.073796	1.073796	1.073796	1.073796
	$\hat{\beta}_r\ (d)$	1.075561	1.075114	1.07467	1.074231	1.073795	1.0736
	$\hat{\beta}_r\ (k,d)$	1.073774	1.07373	1.073687	1.073644	1.0736	1.073581
0.3	$\hat{\beta}\ (k)$	5.435106	5.435106	5.435106	5.435106	5.435106	5.435106
	$\hat{\beta}\ (d)$	5.489926	5.454455	5.433187	5.419196	5.409376	5.405864
	$\hat{\beta}\ (k,d)$	5.430718	5.423201	5.417009	5.411833	5.407449	5.40569
	$\hat{\beta}_r\ (k)$	1.074232	1.074232	1.074232	1.074232	1.074232	1.074232
	$\hat{\beta}_r\ (d)$	1.075561	1.075114	1.07467	1.074231	1.073795	1.0736
	$\hat{\beta}_r\ (k,d)$	1.074166	1.074035	1.073904	1.073774	1.073643	1.073585
0.5	$\hat{\beta}\ (k)$	5.456796	5.456796	5.456796	5.456796	5.456796	5.456796
	$\hat{\beta}\ (d)$	5.489926	5.454455	5.433187	5.419196	5.409376	5.405864
	$\hat{\beta}\ (k,d)$	5.447634	5.433335	5.42274	5.414617	5.408214	5.40576
	$\hat{\beta}_r\ (k)$	1.074672	1.074672	1.074672	1.074672	1.074672	1.074672
	$\hat{\beta}_r\ (d)$	1.075561	1.075114	1.07467	1.074231	1.073795	1.0736
	$\hat{\beta}_r\ (k,d)$	1.074561	1.074341	1.074122	1.073904	1.073687	1.073589
0.7	$\hat{\beta}\ (k)$	5.480049	5.480049	5.480049	5.480049	5.480049	5.480049
	$\hat{\beta}\ (d)$	5.489926	5.454455	5.433187	5.419196	5.409376	5.405864
	$\hat{\beta}\ (k,d)$	5.464572	5.442419	5.427455	5.416755	5.40877	5.40581
	$\hat{\beta}_r\ (k)$	1.075115	1.075115	1.075115	1.075115	1.075115	1.075115
	$\hat{\beta}_r\ (d)$	1.075561	1.075114	1.07467	1.074231	1.073795	1.0736
	$\hat{\beta}_r\ (k,d)$	1.074959	1.074649	1.074341	1.074034	1.07373	1.073594
0.9	$\hat{\beta}\ (k)$	5.504865	5.504865	5.504865	5.504865	5.504865	5.504865
	$\hat{\beta}\ (d)$	5.489926	5.454455	5.433187	5.419196	5.409376	5.405864
	$\hat{\beta}\ (k,d)$	5.481488	5.450632	5.431424	5.418462	5.409197	5.405848
	$\hat{\beta}_r\ (k)$	1.075562	1.075562	1.075562	1.075562	1.075562	1.075562
	$\hat{\beta}_r\ (d)$	1.075561	1.075114	1.07467	1.074231	1.073795	1.0736
	$\hat{\beta}_r\ (k,d)$	1.07536	1.074958	1.07456	1.074165	1.073773	1.073598
2	$\hat{\beta}\ (k)$	5.669295	5.669295	5.669295	5.669295	5.669295	5.669295
	$\hat{\beta}\ (d)$	5.489926	5.454455	5.433187	5.419196	5.409376	5.405864
	$\hat{\beta}\ (k,d)$	5.572832	5.485116	5.44579	5.424093	5.410525	5.405964
	$\hat{\beta}_r\ (k)$	1.078077	1.078077	1.078077	1.078077	1.078077	1.078077
	$\hat{\beta}_r\ (d)$	1.075561	1.075114	1.07467	1.074231	1.073795	1.0736
	$\hat{\beta}_r\ (k,d)$	1.077609	1.076686	1.075779	1.074887	1.074011	1.073622

Table 4
Simulated Average Mahalanobis Loss in Misspecified Model for Different ρ when $\sigma^2 = 1$

	d=0.1						*d*=0.9					
	ρ=0.50	ρ =0.70	ρ =0.80	ρ =0.90	ρ =0.95	ρ =0.99	ρ=0.50	ρ =0.70	ρ =0.80	ρ =0.90	ρ =0.95	ρ =0.99
	***k*=0.1**											
$\hat{\beta}$	7.330753	8.789499	8.962863	8.377061	9.125653	8.318031	7.330753	8.789499	8.962863	8.377061	9.125653	8.318031
$\hat{\beta}\ (k)$	7.316974	8.76604	8.937024	8.345566	9.073971	8.413538	7.316974	8.76604	8.937024	8.345566	9.073971	8.413538
$\hat{\beta}\ (d)$	7.211929	8.584243	8.738882	8.111394	8.702021	9.167709	7.31751	8.767242	8.938873	8.349492	9.085643	8.356931
$\hat{\beta}\ (k,d)$	7.318352	8.768393	8.93962	8.348744	9.079255	8.402417	7.329375	8.78716	8.960291	8.37394	9.120598	8.326203
$\hat{\beta}_r$	1.093778	1.050944	1.05105	1.354731	1.048431	1.253865	1.093778	1.050944	1.05105	1.354731	1.048431	1.253865
$\hat{\beta}_r\ (k)$	1.095344	1.050729	1.050816	1.352287	1.048264	1.255733	1.095344	1.050729	1.050816	1.352287	1.048264	1.255733
$\hat{\beta}_r\ (d)$	1.10884	1.049667	1.049416	1.333101	1.047244	1.270966	1.095334	1.05073	1.050817	1.352295	1.048265	1.255727
$\hat{\beta}_r\ (k,d)$	1.095186	1.05075	1.050839	1.352531	1.048281	1.255546	1.093933	1.050922	1.051026	1.354486	1.048414	1.254052
***k*=0.5**												
$\hat{\beta}$	7.330753	8.789499	8.962863	8.377061	9.125653	8.318031	7.330753	8.789499	8.962863	8.377061	9.125653	8.318031
$\hat{\beta}\ (k)$	7.263153	8.673497	8.835522	8.223383	8.87508	8.834643	7.263153	8.673497	8.835522	8.223383	8.87508	8.834643
$\hat{\beta}\ (d)$	7.211929	8.584243	8.738882	8.111394	8.702021	9.167709	7.31751	8.767242	8.938873	8.349492	9.085643	8.356931
$\hat{\beta}\ (k,d)$	7.269903	8.685263	8.848535	8.239411	8.902826	8.742344	7.323984	8.778063	8.950402	8.362324	9.103028	8.345322
$\hat{\beta}_r$	1.093778	1.050944	1.05105	1.354731	1.048431	1.253865	1.093778	1.050944	1.05105	1.354731	1.048431	1.253865
$\hat{\beta}_r\ (k)$	1.101879	1.050052	1.050011	1.342612	1.047685	1.263286	1.101879	1.050052	1.050011	1.342612	1.047685	1.263286
$\hat{\beta}_r\ (d)$	1.10884	1.049667	1.049416	1.333101	1.047244	1.270966	1.095334	1.05073	1.050817	1.352295	1.048265	1.255727
$\hat{\beta}_r\ (k,d)$	1.101035	1.050121	1.050101	1.343815	1.04775	1.262333	1.094555	1.050834	1.050931	1.35351	1.048346	1.254797

Table 4 (Contd….)

	d=0.1						d=0.9					
	ρ=0.50	ρ =0.70	ρ =0.80	ρ =0.90	ρ =0.95	ρ =0.99	ρ=0.50	ρ =0.70	ρ =0.80	ρ =0.90	ρ =0.95	ρ =0.99
	k=0.9											
$\hat{\beta}$	7.330753	8.789499	8.962863	8.377061	9.125653	8.318031	7.330753	8.789499	8.962863	8.377061	9.125653	8.318031
$\hat{\beta}\ (k)$	7.211408	8.583025	8.736989	8.107272	8.68873	9.318271	7.211408	8.583025	8.736989	8.107272	8.68873	9.318271
$\hat{\beta}\ (d)$	7.211929	8.584243	8.738882	8.111394	8.702021	9.167709	7.31751	8.767242	8.938873	8.349492	9.085643	8.356931
$\hat{\beta}\ (k,d)$	7.223295	8.604192	8.760442	8.136243	8.740489	9.082874	7.318782	8.769359	8.941109	8.351925	9.08879	8.355147
$\hat{\beta}_r$	1.093778	1.050944	1.05105	1.354731	1.048431	1.253865	1.093778	1.050944	1.05105	1.354731	1.048431	1.253865
$\hat{\beta}_r\ (k)$	1.108852	1.049667	1.049416	1.333093	1.047244	1.270972	1.108852	1.049667	1.049416	1.333093	1.047244	1.270972
$\hat{\beta}_r\ (d)$	1.10884	1.049667	1.049416	1.333101	1.047244	1.270966	1.095334	1.05073	1.050817	1.352295	1.048265	1.255727
$\hat{\beta}_r\ (k,d)$	1.107236	1.049728	1.049532	1.335228	1.047331	1.269226	1.095178	1.05075	1.050839	1.352538	1.048281	1.255541
	k=2											
$\hat{\beta}$	7.330753	8.789499	8.962863	8.377061	9.125653	8.318031	7.330753	8.789499	8.962863	8.377061	9.125653	8.318031
$\hat{\beta}\ (k)$	7.07981	8.344909	8.481333	7.819284	8.24093	10.97063	7.07981	8.344909	8.481333	7.819284	8.24093	10.97063
$\hat{\beta}\ (d)$	7.211929	8.584243	8.738882	8.111394	8.702021	9.167709	7.31751	8.767242	8.938873	8.349492	9.085643	8.356931
$\hat{\beta}\ (k,d)$	7.104497	8.391688	8.533267	7.882951	8.36096	10.0004	7.305376	8.747257	8.918224	8.328164	9.060684	8.368253
$\hat{\beta}_r$	1.093778	1.050944	1.05105	1.354731	1.048431	1.253865	1.093778	1.050944	1.05105	1.354731	1.048431	1.253865
$\hat{\beta}_r\ (k)$	1.130288	1.050113	1.048856	1.307721	1.046739	1.292796	1.130288	1.050113	1.048856	1.307721	1.046739	1.292796
$\hat{\beta}_r\ (d)$	1.10884	1.049667	1.049416	1.333101	1.047244	1.270966	1.095334	1.05073	1.050817	1.352295	1.048265	1.255727
$\hat{\beta}_r\ (k,d)$	1.126092	1.049866	1.04884	1.312276	1.046753	1.288729	1.096896	1.050536	1.050598	1.349885	1.048108	1.257586

5. NUMERICAL EXAMPLE

This section illustrates our theoretical results by using a data set on Total National Research and Development Expenditures as a Per cent of Gross National Product originally due to Gruber (1998) also analyzed by Zhong and Yang (2007) and Chandra and Tyagi (2017). It represents the relationship between the dependent variable Y, the percents spent by the U.S., and the four other independent variables X_1, X_2, X_3, and X_4, representing the percent spent by France, West Germany, Japan and the former Soviet Union, respectively. The OLS estimator of $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)'$ is given as $\hat{\beta} = (0.6455, 0.0896, 0.1436, 0.1526)'$. We obtain the eigenvalues of $X'X$ matrix as 302.9626, 0.7283, 0.0446, and 0.0345, which gives the condition number approximately equal to 8,776.382. Hence, the design matrix is ill-conditioned.

Now, let us consider that the investigator has omitted $Z = [X_4]$ mistakenly, which results in misspecified model (2.2) with X matrix having 3 variables X_1, X_2 and X_3. The eigen values of the X matrix in misspecified model are $\lambda_1 = 161.38584077$, $\lambda_2 = 0.10961836$ and $\lambda_3 = 0.04454088$ and the condition number is 3623.32, which indicates ill-conditioned design matrix in the misspecified model. The OLS estimators of β, γ and σ^2 in model (2.2) is obtained as $\hat{\beta}^* = (0.80878236,\ 0.41402294,\ -0.09630492)'$, $\hat{\gamma} = \hat{\beta}_4 = 0.1526$, $\hat{\sigma}^2 = 0.002745$ respectively and we chose $r = 2$. Yang and Huang (2016) suggested the optimum value of k so that the $r-k$ class estimator perform well in average loss sence under Mahalanobis loss and not bad in MSE sense and gave the following method:

$$k_{opt} = \min\{k_0, k_1, k_2\},$$

where $k_0 = \sqrt{\frac{\sigma^2\,(p-r)}{\beta' T_r \Lambda_r^{-1} T_r' \beta}}$, k_1 denotes the positive solution of $\partial MSE\left(\hat{\beta}_r^*(k)\right)/\partial k = 0$ and $k_2 = \frac{\sigma^2}{\gamma_{\max}^2}$ with $\gamma_{\max}$ being the maximum element of $\gamma = T'\beta$. Morover, the authors proposed the optimum value of d for which the r-d class estimator performs well in the average loss sense under the Mahalanobis loss and not bad in MSE sense, and the proposed value is given as

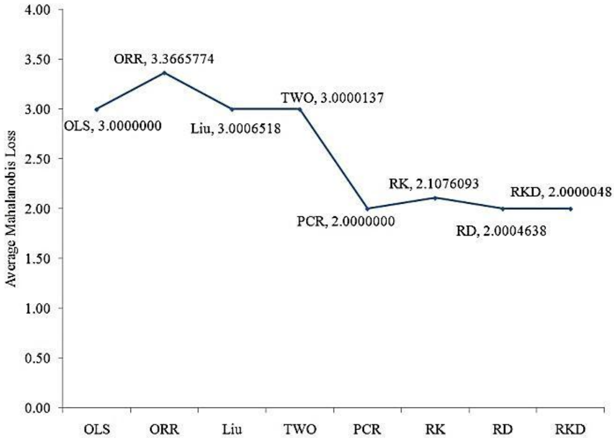
$$d_{opt} = \max\{d_0, d_1, d_2\},$$

where d_0 is the solution of $\sqrt{\frac{p-r}{\beta' T_r S_r^{-1}(d) \Lambda_r S_r^{-1}(d) T_r' \beta}} = \frac{1-d}{\sigma}$, $d_1 = max\left(0, \frac{\gamma_j^2 - \sigma^2}{\sigma^2/\lambda_j + \gamma_j^2}\right)$, $j = 1,2, \ldots r$, $d_2 = \frac{\sum_{j=1}^{r}\left(\gamma_j^2 - \sigma^2\right)/(\lambda_j + 1)^2}{\sum_{j=1}^{r}\left(\lambda_j \gamma_j^2 + \sigma^2\right)/\lambda_j(\lambda_j + 1)^2}$ is the solution of $\partial MSE\left(\hat{\beta}_r^*(d)\right)/\partial d = 0$ and γ_j is the j-th element of γ.

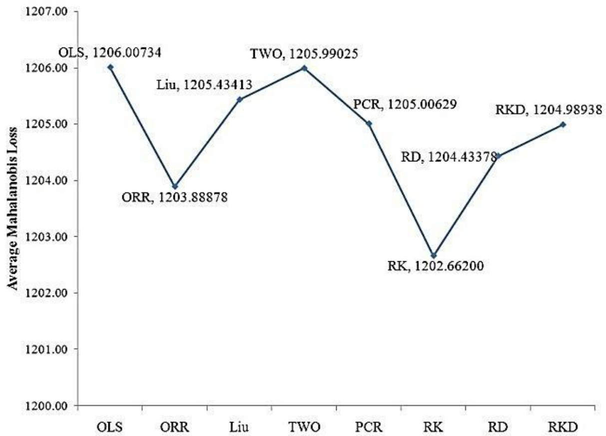
Hence the optimum values of k and d are obtained to be 0.0117 and 0.9926, respectively. The estimated values of β and corresponding average Mahalanobis loss for each estimator under study are presented in Table 5 for true and misspecified model both and Figure 1 represents the average Mahalanobis losses of the estimators in the two models.

Table 5: Estimated Values of Regression Coefficients and Average Malahanobis Loss (Avg. ML) for True and Misspecified Model

	True Model					Misspecified Model			
	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$\boldsymbol{\beta_3}$	$\boldsymbol{\beta_4}$	**Avg. ML**	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$\boldsymbol{\beta_3}$	**Avg. ML**
$\hat{\beta}$	0.6455	0.0896	0.1436	0.1526	3.0000000	0.8088	0.4140	-0.0963	1206.00734
$\hat{\beta}\ (k)$	0.5511	0.1156	0.1800	0.1633	3.3665774	0.7155	0.4387	-0.0427	1203.88878
$\hat{\beta}\ (d)$	0.6422	0.0905	0.1442	0.1534	3.0006518	0.8053	0.4139	-0.0932	1205.43413
$\hat{\beta}\ (k,d)$	0.6448	0.0898	0.1438	0.1527	3.0000137	0.8081	0.4142	-0.0959	1205.99025
$\hat{\beta}_r$	0.2100	0.2401	0.3047	0.1861	2.0000000	0.4090	0.6627	-0.0161	1205.00629
$\hat{\beta}_r\ (k)$	0.2092	0.2395	0.3029	0.1880	2.1076093	0.3994	0.6354	0.0207	1202.66200
$\hat{\beta}_r\ (d)$	0.2098	0.2399	0.3042	0.1866	2.0004638	0.4083	0.6608	-0.0136	1204.43378
$\hat{\beta}_r\ (k,d)$	0.2100	0.2401	0.3047	0.1861	2.0000048	0.4089	0.6625	-0.0159	1204.98938



(a)



(b)

Figure 1: Average Mahalanobis Loss of the estimators
(a) In Case of No Misspecification, (b) When there is Misspecification

From Table 5, we can see the sign of β_3 has changed in the misspecified model from positive to negative which gives an evidence of the well established result that the omission affects the estimation of parameters. Further, looking at the average Mahalanobis losses, a drastic increase in the average Mahalanobis loss is noticed in the misspecified model as compared to the true model. From Figure 1, we see that there is not much difference in the average losses of the PCR, $r-k$ class estimator, $r-d$ class estimator and the $r-(k,d)$ class estimator. Further, the $r-(k,d)$ class estimator performs equally well to the $r-d$ class estimator and the PCR estimator, and outperforms the others in true model. However, from Table 5, the differences in the average losses of the $r-(k,d)$ class estimator with the $r-d$ class and PCR estimators can be noticed only after forth and sixth decimal places respectively.

On the other hand, from the results stated in Table 5 for the misspecified model, we see that the $r-(k,d)$ class estimator is superior to the two-parameter class estimator, which is consistent with the Theorem 3.5. Further, from Figure 1, it is clear that the $r-(k,d)$ class estimator outperforms the OLS, Liu, two-parameter class and the PCR estimators, though it does not perform better than the ORR, the $r-k$ class and $r-d$ class estimators. Moreover, the theoretical findings obtained in this study support the numerical results stated in Table 5. To state a few, we verify the conditions for the dominance of the $r-(k,d)$ class estimator over the OLS and PCR estimators stated in Theorems 3.1 and 3.2 respectively. In this data, $\mathrm{T}_r'\beta=(\alpha_1,\alpha_2)$ is (-0.6113364, -0.4827003) and $\mathrm{T}_r'X'Z\gamma=(\eta_1,\eta_2)$ is (-23.02250243, -0.04507051) and hence $\alpha_i\eta_i>0$ for all $i=1,2$. Since the optimum values of k and d are 0.0117 and 0.9926 respectively, the value of $k\,(1-d)$ is obtained to be 0.00008709193. Further, the values of $2\,(\lambda_i+kd)\alpha_i\eta_i/\lambda_i\alpha_i^2$ are found to be 75.3240653, 0.2066683 for $i=1,2$, respectively, hence conditions for the superiority of the $r-(k,d)$ class estimator over the OLS and PCR estimators are satisfied. Similarly other conditions can also be verified.

6. CONCLUDING REMARKS

In this paper, the consequences of omission of the relevant variables on the superiority of the $r-(k,d)$ class estimator over the other estimators which can be obtained as special cases by substituting suitable values of r,k and d have been studied under the average Mahalanobis loss criterion. It has been seen that the misspecification due to omission of relevant variables may increase or decrease the average Mahalanobis loss depending on the values of certain parameters. Further, it is quite interesting to note that in case of no misspecification the PCR estimator is always superior to the $r-(k,d)$ class estimator, whereas in the misspecified model the $r-(k,d)$ class estimator may dominate the PCR estimator under certain conditions. A Monte Carlo simulation study and a numerical example are also provided which support the theoretical findings.

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