Pak. J. Statist.
2017 Vol. 33(2), 117-128

ON THE EDGEWORTH TYPE EXPANSIONS FOR THE DISTRIBUTION OF EXTREME-SPACINGS STATISTICS

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ABSTRACT

In this article we study the Edgeworth type Expansion of a statistics based on uniform spacings. There is a fairly extensive literature devoted to studying the distribution of random variable based on uniform spacings. One of them is the Extreme spacings Statistics. We aim to approximate the distribution of Extreme spacing statistics by two well-known Edgeworth type expansions. For the purpose of comparison the simulated values of both the expansions are tabulated and graphed.

KEY WORDS

Edgeworth Expansions, Goodness-of-fit, Random Variable, Spacings, Extreme Spacings.

1. INTRODUCTION

Let $X_1', X_2', ..., X_n'$ be an ordered (ascending form) sample from certain population. The population is assumed to be continuous with cumulative distribution function (cdf) F and probability density function (pdf) $f(x)$. The goodness-of-fit problem is to test that this sample is taken from a known distribution against the alternative that it is not the specified one. To reduce the support of F to [0,1] it is well known that the data is transformed via the probability integral transformation $U = F(X')$. By this way the specified cdf reduces to that of a uniform random variable on [0,1]. With notations $X_0' = 0$ and $X_{n+1}' = 1$, sample spacings are defined as $D_j = X_j - X_{j-1}, j = 1, 2, ..., n+1$.

Let $f : [0, \infty) \to R$ be a fixed non-linear measurable function and define the statistics based on spacings as

$$R_n = \sum_{j=1}^{n+1} f_j\left((n+1)D_{j,n}\right), \qquad n = 1, 2, ..., \tag{1.1}$$

When $f_j(x) = f(x)$ then the random variable R_n is symmetric. The statistics R_n is extensively studied by the researchers for different $f(x)$ as kernel function, see Pyke (1965). For example, the statistics R_n is called linear when $f_j(x) = l(x)\, j$ where $l(x)$

is real function defined on [0,1], see for example, Holst and Rao (1981), Morgan and Jammalamadaka (1981), Jammalamadaka and Goria (2004), Mirakhmadov and Naeem (2008a) and Naeem (2015b). When $f(x)=x^2$, it is the classical case of Greenwood statistics (cf. Greenwood (1946)). The introduction of Greenwood statistics enhanced the interest of applied statisticians in testing goodness-of-fit problems based on uniform spacings. The statistics is called log-spacings statistic if $f(x)=\log x$, see Darling (1953), Morgan and Jammalamadaka(1981), Deheuvels and Derzko (2003) and Naeem (2016). The random variable R_n is entropy-type spacings statistics if $f(x)=x\log x$, see for instance, Jammalamadaka and Tiwari (1987), Jammalamadaka et al. (1989) and Bartoszewicz (1995). The random variable R_n is generalized Rao's statistic when $f(x)=|x-k|^r$, $r>0$, the case $r=2$ coincide with Greenwood (1946) statistic, and $r=1$ is Rao's spacings statistic see Rao (1972), also see Del Pino (1979) and Mirakhmedov and Naeem (2008a). The random variable is used for circular data when $f(x)=\sin x$, see for example, Nagayev and Goldfield (1989) and Naeem (2008) see also Naeem (2015a). The random variable R_n is called Extreme-spacings statistic when $f(x)=\{1(x\le\alpha_n)+1(x\ge\beta_n)\}$ where $1(A)$ is indicator function, α_n and β_n are constants see, for example, Jammalamadaka and Wells (1988) and Mirakhmedov and Naeem (2008b). A number of statistical problems related to the distribution of $f(x)=\{1(x\le\alpha_n)+1(x\ge\beta_n)\}$ have been discussed in Darling (1953).

The research articles (cf. Darling (1953), Kale (1969) and Dishevels (1985)) provide a unified treatment to the distributions of random variable R_n. We refer to Mirakhmedov and Klanderov (1998) and Mirakhmedov (2005), where estimates of the remainder term in the Central Limit Theorem (CLT) for the random variable R_n are obtained. Also it is worth noticing that the probability of large deviations of random variable R_n, a problem less investigated earlier, is proved by Mirakhmedov (2006). Sometimes the exact distribution of the random variable in tractable form is not available. Even if it exists often its rate of convergence to the normal form is very slow see for example Does et al. (1988). That is the reason the researchers have shown considerable interest into the asymptotic distribution theory for the statistics of type (1.1). In the course of approximation Edgeworth type expansion is a better choice as it involves the first four moments of the statistics while in the normal approximation only the mean and variance play role. The error is controlled in the Edgeworth approximation which gives it an edge and is considered as a true asymptotic expansion. The authors of (Bhattacharya and Ghosh (1978)) have proved the validity of formal Edgeworth expansion under suitable assumptions. For these reasons, the distribution function of (1.1) may be approximated through Edgeworth series comfortably and one can get better results. Some authors calculated Edgeworth expansion of spacing statistics for small to moderate sample sizes; see for example, Ghosh and Jammalamadaka (1998). The Edgeworth series approximation for large sample sizes is available in many articles see for example Kallenberg (1993) and Naeem (2015 a,b). In their paper Does et al. (1987) have established Edgeworth expansions for statistics (1.1) under a natural moment assumption and an appropriate version of Cramer- type condition. They have shown that a Cramer-type condition holds under an easily verifiable and mild assumption on the function f: if $(c,d)\subset(0,\infty)$, is an open interval on which f is almost everywhere differentiable and the derivative of f is not essentially constant on the prescribed interval then the Cramer-type condition is satisfied. By using the characterization of Does et al. (1987), we aim to find the Edgeworth type expansion for random variable R_n when $f(x)=\{1(x\le\alpha_n)+1(x\ge\beta_n)\}$ we also aim to compare our result with nother well-known Edgeworth expansion which is based on first four moments, see for example, Ghosh and Jammalamadaka (1998). The make-up of the paper is, in section (Asymptotic Normality) we discuss the limit theorem for our statistics, in section (Extreme Spacings Statistics) we discuss our statistics, in section (Edgeworth type Expansion) we formulate our theorem, in section (Proof of Theorem-1) we will prove Theorem-1 which is the main result of this article finally we reproduce the proposition of Ghosh and Jammalamadaka (1998). The results obtained from Theorem-1 and mentioned proposition are tabulated and graphed for the purpose of comparison.

2. ASYMPTOTIC NORMALITY

The asymptotic normality and Cramer's type large deviation theorem for statistics of type (1.1) have been obtained by Mirakhmedov (2005) and Mirakhmedov et al. (2007) for the sum of functions of uniform spacings (i.e. under null hypothesis). Note that under alternatives converging to uniform null hypothesis the spacings D_j can be reduced to uniform spacings for details see Mirakhmedov and Naeem (2008a).

Let X_j, $j=1,2,...,n+1$ be exponential random variables with expectation 1 and f be a fixed real- valued measurable function defined on R^+ then by the well know characterization of Le Cam (1958) we argue that for $n=1,2,...$ if $W_n=\sum_{j=1}^{n+1} f\left(X_j\right)$ then

$$\mathcal{L}\left(R_n\right)=\mathcal{L}\left(W_n\Big/\sum_{j=1}^{n+1}\left(X_j-1\right)=0\right)$$

where $\mathcal{L}(X)$ means the distribution of random vector X. Consider the moments

$$R_n(X)=\sum_{j=1}^{n+1} f_j\left(X_j\right),\ S_n=X_1+X_2+...+X_n,\ \rho=corr\left(R_n(X),S_n\right),$$

$$g_j(u)=f_j(u)-Ef_j(u)-(u-1)\rho\sqrt{\frac{\operatorname{var} R_n(X)}{n}},\ H_n(D)=\sum_{j=1}^{n+1} g_j\left(nD_j\right),$$

$$H_n(X)=\sum_{j=1}^{n+1} g_j\left(X_j\right),\ \text{and}\ A_n=\sum_{j=1}^{n+1} f_j\left(X_j\right).$$

Note that $E\left(H_n(X)\right)=0$, $\sigma_n^2 \equiv VarR_n(X)=\left(1-\rho^2\right)Var\left(R_n(X)\right)$ and $Cov\left(H_n(X),S_n\right)=0.$ Also it is obvious that $H_n(X)=R_n(X)-E\left(R_n(X)\right).$ Clearly one may consider $H_n(X)$ instead of $R_n(X)$. From the definition of σ_n^2, It is clear that $\sigma_n^2=0$ if and only if $f_j(X)=CX+\lambda_j$, $j=1,2,..,n+1$, where constants λ_j are arbitrary and C does not depend on j for all $j=1,2,..,n+1$. We note that $f_j(x)$ are random functions. In such cases we suppose that $f_1(x), f_2(x),..., f_{n+1}(x)$ is sequence of independent random variables not depending on D or X.

Theorem-2.1:

If $\frac{1}{\sigma_N^3}\sum_{j=1}^{n+1} E\left|g_j\left(X_j\right)\right|^3 \to 0$, as $n\to\infty$, then the random variable R_n has asymptotically normal distribution with expectation A_n and variance σ_n^2. Theorem 2.1 is the corollary 2 of Mirakhmedov (2005).

3. EXTREME SPACINGS STATISTIC

Consider the statistics $V_n\left(\alpha_n,\beta_n\right)$ representing the number of those spacings that satisfy $0<\alpha_n<D_j<\beta_n<1$ for $j=1,2,...,n+1$. Then we define the following random variable

$$V_n\left(\alpha_n,\beta_n\right)=\sum_{j=1}^{n+1} f\left(\left(n+1\right)D_j\right),\quad n=1,2,... \tag{3.1}$$

The statistics given in (3.1) is a special case of (1.1) with kernel function $f(x)=\left\{1\left(x\le\alpha_n\right)+1\left(x\ge\beta_n\right)\right\}$ when $x\in\left(\alpha_n,\beta_n\right)$ and $f(x)=0$ elsewhere. Now we fix and write $\alpha_n=\frac{\alpha}{n}$ and $\beta_n=\frac{\beta}{n}$. The statistic (3.1) is further specialized by these fixed α and β and will be represented by $V_n(\alpha,\beta)$ withi**n** rest of the article this specialized random variable is considered. Therefore, by Theorem 2.1 we get

Theorem 3.1:

The Statistics $V_n(\alpha,\beta)$ has asymptotically normal distribution with expectation nA and variance $n\sigma^2$ as $n\to\infty$, where $A=1-e^{-\alpha}+e^{-\beta}$ and $\sigma^2=\left(e^{-\alpha}-e^{-\beta}\right)-\left(e^{-\alpha}-e^{-\beta}\right)^2-\left(\alpha e^{-\alpha}-\beta e^{-\beta}\right)^2$.

Proof:

It is obvious that $E\left(f^2(x)\right)<\infty$, so by Theorem 2.1 statistic $V_n(\alpha,\beta)$ is asymptotically normal with parameters $nE\left(f(x)\right)$ and $n\left(1-\rho^2\right)\mathrm{var}\left(f(x)\right)$ where ρ is the correlation between $f(x)$ and x. By direct calculations it is easy to find $A = E(f(x))$, $(1-\rho^2)$ and $\text{var}(f(x))$.

The asymptotic normality for more general form of $V_n(\alpha,\beta)$ is discussed in our earlier papers of which the above stated Theorem 3.1 is just a special case, see for example, Mirakhmedov and Naeem (2008 a,b). The paper by Jammalamadaka and Wells (1988) has also shown the asymptotic normality of $V_n(\alpha,\beta)$ in details.

4. EDGEWORTH TYPE EXPANSION

Different authors derived asymptotic expansions of spacing statistics of type (1.1) under different conditions see, for example, Kallenberg (1993) and Ghosh & Jammalamadaka (1998). In particular, we refer to Does et al. (1987) in which the authors established a general formula for the Edgeworth expansions of spacings based statistics under a natural moment assumption and an appropriate version of Cramer's- type condition. We state two results from Does et al. (1987) that we use to derive the Edgeworth type expansion of $V_n(\alpha,\beta)$. We have

Lemma-1 (Does et al. (1987)):

Let $g:[0,\infty)\to R$ be a non-linear measurable function whose derivative exists and is not necessarily constant on $(c,d)\subset(0,\infty)$ such that $E\left(g^4(X)\right)<\infty$ where X is distributed exponentially with parameter 1. The random variable R_n is the sum of all g functions of norm uniform spacings. If F_n is the distribution function of $\left(R_n - ER_n \big/ \sqrt{\text{var}\, R_n}\right)$ and $\tilde{F}_n$ is the Edgeworth type expansion of R_n then we have $\lim_{n\to\infty} n \sup_{x\in R}\left|F_n(x)-\tilde{F}_n(x)\right| = 0$.

This Lemma forms the basic result for deriving Edgeworth type expansion of spacings statistics of the form (3.1).

Lemma-2 (Does et al. (1987)):

Let x be a random variable taking values in R^m, the distribution of which is absolutely continuous on some Borel set B with $P(X\in B)>0$. Let $h: R^m \to R^k$ be a measurable function which is Lebesgue almost everywhere differentiable on B with the $k\times m$ matrix h' as differential. If all $\gamma\in\left(R^k \setminus \{0\}\right)$ satisfy $P\left\{\left(h'(X)\right)^T \gamma = 0 \big/ X\in B\right\}<1$.

Then the inequality $\underset{|\gamma|\to\infty}{Lim} \sup\left|Ee^{i\gamma^T h(X)}\right|<1$ holds.

This Lemma provides the necessary condition for the application of Lemma-1. We formulate our Theorem as under.

Let $\Phi(x)$ be the standard normal distribution, $\phi(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ and

$$\tilde{F}_n(x)=\Phi(x)-\phi(x)\left[n^{\frac{-1}{2}}\left\{\frac{1}{6}\left(\frac{589}{25000}\right)\left(x^2-1\right)-\frac{829}{1250}\right\}\right.$$

$$+n^{-1}\left\{\frac{1}{24}\left(\frac{12011}{25000}\right)\left(x^3-3x\right)+\left(\frac{1}{72}\right)\left(\frac{-589}{2500}\right)^2\left(x^5-10x^3+15x\right)\right.$$

$$\left.\left.+\frac{1}{8}\left(-4\left(-\frac{589}{2500}\right)\left(\frac{-829}{1250}\right)+\left(\frac{-133721}{5000}\right)\right)x+\left(\frac{1}{6}\left(-\frac{589}{2500}\right)\left(\frac{-829}{1250}\right)\right)x^3\right\}\right]$$

or we can write

$$\tilde{F}_n(x)=\Phi(x)-\phi(x)\left[n^{-\frac{1}{2}}\left\{\frac{589}{15000}\left(1-x^2\right)-\frac{829}{1250}\right\}+n^{-1}\left\{\left(\frac{1001}{5000}\right)\left(x^3-3x\right)\right.\right.$$

$$\left.\left.+\frac{37}{48000}\left(x^5-10x^3+15x\right)+\left(\frac{8553}{2500}\right)x-\frac{521}{20000}x^3\right\}\right] \quad (4.1)$$

Theorem-1:

Let $\nabla_n=n-V_n$ with $\tilde{F}_n(x)$ as in (4.1) while $\mu=E\left(V_n(\alpha,\beta)\right)=1-e^{-\alpha}+e^{-\beta}$ and $\sigma_n^2=\operatorname{var}\left(V_n(\alpha,\beta)\right)=\left(e^{-\alpha}-e^{-\beta}\right)-\left(e^{-\alpha}-e^{-\beta}\right)^2-\left(\alpha e^{-\alpha}-\beta e^{-\beta}\right)^2$. Then we have the following $P\left\{\left(\nabla_n-\mu\right)\sigma^{-1}\le x\right\}=\tilde{F}_n(x)+o\left(\frac{1}{n}\right)$, $x\in R$ as $n\to\infty$.

Proof of Theorem-1:

It is to be noted that all the conditions settled in lemma-1 and Lemma-2 are satisfied by the statistic given in (3.1). Therefore, if $F_n(x)$ is the distribution of $\tilde{V}_n=\left(V_n-EV_n\big/\sqrt{\operatorname{var}V_n}\right)$ and $\tilde{F}_n(x)$ is as in (4.1) then

$$\lim_{n\to\infty} n\sup_{x\in R}\left|F_n(x)-\tilde{F}_n(x)\right|=\mathrm{O}(1).$$

We replace the function $f(x)$ by $\tilde{f}(x)=\left(f(x)-\mu-\tau(x-1)\right)\left(\sigma^2-\tau^2\right)^{-1/2}$ which is merely a sort of centralization and does not affect the distribution of $F_n(x)$. It is well known that the efficiency of the random variable $V_n(\alpha,\beta)$ is maximum when we set $\alpha=0.7355$ and $\beta=4.3205$ see, for example, Jammalamadaka and Wells (1988) and Mirakhmedov and Naeem (2008b). By direct calculation, using fixed values of α and β, we get different parameters as

$$\mu = 267/500,\ \sigma^2 = 311/1250,\ \tau^2 = 871/1000,$$

$$\kappa_3 = E\left(\tilde{f}(x)\right)^3 = \text{-}589/2500,\ a = -\frac{1}{2}E\left(\tilde{f}(x)(x-1)^2\right) = 829/1250,$$

$$\kappa_4 = E\left(\tilde{f}(x)\right)^4 - 3 - 3\left(E\ \tilde{f}^2(x)(x-1)\right)^2 = 12011/2500,$$

$$b = 3\left(E\ \tilde{f}(x)(x-1)^2\right)^2 - 2E\ \tilde{f}^2(x)(x-1)^2 + 4E\ \tilde{f}(x)(x-1)^3 + 6 = 133721/5000,$$

so that the Edgeworth type expansion $\tilde{F}_n(x)$ of function $\tilde{f}(x)$ is as given in (4.1).

Note that $E\left(g^4(X)\right) = 421/10000 < \infty$ so that the first condition of Lemma-1 is satisfied. By taking $m=1$, $k=1$, $h(x) = \left(x, \frac{f(x) + 0.2951x - 0.8291}{0.1617}\right)$ and $B = (0, \infty)$ in Lemma-2 and let $\gamma = (c, d)$ then

$$(h(x))^T.\gamma = \left[x\ \frac{f(x) + 0.2951x - 0.8291}{0.1617}\right]\left[c\ d\right]^T = cx + \frac{f(x) + 0.2951x - 0.8291}{0.1617}d.$$

where X^T means the transpose of X. For $(h(x))^T.\gamma = 0$, three cases arises $(i)\ c = 0,\ d \neq 0$ $(ii)\ c \neq 0,\ d = 0$ $(iii)\ c \neq 0, d \neq 0.$ For all the three possible cases

$$P\left\{(h(x))^T.\gamma = 0 / x \in B\right\} < 1.$$

So if $Q(s,t)$ is the characteristic function of $\left(x, \tilde{f}(x)\right)$ then by lemma-2 $\lim_{(s,t)\to\infty} \sup\left|Q(s,t)\right| < 1$. Hence by lemma-1 $\lim_{n\to\infty} n \sup_{x\in R}\left|F_n(x) - \tilde{F}_n(x)\right| = \mathrm{O}(1)$, where $F_n(x)$ is the distribution of $(\nabla_n - \mu)\sigma^{-1}$ that is

$$P\left\{(\nabla_n - \mu)\sigma^{-1} \leq x\right\} = \tilde{F}_n(x) + o\left(\frac{1}{n}\right),\quad x \in R \text{ as } n \to \infty.$$

This complete the proof.

5. PROPOSITION OF GHOSH

A different kind of Edgeworth expansion correct to $o\left(\frac{1}{n}\right)$ for any asymptotic normal statistic can be obtained as follows:

Lemma-3 (Ghosh and Jammalamadaka (1998))

Suppose $G_n(x)$ is a statistics with an asymptotic normal distribution and $\tilde{S}_n = \left(G_n - EG_n \big/ \sqrt{\operatorname{var} G_n}\right)$. Let κ_3 and κ_4 be the third and fourth moments respectively such that $\kappa_3 = O\left(n^{-1/2}\right)$, $\kappa_4 - 3 = O\left(n^{-1}\right)$ and higher order cumulants of $\tilde{S}_n$ are $O\left(n^{-a}\right)$, $a > 1$ then we have

$$P\left(\tilde{S}_n \le x\right) = \Phi(x) - \phi(x)\left[\frac{\kappa_3\left(x^2-1\right)}{6} + \frac{(\kappa_4-3)\left(x^3-3x\right)}{24} + \frac{\kappa_3^2\left(x^5-10x^3+15x\right)}{72} + o\left(n^{-1}\right)\right].$$

The result given in Lemma-3 is the Proposition 3.2 of Ghosh and Jammalamadaka (1998). Let us consider again the random variable $\tilde{V}_n = \left(V_n - EV_n \big/ \sqrt{\operatorname{var} V_n}\right)$ then by Lemma-3 the Edgeworth expansion $F_n^*(x)$ of random variable $\tilde{V}_n$ is given by

$$F_n^*(x) = \Phi(x) - \phi(x)\left[\frac{1}{6\sqrt{n}}\left\{\kappa_3\left(x^2-1\right)\right\} + \frac{1}{72n}\left\{3(\kappa_4-3)\right\}\left\{\left(x^3-3x\right) + \kappa_3^2\left(x^5-10x^3+15x\right)\right\} + o\left(n^{-1}\right)\right] \quad (5.1)$$

Again we are using the fixed values of $\alpha = 0.7355$ and $\beta = 4.3205$, we approximate the first four raw moments, skewness and kurtosis given as under

$$\mu_1' = 1 + G_1(\alpha,1) - G_1(\beta,1) = 1 - e^{-\alpha} + e^{-\beta} = 267/500,$$

since $\sigma_n^2 = \operatorname{var} V_n(\alpha,\beta) = \left(e^{-\alpha} - e^{-\beta}\right) - \left(e^{-\alpha} - e^{-\beta}\right)^2 - \left(\alpha e^{-\alpha} - \beta e^{-\beta}\right)^2$ so we get

$$\mu_2' = \sigma^2 + (\mu_1')^2 = 447/1000,$$
$$\mu_3' = E\left(X^3\right) = 467/2000,$$
$$\mu_4' = E\left(X^4\right) = 951/2500.$$

By using the standard relations for skewness and kurtosis, we have obtained the following

$$\kappa_3 = -89/500 \text{ and } \kappa_4 = 161/400$$

where $\Gamma(k)G_k(\alpha,l)=\int_0^x u^{l-1}e^{-u}\,du$ and $k=1,2,3,\ldots$. This reduce (5.1) to

$$F_n^*(x)=\Phi(x)-\phi(x)\left(\frac{10^{-4}}{n}\right)\left[\frac{\sqrt{n}}{297}\left(1-x^2\right)+1082\left(x^3-3x\right)+8\left(x^5-10x^3+15x\right)\right] \tag{5.2}$$

The two Edgeworth expansions i.e. $\tilde{F}_n(x)$ and $F_n^*(x)$ are calculated using Mathematica for $n=10,20,30,50,70,100 \text{ and } 300$ in the region $|x|\le 3$ and the values are tabulated as under.

Table 1
Values of $\tilde{F}_n(x)$ and $F_n^*(x)$ for Different "*n*"

x	-3	-2.5	-2	-1.5	-1	-.5	0	.5	1	1.5	2	2.5	3
$\tilde{F}_{10}$	.008	.003	.076	.161	.284	.430	.579	.712	.818	.897	.950	.980	.994
F_{10}^*	.001	.005	.023	.070	.164	.312	.496	.684	.836	.933	.980	.996	1.00
$\tilde{F}_{20}$	.005	.019	.053	.120	.232	.384	.556	.716	.840	.921	.966	.988	.997
F_{20}^*	.001	.006	.024	.069	.162	.310	.497	.688	.838	.933	.979	.995	1.00
$\tilde{F}_{30}$	.004	.015	.044	.106	.213	.367	.545	.716	.846	.928	.971	.991	.998
F_{30}^*	.001	.006	.024	.068	.161	.309	.498	.689	.839	.933	.978	.995	.999
$\tilde{F}_{50}$	.003	.012	.037	.093	.196	.351	.535	.713	.849	.933	.975	.992	.998
F_{50}^*	,001	.006	.023	.068	.160	.309	.498	.690	.840	.933	.978	.995	.999
$\tilde{F}_{70}$	.002	.011	.035	.087	.189	.343	.530	.711	.850	.935	.976	.993	.999
F_{70}^*	.001	.006	.023	.068	.160	.309	.499	.690	.840	.933	.978	.995	.999
$\tilde{F}_{100}$	.002	.009	.031	.083	.182	.336	.525	.709	.850	.936	.977	.994	.999
F_{100}^*	.001	.006	.023	.068	.160	.309	.499	.691	.840	.933	.978	.994	.999
$\tilde{F}_{150}$	.002	.009	.027	.079	.177	.331	.520	.707	.849	.936	.978	.994	.999
F_{150}^*	.001	.006	.023	.068	.159	.309	.499	.691	.841	.933	.977	.994	.999
$\tilde{F}_{250}$	.001	.008	.027	.075	.172	.325	.516	.704	.848	.936	.978	.994	.999
F_{250}^*	.001	.006	.023	.067	.159	.309	.499	.691	.841	.933	.977	.994	.999
$\tilde{F}_{300}$	.001	.008	.027	.075	.171	.324	.514	.703	.848	.936	.978	.994	.999
F_{300}^*	.001	.006	.023	.067	.159	.309	.499	.691	.841	.933	.977	.994	.999
Φ	.001	.006	.023	.067	.159	.309	.500	.692	.841	.933	.977	.994	.999

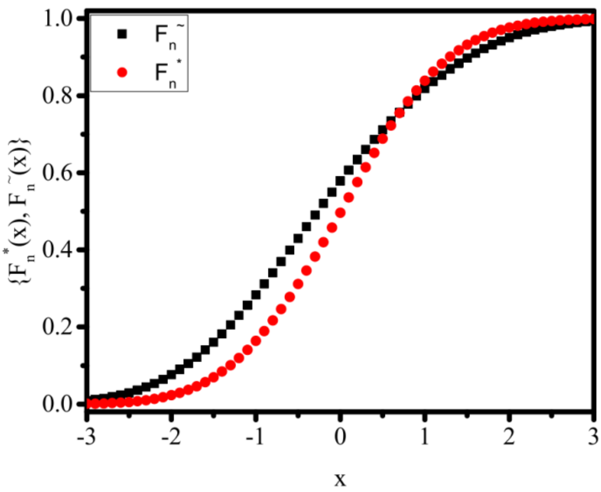

Figure 1: The Profiles of F_n^* and $\tilde{F}_n$ versus *x*-axis

CONCLUSION

From the table as well as graph we observe that the Edgeworth expansion given in the proposition 3.2 of Ghosh and Jammalamadaka (1998) perform better than the one proposed by Does et al. (1987). Perhaps, because of the conditions used by Does et al. (1987) are not strong enough. While in the former expansion the exact first four moments are utilized. Even then later expansion is efficient enough and its importance cannot be ignored.

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