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THE MONOTONIC PROPERTIES OF (p, a)-GENERALIZED SIGMOID FUNCTION WITH APPLICATION

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ABSTRACT

In this paper, we define a new (p, a)-generalized sigmoid function, and prove some analytic properties. By using analytical techniques, several inequalities involving the generalized function are established. Some of these inequalities connect the generalized sigmoid function to the softplus function. Finally, some statistical properties of the generalized sigmoid function are studied.

KEY WORDS

Generalized sigmoid function; Inequality; Statistical properties.

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1. INTRODUCTION

The classical sigmoid function, which is universally known as the standard logistic function is defined by

$$S(x) = \frac{e^x}{1 + e^x} = \frac{1}{1 + e^{-x}} \tag{1.1}$$

$$= \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2}\right), x \in (-\infty, +\infty) \tag{1.2}$$

The sigmoid function plays an important role in many scientific disciplines including probability and statistics, biology, demography, machine learning, population dynamics, ecology, and mathematical psychology (see Barry (2017), Costarelli and Spigler (2013a), Kyurkchiev (2016), Kyurkchiev and Markov (2016)).

Especially, in the business field, the function has been used to study performance growth in manufacturing and service management (see Jonas (2007)). As well, in the artificial neural networks, the function plays as an activation function at the output of each neuron (see Basterretxea, Tarela and Del Campo (2004), Centin, Temurtas and Gulgonul (2015), Chen, Cao and Hu (2015), Chen and Cao (2009), Costarelli and Spigler (2013b), Elliott (1993), Guliyev and Ismailov (2016), Minai and Williams (1993)). And in the artificial neural networks sometimes non-smooth functions are used instead for efficiency, which are known as hard sigmoids (see Yingxin Guo (2010), Yingxin Guo (2013), Yingxin Guo (2009)). Another well-known application is in the medicine field, where the function is applied to study pharmacokinetic reactions or model the growth of tumors (see Kyurkchiev (2016)). It is also utilized in forestry. For instance, in Coble and Lee (2006) a general form of the function is applied to forecast the site index of unmanaged loblolly and slash pine plantations in East Texas. Moreover, it has been played an important role in computer graphics or image processing to enhance image contrast (see Cyganek and Socha (2012), Hassan and Akamatsu (2004)). Recently, many statisticians have paid their attention to novel produced families of continuous distributions to expand fresh models (see Almarashi et al., (2019), Kyurkchiev, Iliev and Markov (2017), and the references therein).

The important roles of the function make its properties be interesting and meaningful. In the recent work Ezeafulukwe, Darus and Abidemi (2018), the authors studied some analytic properties of the function such as convexity and starlikeness in a unit disc. In Nantomah (2019), the author studied the properties such as subadditivity, inequalities, convexity and super multiplicativity of the sigmoid function.

In this paper, the subadditivity, inequalities, convexity and super multiplicativity as well as other properties related to (p, a)-generalized sigmoid function are studied. Meanwhile, the two-parameter sigmoid function can be considered as a distribution function and some generalized statistical properties of the distribution are discussed.

2. MAIN RESULTS

The (p, a)-generalized sigmoid function is defined as

$$S_{p,a}(x) = \frac{\mathrm{e}^{p(x-a)}}{1+\mathrm{e}^{p(x-a)}} = \frac{1}{1+\mathrm{e}^{-p(x-a)}} \tag{2.1}$$

$$= \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{p(x-a)}{2}\right), x \in (-\infty, +\infty). \tag{2.2}$$

where $p > 0$ and $a \in (-\infty, +\infty)$ are real numbers. The first and second derivatives of the general sigmoid function are given as

$$S'_{p,a}(x) = \frac{p\mathrm{e}^{p(x-a)}}{(1+\mathrm{e}^{p(x-a)})^2} = pS_{p,a}(x)\,\left(1 - S_{p,a}(x)\right), \tag{2.3}$$

$$S''_{p,a}(x) = \frac{p^2\mathrm{e}^{p(x-a)}\left(1-\mathrm{e}^{p(x-a)}\right)}{(1+\mathrm{e}^{p(x-a)})^3} \tag{2.4}$$

$$= p^2 S_{p,a}(x)\,\left(1 - S_{p,a}(x)\right)\left(1 - 2S_{p,a}(x)\right), \tag{2.5}$$

for $x \in (-\infty, +\infty)$.

In view of (2.3), $S_{p,a}(x)$ is positive and increasing on $(-\infty, +\infty)$ and $y = S_{p,a}(x)$ is a solution to the initial value problem $\begin{cases} \frac{dy}{dx} = py(1-y), \\ y(a) = 0.5. \end{cases}$

In view of (2.4), $S'_{p,a}(x)$ is decreasing on $(a, +\infty)$ and increasing on $(-\infty, a)$. Moreover, the (p, a)-generalized sigmoid function have the following features.

$$\lim_{x\to+\infty} S_{p,a}(x) = 1, \tag{2.6}$$

$$\lim_{x\to a} S_{p,a}(x) = \frac{1}{2}, \tag{2.7}$$

$$\lim_{x\to-\infty} S_{p,a}(x) = 0, \tag{2.8}$$

$$\lim_{x\to-\infty} S'_{p,a}(x) = 0, \tag{2.9}$$

$$\lim_{x\to a} S'_{p,a}(x) = \frac{p}{4}, \tag{2.10}$$

$$\int S_{p,a}(x)dx = \frac{1}{p}\log\left(1 + \mathrm{e}^{p(x-a)}\right) + C \tag{2.11}$$

with C is an integration constant. The function $\frac{1}{p}\log\left(1+\mathrm{e}^{p(x-a)}\right)$ is defined as the (p,a)-general softplus function. The derivative of (2.11) gives the (p,a)-generalized sigmoid function.

Theorem 2.1:

For $p > 0$ and $a \in [0, +\infty)$, the (p,a)-generalized sigmoid function $S_{p,a}(x)$ is sub-additive on $(-\infty, 0) \cup (a, +\infty)$. Namely, the function satisfies the inequality

$$S_{p,a}(x+y) < S_{p,a}(x) + S_{p,a}(y) \text{ , for } x, y \in (-\infty, 0) \text{ and } x, y \in (a, +\infty). \tag{2.12}$$

Proof:

We need to prove the case $x, y \in (-\infty, 0)$ and the case $x, y \in (a, +\infty)$ respectively.

For $p > 0$ and $a \in (0, +\infty)$, let

$$\mathcal{A}(x,y) = S_{p,a}(x+y) - S_{p,a}(x) - S_{p,a}(y) \tag{2.13}$$

$$= \frac{\mathrm{e}^{p(x+y-a)}}{1+\mathrm{e}^{p(x+y-a)}} - \frac{\mathrm{e}^{p(x-a)}}{1+\mathrm{e}^{p(x-a)}} - \frac{\mathrm{e}^{p(y-a)}}{1+\mathrm{e}^{p(y-a)}}. \tag{2.14}$$

For any fixed y, we obtain

$$\frac{\partial}{\partial x}\mathcal{A}(x,y) = \frac{p\mathrm{e}^{p(x+y-a)}}{(1+\mathrm{e}^{p(x+y-a)})^2} - \frac{p\mathrm{e}^{p(x-a)}}{(1+\mathrm{e}^{p(x-a)})^2}. \tag{2.15}$$

For $S'_{p,a}(x)$ is decreasing on $(a, +\infty)$, hence $\mathcal{A}(x,y)$ is decreasing on $(a, +\infty)$. Then for $x, y \in (a, +\infty)$ we can have

$$\mathcal{A}(x,y) < \mathcal{A}(x,a) = \lim_{x\to a}\mathcal{A}(x,a) < A(a,a) = -\frac{1}{1+\mathrm{e}^{pa}} < 0. \tag{2.16}$$

For $S'_{p,a}(x)$ is increasing on $(-\infty, a)$ and $a > 0$, then $\mathcal{A}(x,y)$ is increasing on $(-\infty, 0)$. Hence on $(-\infty, 0)$ we can have

$$\mathcal{A}(x,y) < \mathcal{A}(x,0) = \lim_{x\to 0}\mathcal{A}(x,0) < A(0,0) = -\frac{1}{1+\mathrm{e}^{pa}} < 0. \tag{2.17}$$

Remark 1:

In Theorem (2.1), if $a = 0$, then $S_{p,a}(x)$ is sub-additive on $(-\infty, +\infty)$.

Theorem 2.2:

For $p>0$, $a\in(-\infty,+\infty)$ and $b\in(0,+\infty)$, the (p,a)-generalized sigmoid function $S_{p,a}(x)$ satisfies the following inequalities.

$$1<\frac{S_{p,a}(x+b)}{S_{p,a}(x)}<\mathrm{e}^{pb},x\in(-\infty,+\infty), \tag{2.18}$$

$$\frac{2\mathrm{e}^{pb}}{1+\mathrm{e}^{pb}}<\frac{S_{p,a}(x+b)}{S_{p,a}(x)}<\mathrm{e}^{pb},x\in(-\infty,a), \tag{2.19}$$

$$1<\frac{S_{p,a}(x+b)}{S_{p,a}(x)}<\frac{2\mathrm{e}^{pb}}{1+\mathrm{e}^{pb}},x\in(a,+\infty). \tag{2.20}$$

Proof:

Since

$$1<\left(\frac{S_{p,a}^{'}(x)}{S_{p,a}(x)}\right)^{'}<-\frac{p^2\mathrm{e}^{p(x-a)}}{(1+\mathrm{e}^{p(x-a)})^2}<0,x\in(-\infty,+\infty). \tag{2.21}$$

Hence, the function $\frac{S_{p,a}^{'}(x)}{S_{p,a}(x)}$ is decreasing on $(-\infty,+\infty)$. Let

$$\mathcal{B}(x)=\frac{S_{p,a}(x+b)}{S_{p,a}(x)},x\in(-\infty,+\infty)$$

and $v(x)=\log\mathcal{B}(x)$. So

$$v^{'}(x)=\frac{S_{p,a}^{'}(x+b)}{S_{p,a}(x+b)}-\frac{S_{p,a}^{'}(x)}{S_{p,a}(x)}<0, \tag{2.22}$$

Thus $v(x)$ and $\mathcal{B}(x)$ are decreasing. Hence,

$$1=\lim_{x\to+\infty}\mathcal{B}(x)<\mathcal{B}<\lim_{x\to-\infty}\mathcal{B}(x)=\mathrm{e}^{pb},x\in(-\infty,+\infty). \tag{2.23}$$

$$\frac{2\mathrm{e}^{pb}}{1+\mathrm{e}^{pb}}=\lim_{x\to a}\mathcal{B}(x)<\mathcal{B}(x)<\lim_{x\to-\infty}\mathcal{B}(x)=\mathrm{e}^{pb},x\in(-\infty,a). \tag{2.24}$$

$$1=\lim_{x\to+\infty}\mathcal{B}(x)<\mathcal{B}(x)<\lim_{x\to a}\mathcal{B}(x)=\frac{2\mathrm{e}^{pb}}{1+\mathrm{e}^{pb}},x\in(a,+\infty). \tag{2.25}$$

Corollary 2.3:

For $p>0$ and $a\in(-\infty,+\infty)$, the (p,a)-generalized sigmoid function $S_{p,a}(x)$ satisfies the following inequalities.

$$1<\frac{S_{p,a}\left(x+\frac{1}{p}\right)}{S_{p,a}(x)}<\mathrm{e},x\in(-\infty,+\infty),$$

$$\frac{2\mathrm{e}}{1+\mathrm{e}}<\frac{S_{p,a}\left(x+\frac{1}{p}\right)}{S_{p,a}(x)}<\mathrm{e},x\in(-\infty,a),$$

$$1 < \frac{S_{p,a}\left(x + \frac{1}{p}\right)}{S_{p,a}(x)} < \frac{2\mathrm{e}}{1 + \mathrm{e}}, x \in (a, +\infty).$$

Theorem 2.4:

For $p > 0$ and $a \in [0, +\infty)$, the (p, a)-generalized sigmoid function $S_{p,a}(x)$ is AH-concave on $(a, +\infty)$. Namely,

$$S_{p,a}\left(\frac{x+y}{2}\right) \geq \frac{2S_{p,a}(x)S_{p,a}(y)}{S_{p,a}(x) + S_{p,a}(y)}, x \in (a, +\infty). \tag{2.26}$$

Proof:

For

$$\left(\frac{S'_{p,a}(x)}{S^2_{p,a}(x)}\right)' = -p^2\mathrm{e}^{-p(x-a)} < 0,$$

by Corollary 2.5 in Anderson, Vamanamurthy and Vuorinen (2007) the desired result is proved.

Theorem 2.5:

For $p > 0$ and $a \in (-\infty, +\infty)$, the (p, a)-generalized function $S_{p,a}(x)$ is logarithmically concave on $(-\infty, +\infty)$. Namely, for all $x, y \in (-\infty, +\infty)$, $r > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$, the following inequality holds.

$$S_{p,a}\left(\frac{x}{r} + \frac{y}{s}\right) \geq \left[S_{p,a}(x)\right]^{\frac{1}{r}}\left[S_{p,a}(y)\right]^{\frac{1}{s}}. \tag{2.27}$$

Proof:

Let $D(x) = \log S_{p,a}(x) = p(x-a) - \log(1 + \mathrm{e}^{p(x-a)})$, then

$$D''_{p,a}(x) = -\frac{p^2\mathrm{e}^{p(x-a)}}{(1 + \mathrm{e}^{p(x-a)})^2} < 0, \tag{2.28}$$

which indicates the inequality (2.27) is right.

Corollary 2.6:

For $p > 0$ and $a \in (-\infty, +\infty)$, the inequalities hold.

$$S''_{p,a}(x)S_{p,a}(x) - \left(S'_{p,a}(x)\right)^2 \leq 0, x \in (-\infty, +\infty), \tag{2.29}$$

$$S_{p,a}(a+u)S_{p,a}(a-u) \leq \frac{1}{4}, u \in (-\infty, +\infty). \tag{2.30}$$

Proof:

Inequality (2.29) can be deduced from the logarithmic concavity of $S_{p,a}(x)$. Let $r = s = 2$, $x = a - u$ and $y = a - u$ in (2.27), which yields (2.30) is right.

Theorem 2.7:

For $p > 0$ and $a \in (-\infty, +\infty)$, the (p, a)-generalized sigmoid function $S_{p,a}(x)$ holds the following inequalities.

$$S_{p,a}^2(x+y) \le S_{p,a}(x)S_{p,a}(y), x, y \in [0, +\infty), \tag{2.31}$$

$$S_{p,a}^2(x+y) \ge S_{p,a}(x)S_{p,a}(y), x, y \in (-\infty, 0]. \tag{2.32}$$

Equality holds if $x = y = 0$.

Proof:

For $x, y \in [0, +\infty)$, thus $x + y \ge x$ and $x + y \ge y$. Since $S_{p,a}(x)$ is increasing, hence

$$S_{p,a}(x+y) \ge S_{p,a}(x) > 0, \tag{2.33}$$

$$S_{p,a}(x+y) \ge S_{p,a}(y) > 0. \tag{2.34}$$

Product (2.33) and (2.34), which demonstrates inequality (2.31) is right. Using the similar method, (2.32) can be proved.

Theorem 2.8:

For $p > 0$ and $a \in (-\infty, +\infty)$, the (p, a)-generalized sigmoid function $S_{p,a}(x)$ satisfies the following inequalities.

$$S_{p,a}^2(xy) \le S_{p,a}(x)S_{p,a}(y), x, y \in (0,1], \tag{2.35}$$

$$S_{p,a}^2(xy) \ge S_{p,a}(x)S_{p,a}(y), x, y \in [1, +\infty). \tag{2.36}$$

Equality holds if $x = y = 1$.

Proof:

For $x, y \in (0,1]$, thus $xy \le x$ and $xy \le y$. As $S_{p,a}(x)$ is increasing, then

$$S_{p,a}(x) \ge S_{p,a}(xy) > 0, \tag{2.37}$$

$$S_{p,a}(y) \ge S_{p,a}(xy) > 0. \tag{2.38}$$

Product (2.37) and (2.38), which indicates inequality (2.35) holds. Using the similar method, we can get (2.36).

Theorem 2.9:

For $p > 0$ and $a \in (-\infty, +\infty)$, the (p, a)-generalized sigmoid function $S_{p,a}(x)$ is supermultiplicative on $(1, +\infty)$. Namely,

$$S_{p,a}(xy) > S_{p,a}(x)S_{p,a}(y), x, y \in (-\infty, +\infty).$$

Proof:

For $0 < S_{p,a}(u) < 1$, then $S_{p,a}^2(u) < S_{p,a}(u)$ on $u \in (-\infty, +\infty)$. As $S_{p,a}(x)$ is increasing, and $xy > x, xy > y$ on $(1, +\infty)$. Hence, $S_{p,a}(xy) > S_{p,a}^2(xy) > S_{p,a}(x)S_{p,a}(y)$ on $(1, +\infty)$.

In the following, some sharp inequalities connecting the (p, a)-generalized sigmoid and the softplus functions are studied.

Theorem 2.10:

For $p > 0$ and $a \in (-\infty, +\infty)$, the following inequalities hold.

$$\frac{e^{p(x-a)}}{1+e^{p(x-a)}} < \log\left(1+e^{p(x-a)}\right) < \log 2 - \frac{1}{2} + \frac{e^{p(x-a)}}{1+e^{p(x-a)}}, x \in (-\infty, a), \tag{2.39}$$

$$\log 2 - \frac{1}{2} + \frac{e^{p(x-a)}}{1+e^{p(x-a)}} < \log\left(1+e^{p(x-a)}\right), x \in (a, +\infty), \tag{2.40}$$

$$\frac{e^{p(x-a)}}{1+e^{p(x-a)}} < \log\left(1+e^{p(x-a)}\right), x \in (-\infty, +\infty). \tag{2.41}$$

Proof:

Let

$$G(x) = \log\left(1+e^{p(x-a)}\right) - \frac{e^{p(x-a)}}{1+e^{p(x-a)}}, x \in (-\infty, +\infty), \tag{2.42}$$

Hence, for

$$G'(x) = \frac{pe^{p(x-a)}}{1+e^{p(x-a)}}\left(1 - \frac{1}{1+e^{p(x-a)}}\right) = p\left(\frac{e^{p(x-a)}}{1+e^{p(x-a)}}\right)^2 > 0 \\ > 0, x \in (-\infty, +\infty). \tag{2.43}$$

So $G(x)$ is increasing on $x \in (-\infty, +\infty)$. Hence for $x \in (-\infty, a)$,

$$0 = \lim_{x\to-\infty} G(x) < G(x) < \lim_{x\to a} G(x) = \log 2 - \frac{1}{2}, \tag{2.44}$$

which illustrates inequality (2.39) holds. For $x \in (a, +\infty)$,

$$\log 2 - \frac{1}{2} = \lim_{x\to a} G(x) < G(x) < \lim_{x\to+\infty} G(x) < +\infty, \tag{2.45}$$

which indicates inequality (2.40) is right. For $x \in (-\infty, +\infty)$,

$$0 = \lim_{x\to-\infty} G(x) < G(x) < \lim_{x\to+\infty} G(x) < +\infty, \tag{2.46}$$

which demonstrates inequality (2.41) satisfies.

Theorem 2.11:

For $p > 0$, $a \in (-\infty, +\infty)$ and $x \in (-\infty, +\infty)$, the inequality

$$e^{p(x-a)} - \log\left(1+e^{p(x-a)}\right) > 0 \tag{2.47}$$

holds.

Proof:

Let

$$W(x) = e^{p(x-a)} - \log\left(1+e^{p(x-a)}\right), x \in (-\infty, +\infty), \tag{2.48}$$

and

$$W'(x) = pe^{p(x-a)}\left(1 - \frac{1}{1+e^{p(x-a)}}\right) = \frac{pe^{2p(x-a)}}{1+e^{p(x-a)}} > 0, \tag{2.49}$$

which indicates $W(x)$ is increasing on $(-\infty, +\infty)$. Hence, we get

$$0 = \lim_{x\to-\infty} W(x) < W(x) < \lim_{x\to+\infty}\left[\mathrm{e}^{p(x-a)} - \log\left(1 + \mathrm{e}^{p(x-a)}\right)\right], \tag{2.50}$$

which demonstrates inequality (2.47) is right.

Theorem 2.12:

For $p > 0$, $a \in (-\infty, \infty)$ and $x \in (-\infty, \infty)$, then $q(x) = \left(1 + \mathrm{e}^{p(x-a)}\right)^{\frac{1}{\mathrm{e}^{p(x-a)}}}$ is decreasing, $k(x) = \left(1 + \mathrm{e}^{p(x-a)}\right)^{1+\frac{1}{\mathrm{e}^{p(x-a)}}}$ is increasing and the following inequalities hold.

$$(\log 2)\mathrm{e}^{p(x-a)} < \log\left(1 + \mathrm{e}^{p(x-a)}\right) < \mathrm{e}^{p(x-a)}, x \in (-\infty, a), \tag{2.51}$$

$$\frac{\mathrm{e}^{p(x-a)}}{1 + \mathrm{e}^{p(x-a)}} < \log\left(1 + \mathrm{e}^{p(x-a)}\right) < (2\log 2)\frac{\mathrm{e}^{p(x-a)}}{1 + \mathrm{e}^{p(x-a)}}, x \in (-\infty, a), \tag{2.52}$$

$$\frac{\mathrm{e}^{p(x-a)}}{1 + \mathrm{e}^{p(x-a)}} < \log\left(1 + \mathrm{e}^{p(x-a)}\right) < \mathrm{e}^{p(x-a)}, x \in (-\infty, +\infty). \tag{2.53}$$

Proof:

For $(-\infty, +\infty)$, let

$$J(x) = \log q(x) = \frac{\log\left(1 + \mathrm{e}^{p(x-a)}\right)}{\mathrm{e}^{p(x-a)}},$$

and

$$K(x) = \log k(x) = \left(1 + \frac{1}{\mathrm{e}^{p(x-a)}}\right)\log\left(1 + \mathrm{e}^{p(x-a)}\right).$$

In view of (2.41),

$$J'(x) = \frac{p}{1 + \mathrm{e}^{p(x-a)}} - \frac{p\log\left(1 + \mathrm{e}^{p(x-a)}\right)}{\mathrm{e}^{p(x-a)}} = \frac{p}{\mathrm{e}^{p(x-a)}}\left(\frac{\mathrm{e}^{p(x-a)}}{1 + \mathrm{e}^{p(x-a)}} - \log\left(1 + \mathrm{e}^{p(x-a)}\right)\right) < 0,$$

So $J(x)$ is decreasing and correspondingly $q(x)$ is decreasing, too.

In view of (2.47),

$$K'(x) = \frac{p}{\mathrm{e}^{p(x-a)}}\left(\mathrm{e}^{p(x-a)} - \log\left(1 + \mathrm{e}^{p(x-a)}\right)\right) > 0,$$

Hence $K(x)$ is increasing and correspondingly $k(x)$ is increasing, too. Furthermore,

$$J(a) = \log 2, K(a) = 2\log 2, \tag{2.54}$$

$$\lim_{x\to-\infty} J(x) = \lim_{x\to-\infty}\frac{\log\left(1 + \mathrm{e}^{p(x-a)}\right)}{\mathrm{e}^{p(x-a)}} = \lim_{x\to-\infty}\frac{1}{1 + \mathrm{e}^{p(x-a)}} = 1, \tag{2.55}$$

$$\lim_{x\to+\infty} J(x) = \lim_{x\to+\infty}\frac{\log\left(1 + \mathrm{e}^{p(x-a)}\right)}{\mathrm{e}^{p(x-a)}} = \lim_{x\to+\infty}\frac{1}{1 + \mathrm{e}^{p(x-a)}} = 0, \tag{2.56}$$

$$\lim_{x\to-\infty} K(x) = \lim_{x\to-\infty} \frac{\log\left(1+\mathrm{e}^{p(x-a)}\right)}{\frac{\mathrm{e}^{p(x-a)}}{1+\mathrm{e}^{p(x-a)}}} = \lim_{x\to-\infty}\left(1+\mathrm{e}^{p(x-a)}\right) = 1, \tag{2.57}$$

$$\lim_{x\to+\infty} K(x) = \lim_{x\to+\infty} \frac{\log\left(1+\mathrm{e}^{p(x-a)}\right)}{\frac{\mathrm{e}^{p(x-a)}}{1+\mathrm{e}^{p(x-a)}}} = \lim_{x\to+\infty}\left(1+\mathrm{e}^{p(x-a)}\right) = +\infty. \tag{2.58}$$

For $(-\infty, a)$, we obtain

$$\log 2 = J(a) < J(x) < \lim_{x\to-\infty} J(x) = 1,$$

which indicates inequality (2.51) holds. For $x \in (-\infty, a)$, we obtain

$$1 = \lim_{x\to-\infty} K(x) < K(x) < K(a) = 2\log 2,$$

which illustrates inequality (2.52) is right. For $x \in (-\infty, +\infty)$, we obtain

$$0 = \lim_{x\to+\infty} J(x) < J(x) < \lim_{x\to-\infty} J(x) = 1,$$

which demonstrates

$$\log\left(1+\mathrm{e}^{p(x-a)}\right) < \mathrm{e}^{p(x-a)}.$$

As well, we get

$$1 = \lim_{x\to-\infty} K(x) < K(x) < \lim_{x\to+\infty} K(x) = +\infty, \tag{2.59}$$

which again indicates

$$\frac{\mathrm{e}^{p(x-a)}}{1+\mathrm{e}^{p(x-a)}} < \log\left(1+\mathrm{e}^{p(x-a)}\right). \tag{2.60}$$

By (2.59) and (2.61), the inequality (2.53) is proved.

Theorem 2.13:

For $p > 0$ and $a \in (-\infty, +\infty)$, set

$$\mathcal{F}(x) = \frac{\mathrm{e}^{p(x-a)}\log\left(1+\mathrm{e}^{p(x-a)}\right)}{\mathrm{e}^{p(x-a)} - \log\left(1+\mathrm{e}^{p(x-a)}\right)}, x \in (-\infty, a). \tag{2.61}$$

Then, $\mathcal{F}(x)$ is increasing and satisfies the inequality

$$2 < \frac{\mathrm{e}^{p(x-a)}\log\left(1+\mathrm{e}^{p(x-a)}\right)}{\mathrm{e}^{p(x-a)} - \log\left(1+\mathrm{e}^{p(x-a)}\right)} < \frac{\log 2}{1-\log 2} \approx 2.2588915. \tag{2.62}$$

Proof:

Firstly,

$$\lim_{x\to a} \mathcal{F}(x) = \frac{\log 2}{1-\log 2}, \tag{2.63}$$

and

$$\lim_{x\to-\infty}\mathcal{F}(x)=\lim_{x\to-\infty}\frac{\mathrm{e}^{p(x-a)}\log\left(1+\mathrm{e}^{p(x-a)}\right)}{\mathrm{e}^{p(x-a)}-\log(1+\mathrm{e}^{p(x-a)})}$$
$$=\lim_{x\to-\infty}\frac{p\mathrm{e}^{p(x-a)}\log\left(1+\mathrm{e}^{p(x-a)}\right)+p\frac{\mathrm{e}^{2p(x-a)}}{1+\mathrm{e}^{p(x-a)}}}{p\mathrm{e}^{p(x-a)}-p\frac{\mathrm{e}^{p(x-a)}}{1+\mathrm{e}^{p(x-a)}}}$$
$$=\lim_{x\to-\infty}\frac{\log\left(1+\mathrm{e}^{p(x-a)}\right)+\frac{\mathrm{e}^{p(x-a)}}{1+\mathrm{e}^{p(x-a)}}}{1-\frac{1}{1+\mathrm{e}^{p(x-a)}}}$$
$$=\lim_{x\to-\infty}\frac{\frac{\log\left(1+\mathrm{e}^{p(x-a)}\right)}{\mathrm{e}^{p(x-a)}}+\frac{1}{1+\mathrm{e}^{p(x-a)}}}{\frac{1}{1+\mathrm{e}^{p(x-a)}}}=2.$$

Secondly, let

$$N(x)=\mathrm{e}^{p(x-a)}\log\left(1+\mathrm{e}^{p(x-a)}\right)$$

and

$$M(x)=\mathrm{e}^{p(x-a)}-\log\left(1+\mathrm{e}^{p(x-a)}\right),$$

hence, $N(-\infty)=\lim_{x\to-\infty}N(x)=0$ and $M(-\infty)=\lim_{x\to-\infty}M(x)=0$. So,

$$N'(x)=p\mathrm{e}^{p(x-a)}\left[\log\left(1+\mathrm{e}^{p(x-a)}\right)+\frac{\mathrm{e}^{p(x-a)}}{1+\mathrm{e}^{p(x-a)}}\right]>0,$$

and

$$M'(x)=p\mathrm{e}^{p(x-a)}\left[1-\frac{1}{1+\mathrm{e}^{p(x-a)}}\right]=\frac{p\mathrm{e}^{2p(x-a)}}{1+\mathrm{e}^{p(x-a)}}>0.$$

Then

$$\left(\frac{N'(x)}{M'(x)}\right)'=\left(\frac{1+\mathrm{e}^{p(x-a)}}{\mathrm{e}^{p(x-a)}}\log\left(1+\mathrm{e}^{p(x-a)}\right)+1\right)'$$
$$=\frac{p}{\mathrm{e}^{p(x-a)}}\left[\mathrm{e}^{p(x-a)}-\log\left(1+\mathrm{e}^{p(x-a)}\right)\right]>0.$$

Thus $\frac{N'(x)}{M'(x)}$ is increasing, which illustrates $\frac{N(x)}{M(x)}=\mathcal{F}(x)$ is increasing (by the conclusion of Pinelis (2002)).

Finally, for $x\in(-\infty,a)$,

$$2=\lim_{x\to-\infty}\mathcal{F}(x)<F(x)<\lim_{x\to a}\mathcal{F}(x)=\frac{\log 2}{1-\log 2}.$$

Remark 2:

Let $\lambda=\frac{\log 2}{1-\log 2}$, the inequality (2.62) can be written as

$$\log\left(1+\mathrm{e}^{p(x-a)}\right)<\frac{\lambda\mathrm{e}^{p(x-a)}}{\lambda+\mathrm{e}^{p(x-a)}},\qquad(2.64)$$

and by (2.60) the inequality

$$\frac{e^{p(x-a)}}{1+e^{p(x-a)}} < \log\left(1+e^{p(x-a)}\right) < \frac{\lambda e^{p(x-a)}}{\lambda + e^{p(x-a)}} \tag{2.65}$$

is right.

3. THE STATISTICAL PROPERTIES OF THE GENERALIZED SIGMOID FUNCTION

In this section, the (p, a)-generalized sigmoid function

$$F(x,\theta) = S_{p,a}(x) = \frac{e^{p(x-a)}}{1+e^{p(x-a)}}, p > 0, a \in (-\infty, +\infty) \tag{3.1}$$

can be considered as distribution function and the probability density function of the proposed distribution is

$$f(x,\theta) = S'_{p,a}(x) = \frac{e^{p(x-a)}}{(1+e^{p(x-a)})^2}, \tag{3.2}$$

with $\theta = (p, a)$ is the parameter set. The survival function, cumulative hazard rate, hazard rate and reversed hazard rate function of the distribution are defined as the followings

$$\bar{S}(x,\theta) = 1 - F(x,\theta) = \frac{1}{1+e^{p(x-a)}},$$

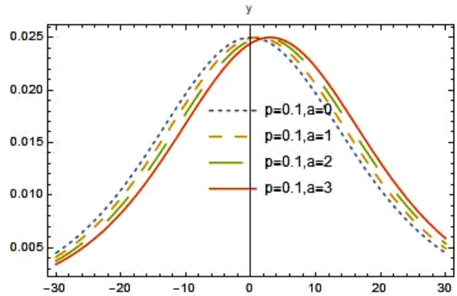
$$\rho(x,\theta) = -\log\left(1 - F(x,\theta)\right) = \log\left(1+e^{p(x-a)}\right),$$

$$h(x,\theta) = \frac{f(x,\theta)}{\bar{S}(x,\theta)} = \frac{pe^{p(x-a)}}{1+e^{p(x-a)}},$$

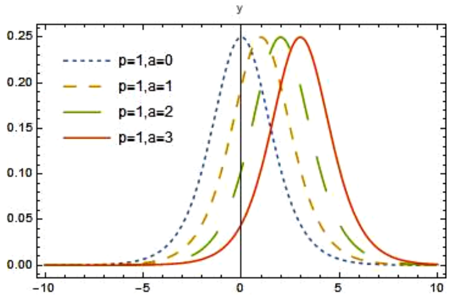
and

$$\tau(x,\theta) = \frac{f(x,\theta)}{F(x,\theta)} = \frac{p}{1+e^{p(x-a)}}.$$

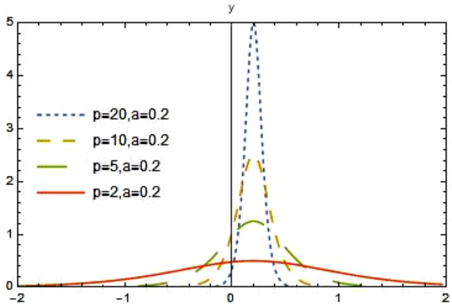
Curves of the probability density function and the hazard rate function of the proposed distribution with some parameter values are respectively showed in Figures 1 and 2. The Figures show that the shapes of the distribution deeply depend on the parameter values. The curves would be uniform to right skewed, which depend on the parameter values.



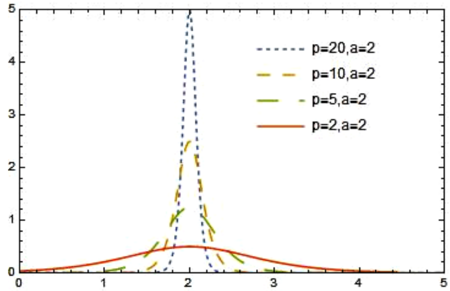
(a) p =0.1



(b) For p =1

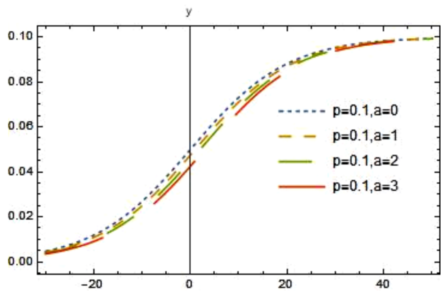


(c) For a =0.2

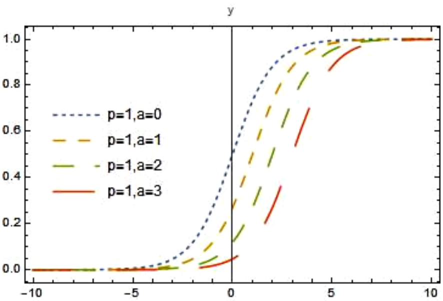


(d) For a =2

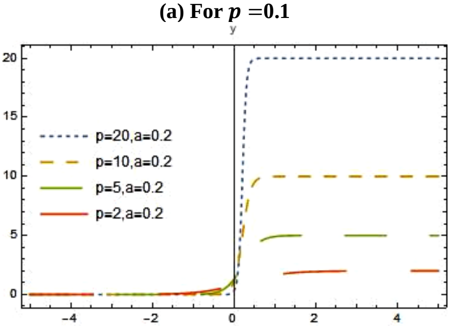
Figure 1: Curves of the Probability Density Function of the Distribution with Some Parameter Values



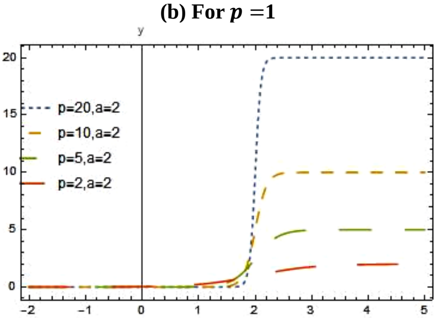
(a) For p =0.1



(b) For p =1



(c) For a =0.2



(d) For a =2

Figure 2: Curves of the Hazard Rate Function of the Proposed Distribution with Some Parameter Values

3.1 Useful Expansions

Applying the general binomial theorem, for $\alpha > 0$ and $|u| > 0$,

$$(1+u)^{-\alpha} = \sum_{\mu=0}^{\infty} (-1)^{\mu} \binom{\alpha+\mu-1}{\mu} u^{\mu}. \tag{3.3}$$

So, by using the formula (3.3) in (3.2), the probability density function of the distribution is changed into

$$f(x,\theta) = \sum_{\mu=0}^{\infty} (-1)^{\mu} \binom{\mu+1}{\mu} p\mathrm{e}^{p(\mu+1)(x-a)} = \sum_{\mu=0}^{\infty} (-1)^{\mu} (\mu+1) p\mathrm{e}^{p(\mu+1)(x-a)}. \tag{3.4}$$

And the cumulative distribution function becomes

$$[F(x,\theta)]^{\delta} = \sum_{\mu=0}^{\infty} (-1)^{\mu} \binom{\delta+\mu-1}{\mu} \mathrm{e}^{p(\mu+\delta)(x-a)}. \tag{3.5}$$

3.2 Quantile and Median

Quantile function has widely been used in simulations, statistical applications and theoretical aspects of probability theory. For example, quantile function can engender simulated random variables for classical or novel continuous distributions in simulation methods. The quantile function, namely, $Q(v) = F^{-1}(v)$ of χ is defined as

$$v = \frac{\mathrm{e}^{p(Q(v)-a)}}{1+\mathrm{e}^{p(Q(v)-a)}}, \tag{3.6}$$

which can be reduced to the following

$$Q(v) = \frac{1}{p} \log \frac{v}{1-v} + a, \tag{3.7}$$

with v is a uniform random variable on the interval $(0,1)$.

Particularly, let $v = 0.5$ in (3.7) then the median can be deduced. Namely, the median is derived as

$$\text{median} = a.$$

3.3 Maximum Likelihood Estimation

The complete samples can determine the maximum likelihood estimates of the unknown parameters for the proposed distribution. Suppose $\chi_1, \chi_2, \cdots, \chi_n$ are observed values from the distribution with parameter set $\theta = (p,a)^T$. The total log-likelihood function for parameters θ is written as

$$\log L(\theta) = n \log p + p \sum_{i=1}^{n} (x_i - a) - 2 \sum_{i=1}^{n} \log\left(1 + \mathrm{e}^{p(x_i-a)}\right). \tag{3.8}$$

The elements of the fractional function $U(\theta) = (U_p, U_a)$ are written as

$$U_p = \frac{n}{p} + \sum_{i=1}^{n}(x_i - a) - 2\sum_{i=1}^{n}\frac{(x_i - a)\mathrm{e}^{p(x_i-a)}}{1 + \mathrm{e}^{p(x_i-a)}} = \frac{n}{p} + \sum_{i=1}^{n}\frac{1 - \mathrm{e}^{p(x_i-a)}}{1 + \mathrm{e}^{p(x_i-a)}}(x_i - a), \tag{3.9}$$

$$U_a = -np + 2\sum_{i=1}^{n}\frac{\mathrm{e}^{p(x_i-a)}}{1 + \mathrm{e}^{p(x_i-a)}}. \tag{3.10}$$

Let the last two equations be zero and solve them, then the maximum likelihood estimates of p and a can be achieved.

4. CONCLUSION

The inequalities, subadditivity, super multiplicativity and other properties related to a double parameters sigmoid function are proved. In fact, the two-parameter sigmoid function can be considered as a distribution function and some statistical properties of the proposed distribution are given.

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