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SOME CHARACTERIZATION RESULTS ON LENGTH-BIASED GENERALIZED INTERVAL ENTROPY FOR LIFETIME DISTRIBUTIONS

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ABSTRACT

In this article, we develop a new “length-biased” shift-dependent generalized uncertainty measure, called weighted generalized interval entropy (WGIE). We discuss a numerical comparison between a generalized entropy (GE) and its weighted version (WGE) which gives the significance of weighted uncertainty measures. Some important characterization results of the proposed measure (WGIE) are also focused. Further, various momentous properties of WGIE and their relationships with other reliability measures are discussed. Finally, for uniform distribution, we study the expressions of all the entropies that are mentioned in this article.

KEYWORDS

Shannon’s entropy, weighted entropy, Length-biased generalized interval uncertainty measure, Characterization results.

1. INTRODUCTION

In recent years, the uncertainty measures have attained a wide interest among researchers. The basic uncertainty measure in the area of information theory was originally developed by Claud Shannon (1948) and hence it is mostly known as Shannon’s entropy. For an absolutely continuous non-negative r.v X with p.d.f $f\left(x\right)$, it is given by

$$H_X\left(f\right)=-\int_0^{\infty} f\left(x\right)\log f\left(x\right)dx=-E\left[\log f\left(X\right)\right]. \tag{1}$$

Throughout this article, X is considered an absolutely continuous non-negative r.v.

It is obvious that the entropy (1) has a great importance in measuring the uncertainty for the r.v X. But, this measure does not consider the subjective information or importance about the goal of events occurrence and hence it is found a major drawback in this measure. As a solution, Belis and Guiasu (1968) introduced a new measure called “length- biased” or weighted entropy, defined as

$$H_{(w,X)}(f) = -\int_0^\infty w(x) f(x) \log f(x) dx$$
$$= -\int_0^\infty x f(x) \log f(x) dx = -E\left[X f(X)\right], \quad (2)$$

where, the coefficient x in the integrand represents the weight function $w(x)$ of the elementary events and hence leads $H_{(w,X)}(f)$ to take the designation of "length-biased" shift-dependent uncertainty measure or weighted entropy.

For the residual lifetime $X_R = \left[X - t \mid X > t\right]$ of a random lifetime X, Ebrahimi (1996) introduced a new measure called residual entropy as

$$H_X(f;t) = -\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx. \quad (3)$$

The notion of the length-biased shift-depend residual uncertainty measure was developed by Di Crescenzo and Longobardi (2006) as follows

$$H_{(w,X)}(f;t) = -\int_t^\infty x \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx. \quad (4)$$

Sometimes, in survival analysis and in life testing, we have to study the system that is surviving at time t_0 and is found to be dead at time t_1. The measure (1) is not an appropriate uncertainty measure for such a system. Therefore, Sunoj et al. (2009) introduced an extension of Shannon's entropy based on a r.v $X_t = \left(X \mid t_0 < X < t_1\right)$, known as interval entropy, defined as

$$IH_X(f;t_0,t_1) = -\int_{t_0}^{t_1} \frac{f(x)}{F(t_1) - F(t_0)} \log \frac{f(x)}{F(t_1) - F(t_0)} dx, \quad (5)$$

where, $(t_0,t_1) \in D = \left\{(t_0,t_1) : F(t_0) < F(t_1)\right\}$.

The concept of the weighted interval entropy was taken into existence by Misagh and Yari (2011) as follows

$$IH_{(w,X)}(f;t_0,t_1) = -\int_{t_0}^{t_1} x \frac{f(x)}{F(t_1) - F(t_0)} \log \frac{f(x)}{F(t_1) - F(t_0)} dx. \quad (6)$$

Various researchers have developed the several generalizations of (1) and herein an effort is made to introduce a new generalization of this measure in the following way

$$H_X^{(\alpha,\beta)}(f) = \frac{\alpha}{\beta(\beta - \alpha)} \left[\int_0^\infty f^{\alpha-\beta+1}(x) dx - 1\right], \; \beta - 1 < \alpha < \beta, \beta \geq 1, \quad (7)$$

where, $\lim_{\substack{\alpha\to 1\\ \beta=1}} H_X^{(\alpha,\beta)}(f) = -\int_0^\infty f(x)\log f(x)\,dx$ is the well-known uncertainty measure given in (1).

Analogous to (5) and on the basis of (7), the generalized interval entropy is defined as

$$IH_X^{(\alpha,\beta)}(f;t_0,t_1) = \frac{\alpha}{\beta(\beta-\alpha)}\left[\int_{t_0}^{t_1}\left(\frac{f(x)}{F(t_1)-F(t_0)}\right)^{\alpha-\beta+1} dx - 1\right],$$

$$\beta-1<\alpha<\beta,\ \beta\geq 1. \qquad (8)$$

In the recent past, various researches like Guiasu (1986), Mirali et al. (2017), Kayal (2017), Khorashadizadeh et al. (2013), Das (2016), Mirali and Baratpour (2017) have developed different weighted and interval uncertainty measures.

The objective of this article is to develop a new "length-biased" shift-dependent generalized interval uncertainty measure and the organization is as follows: In section 2, we discuss the weighted generalized entropy (WGE) and also explore a numerical comparison between this WGE and its shift-independent version (GE). The existence of the weighted generalized interval entropy (WGIE) and its various characterization results are discussed in section 3. Further, in section 4, we present some important properties of WGIE and their relationships with other reliability measures. Also, for uniform distribution, we find the expressions of all the entropies that are mentioned in this article. Some concluding results are given in the final section.

2. WEIGHTED GENERALIZED ENTROPY

In this section, we define the weighted generalized entropy (WGE) and also investigate its general expressions for some particular lifetime distributions.

Analogous to (2) and on the basis of (7), the WGE is defined as

$$H_{(w,X)}^{(\alpha,\beta)}(f) = \frac{\alpha}{\beta(\beta-\alpha)}\left[\int_0^\infty (xf(x))^{\alpha-\beta+1} dx - 1\right],\ \beta-1<\alpha<\beta,\ \beta\geq 1, \qquad (9)$$

where, the coefficient x in the integral denotes the weight function as in (2).

In the following example, we study the comparison between the GE (7) and its weighted version (9).

Example 2.1:

Let the two r.v's X and Y be distributed as follows

$$f_X(x) = \begin{cases} \frac{1}{2},\ 0\leq x\leq 2 \\ 0,\ otherwise \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} \frac{1}{2},\ 2\leq y\leq 4 \\ 0,\ otherwise \end{cases}$$

Table 1 provides the comparison between the GE (7) and WGE (9) with respect to the above distributions. It is shown that for different values of α and β, even though the GE of the r.v X is same as that of Y, but $H_{(w,X)}^{(\alpha,\beta)}(f) \neq H_{(w,Y)}^{(\alpha,\beta)}(f)$.

Table 1
Comparison between GE and WGE

α	β	$H_X^{(\alpha,\beta)}(f)$	$H_Y^{(\alpha,\beta)}(f)$	$H_{(w,X)}^{(\alpha,\beta)}(f)$	$H_{(w,Y)}^{(\alpha,\beta)}(Y)$
0.5	1	0.414	0.414	0.333	1.438
	1.2	0.372	0.372	0.321	0.744
	1.3	0.356	0.356	0.321	0.559
	1.4	0.344	0.344	0.325	0.428
1.5	2.1	0.614	0.614	0.510	1.597
	2.2	0.608	0.608	0.524	1.217
	2.3	0.604	0.604	0.543	0.948
	2.4	0.601	0.601	0.568	0.749
2	2.6	0.661	0.661	0.549	1.720
	2.7	0.661	0.661	0.570	1.322
	2.8	0.662	0.662	0.595	1.038
	2.9	0.664	0.664	0.627	0.827
2.5	2.9	0.689	0.689	0.539	3.317
	3	0.69	0.69	0.556	2.397
	3.1	0.693	0.693	0.576	1.803
	3.2	0.697	0.697	0.601	1.395
3	3.6	0.716	0.716	0.595	1.863
	3.7	0.723	0.723	0.624	1.448
	3.8	0.731	0.731	0.658	1.147
	3.9	0.74	0.74	0.699	0.922

In Table 2, we design the general expressions of $H_{(w,X)}^{(\alpha,\beta)}(f)$ for some lifetime distributions. Also, note that $\Gamma(n,mz) = m^n \int_z^\infty e^{-mx} x^{n-1} dx$ represents an upper incomplete gamma function and $p = \frac{\alpha}{\beta(\beta-\alpha)}$, $r = \alpha - \beta + 1$ respectively.

Table 2

$H_{(w,X)}^{(\alpha,\beta)}(f)$ of Some Lifetime Distributions

Distribution	$f(x)$	x	$pH_{(w,X)}^{(\alpha,\beta)}(f)$
Uniform	$\frac{1}{b-a}$	$a<x<b$	$\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)^{r}}-1$
Exponential	$\mu e^{-\mu x}$	$x\geq 0,\ \mu>0$	$\frac{\Gamma(r+1)}{\mu r^{r+1}}-1$
Weibull	$\frac{1}{\lambda}e^{-\left(\frac{x-\mu}{\lambda}\right)}$	$x>\mu,\lambda>0,\mu>0$	$\frac{\lambda e^{\frac{\mu}{\lambda}r}\Gamma\left(r+1,\frac{\mu}{\lambda}r\right)}{r^{r+1}}-1$
Gamma	$\frac{1}{\Gamma(\theta)}e^{-x}x^{\theta-1}$	$0<x<\infty,\theta>0$	$\frac{\Gamma(\theta r+1)}{(\Gamma(\theta))^{r}r^{\theta r+1}}-1$
Pareto	$\frac{ab^{a}}{x^{a+1}}$	$x\geq b,a>0,b>0$	$\frac{ba^{r}}{ar-1}-1,\ ar>1$
Lomax	$\frac{\mu}{(1+x)^{1+\mu}}$	$x>0,\mu>0$	$\frac{\mu^{r}\Gamma(r+1)\Gamma(r\mu-1)}{\Gamma(r(1+\mu))}-1,\ \mu r>1$

3. WEIGHTED GENERALIZED INTERVAL ENTROPY

In this section, we discuss the weighted version of GIE (8) and is called weighted generalized interval entropy (WGIE). Some of its particular characterization results are also studied.

Definition 3.1:

Analogous to (6), the weighted form of (8) is defined as

$$IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)=\frac{\alpha}{\beta(\beta-\alpha)}\left[\int_{t_0}^{t_1}\left(x\frac{f(x)}{F(t_1)-F(t_0)}\right)^{\alpha-\beta+1}dx-1\right],$$

$$\beta-1<\alpha<\beta,\ \beta\geq 1. \qquad (10)$$

A new way of expressing the WGIE (10) is obtained as follows.

Theorem 3.1:

For all $0<t_0<t_1$, the following equality holds

$$IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)=t_0^{\alpha-\beta+1}IH_{X}^{(\alpha,\beta)}(f;t_0,t_1)+\frac{\alpha}{\beta(\beta-\alpha)}\Bigg[t_0^{\alpha-\beta+1}-1$$

$$+(\alpha-\beta+1)\int_{z=t_0}^{t_1}z^{\alpha-\beta}\left(\frac{F(t_1)-F(z)}{F(t_1)-F(t_0)}\right)^{\alpha-\beta+1}\left(\frac{\beta(\beta-\alpha)}{\alpha}IH_{X}^{(\alpha,\beta)}(f;z,t_1)+1\right)dz\Bigg]. \qquad (11)$$

Proof:

$$\int_{t_0}^{t_1}\left(x\frac{f(x)}{F(t_1)-F(t_0)}\right)^{\alpha-\beta+1}dx=\int_{t_0}^{t_1}\left(\int_0^x(\alpha-\beta+1)z^{\alpha-\beta}dz\right)\left(\frac{f(x)}{F(t_1)-F(t_0)}\right)^{\alpha-\beta+1}dx$$

$$=(\alpha-\beta+1)\int_{t_0}^{t_1}\left[\int_0^{t_0}z^{\alpha-\beta}dz+\int_{t_0}^{x}z^{\alpha-\beta}dz\right]\left(\frac{f(x)}{F(t_1)-F(t_0)}\right)^{\alpha-\beta+1}dx$$

$$=t_0^{\alpha-\beta+1}\int_{t_0}^{t_1}\left(\frac{f(x)}{F(t_1)-F(t_0)}\right)^{\alpha-\beta+1}dx$$

$$+(\alpha-\beta+1)\int_{z=t_0}^{t_1}z^{\alpha-\beta}\left(\int_{x=z}^{t_1}\left(\frac{f(x)}{F(t_1)-F(t_0)}\right)^{\alpha-\beta+1}dx\right)dz\,. \quad (12)$$

Since, from (8), we have

$$\int_{t_0}^{t_1}\left(\frac{f(x)}{F(t_1)-F(t_0)}\right)^{\alpha-\beta+1}dx=\frac{\beta(\beta-\alpha)}{\alpha}IH_X^{(\alpha,\beta)}(f;t_0,t_1)+1\,. \quad (13)$$

and

$$\int_{t_0}^{t_1}f^{\alpha-\beta+1}(x)dx=\left(F(t_1)-F(t_0)\right)^{\alpha-\beta+1}\left(\frac{\beta(\beta-\alpha)}{\alpha}IH_X^{(\alpha,\beta)}(f;t_0,t_1)+1\right). \quad (14)$$

Substituting (12), (13) and (14) in (10), we get the desired result.

Definition 3.2:

Let $f(x)$ and $F(x)$ be the p.d.f and c.d.f of a r.v X, then the GFR's of X are defined by $h_0(t_0,t_1)=\frac{f(t_0)}{F(t_1)-F(t_0)}$ and $h_1(t_0,t_1)=\frac{f(t_1)}{F(t_1)-F(t_0)}$.

Example 3.1:

If the r.v X has

(I) Uniform distribution with p.d.f $f(x)=\frac{1}{m-n}$, $m<x<n$, then

$$IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)=\frac{\alpha}{\beta(\beta-\alpha)}\left[\left(\frac{t_1^{\alpha-\beta+2}-t_0^{\alpha-\beta+2}}{\alpha-\beta+2}\right)h_i^{\alpha-\beta+1}(t_0,t_1)-1\right],\ i=0,1$$

(II) Pareto distribution having p.d.f $f(x)=\frac{mn^m}{x^{m+1}}$, $x\geq n, m>0, n>0$, then

$$IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)=\frac{\alpha}{\beta(\beta-\alpha)}\left[\frac{t_0^{\alpha-\beta+2}h_0^{\alpha-\beta+1}(t_0,t_1)-t_1^{\alpha-\beta+2}h_1^{\alpha-\beta+1}(t_0,t_1)}{m(\alpha-\beta+1)-1}-1\right].$$

In theorem 3.2, we show that $F(x)$ is determined by $H_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)$ uniquely.

Theorem 3.2:

Let X be a r.v having p.d.f $f(x)$ and c.d. $F(x)$. Assume that $IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)<\infty,\ \forall\ \beta-1<\alpha<\beta,\ \beta\geq 1$. Then for each α and β, $IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)$ uniquely determines $F(x)$.

Proof:

Consider two r.v's X and Y having density function $f_X(x)$, $f_Y(y)$, distribution functions $F_X(x)$, $F_Y(y)$ and GFR's $h_i^X(t_0,t_1)$ and $h_i^Y(t_0,t_1)$, $i=0,1$ respectively. Also, let

$$IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)=IH_{(w,Y)}^{(\alpha,\beta)}(f;t_0,t_1),\ \forall\beta-1<\alpha<\beta,\beta\geq 1,0<t_0<t_1 \tag{15}$$

Now, rewriting (10), we have

$$\frac{\beta(\beta-\alpha)}{\alpha}IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)=\int_{t_0}^{t_1}\left(x\frac{f(x)}{F(t_1)-F(t_0)}\right)^{\alpha-\beta+1}dx-1. \tag{16}$$

Differentiating (16) w.r.t t_0 and t_1, we obtain

$$\frac{\beta(\beta-\alpha)}{\alpha}\frac{\partial}{\partial t_0}IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)=(\alpha-\beta+1)h_0^X(t_0,t_1)\int_{t_0}^{t_1}\left(x\frac{f(x)}{F(t_1)-F(t_0)}\right)^{\alpha-\beta+1}dx$$
$$-\left(t_0h_0^X(t_0,t_1)\right)^{\alpha-\beta+1}. \tag{17}$$

and

$$\frac{\beta(\beta-\alpha)}{\alpha}\frac{\partial}{\partial t_1}IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)=\left(t_1h_1^X(t_0,t_1)\right)^{\alpha-\beta+1}$$
$$-(\alpha-\beta+1)h_1^X(t_0,t_1)\int_{t_0}^{t_1}\left(x\frac{f(x)}{F(t_1)-F(t_0)}\right)^{\alpha-\beta+1}dx. \tag{18}$$

where, $h_0^X(t_0,t_1)$ and $h_1^X(t_0,t_1)$ are the GFRs of X. Using (16), (17) and (18) can be rewritten as

$$\left(t_0 h_0^X(t_0,t_1)\right)^{\alpha-\beta+1} - (\alpha-\beta+1) h_0^X(t_0,t_1)\left[\frac{\beta(\beta-\alpha)}{\alpha} IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)+1\right]$$
$$+\frac{\beta(\beta-\alpha)}{\alpha}\frac{\partial}{\partial t_0} IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)=0. \quad (19)$$

and

$$\left(t_1 h_1^X(t_0,t_1)\right)^{\alpha-\beta+1} - (\alpha-\beta+1) h_1^X(t_0,t_1)\left[\frac{\beta(\beta-\alpha)}{\alpha} IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)+1\right]$$
$$-\frac{\beta(\beta-\alpha)}{\alpha}\frac{\partial}{\partial t_0} IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)=0. \quad (20)$$

Using (19) and (20) in (15), we have

$$\left(t_0 h_0^X(t_0,t_1)\right)^{\alpha-\beta+1} - (\alpha-\beta+1) h_0^X(t_0,t_1)\left[\frac{\beta(\beta-\alpha)}{\alpha} IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)+1\right]$$
$$+\frac{\beta(\beta-\alpha)}{\alpha}\frac{\partial}{\partial t_0} IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)$$
$$=\left(t_0 h_0^Y(t_0,t_1)\right)^{\alpha-\beta+1} - (\alpha-\beta+1) h_0^Y(t_0,t_1)\left[\frac{\beta(\beta-\alpha)}{\alpha} IH_{(w,Y)}^{(\alpha,\beta)}(f;t_0,t_1)+1\right]$$
$$+\frac{\beta(\beta-\alpha)}{\alpha}\frac{\partial}{\partial t_0} IH_{(Y,w)}^{(r,s)}(f;t_1,t_2).$$

which leads to

$$h_0^X(t_0,t_1)=h_0^Y(t_0,t_1). \quad (21)$$

Similarly, we can obtain

$$h_1^X(t_0,t_1)=h_1^Y(t_0,t_1). \quad (22)$$

Thus, from (21) and (22), we conclude that $h_i^X(t_0,t_1)=h_i^Y(t_0,t_1)$, $i=0,1$ or $F_X(t)=F_Y(t)$ and hence $IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)$ characterizes $F(x)$ uniquely.

Here, we study the monotonic behavior of $IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)$ with respect to uniform distribution.

For uniform distribution $f(x)=\frac{1}{b}$, $0<x<b$, we have

$$IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)=\frac{\alpha}{\beta(\beta-\alpha)}\left[\frac{t_1^{\alpha-\beta+2}-t_0^{\alpha-\beta+2}}{(\alpha-\beta+2)(t_1-t_0)^{\alpha-\beta+1}}-1\right].$$

Fig. 1 exhibits the monotonic behavior of $IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)$ with respect to t_0 and t_1.

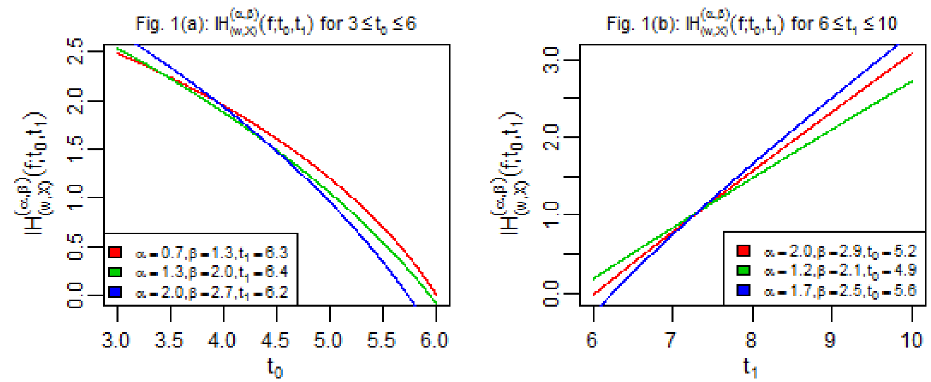

Fig.1: $IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)$ plots of Uniform Distribution with respect to t_0 and t_1.

It is obvious from Fig. 1 that $IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)$ is monotone with respect both t_0 and t_1.

4. PROPERTIES AND INEQUALITIES OF $IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)$

In this section, some of the important properties and inequalities of weighted generalized interval entropy (WGIE) are studied.

Definition 4.1:

A r.v X is said to have smaller WGIE than the other r.v Y (denoted by $X \overset{WGIE}{\leq} Y$) if $IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1) \leq IH_{(w,Y)}^{(\alpha,\beta)}(f;t_0,t_1), \forall\ 0 < t_0 < t_1$.

Definition 4.2:

A r.v X is said to have

- Increasing weighted generalized interval entropy (IWGIE), if for any fixed t_0, $IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)$ is increasing with respect to t_1.
- decreasing weighted generalized interval entropy (DWGIE), if for any fixed t_1, $IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)$ is decreasing with respect to t_0.

In the following theorem, we show that there exists no non-negative r.v having increasing WGIE over the domain $[0,\infty)$.

Theorem 4.1:

The WGIE of a r.v X cannot be an increasing function with respect to t_0 for any fixed t_1.

Proof:

By using L.Hospital's rule, we have

$$\begin{aligned}\lim_{t_0\to t_1} IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1) &= \lim_{t_0\to t_1}\frac{\alpha}{\beta(\beta-\alpha)}\left[\int_{t_0}^{t_1}\left(x\frac{f(x)}{F(t_1)-F(t_0)}\right)^{\alpha-\beta+1}dx-1\right]\\ &= \lim_{t_0\to t_1}\frac{\alpha}{\beta(\beta-\alpha)}\int_{t_0}^{t_1}\left(x\frac{f(x)}{F(t_1)-F(t_0)}\right)^{\alpha-\beta+1}dx-\lim_{t_0\to t_1}\frac{\alpha}{\beta(\beta-\alpha)}\\ &= \frac{\alpha}{\beta(\beta-\alpha)}\lim_{t_0\to t_1}\frac{t_0^{\alpha-\beta+1}}{(\alpha-\beta+1)}\left(\frac{f(t_0)}{F(t_1)-F(t_0)}\right)^{\alpha-\beta}-\frac{\alpha}{\beta(\beta-\alpha)}\\ &= \infty.\end{aligned}$$

Now, on the contrary suppose that $IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)$ is increasing in t_0, then for all $t_0 \le t_1$, $IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1) \le IH_{(w,X)}^{(\alpha,\beta)}(f;t_1,t_1) = \infty$, which is a contradiction to the obvious statement that $IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1) \in \Re, \forall (t_0,t_1) \in D$. Thus, it is clear that $IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)$ is non-increasing in t_0.

Theorem 4.2:

Let the r.v X has DWGIE for all $(t_0,t_1) \in D$, then $IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)$ obtains an upper bound as follows

$$IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1) \le \frac{\alpha}{\beta(\beta-\alpha)}\left[\frac{t_0^{\alpha-\beta+1}}{\alpha-\beta+1}\left(\frac{1+\frac{\partial}{\partial t_0}\mu(t_0,t_1)}{\mu(t_0,t_1)}\right)^{\alpha-\beta}-1\right].$$

where,

$\mu(t_0,t_1)$ is the doubly truncated mean residual life function of X.

Proof:

From (19), we have

$$\frac{\beta(\beta-\alpha)}{\alpha}\frac{\partial}{\partial t_0} IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)$$

$$=\left(\alpha-\beta+1\right)h_0\left(t_0,t_1\right)\left(\frac{\beta\left(\beta-\alpha\right)}{\alpha}IH_{(w,X)}^{(\alpha,\beta)}\left(f;t_0,t_1\right)+1\right)-t_0^{\alpha-\beta+1}h_0^{\alpha-\beta+1}\left(t_0,t_1\right).$$

Since, for any fixed t_1, $IH_{(w,X)}^{(\alpha,\beta)}\left(f;t_0,t_1\right)$ is decreasing w.r.t t_0. Therefore, we have

$$IH_{(w,X)}^{(\alpha,\beta)}\left(f;t_0,t_1\right)\le\frac{\alpha}{\beta\left(\beta-\alpha\right)}\left(\frac{t_0^{\alpha-\beta+1}}{\alpha-\beta+1}h_0^{\alpha-\beta}\left(t_0,t_1\right)-1\right).$$

Substituting, $h_0\left(t_0,t_1\right)=\dfrac{1+\dfrac{\partial}{\partial t_0}\mu\left(t_0,t_1\right)}{\mu\left(t_0,t_1\right)}$ and hence the desired result is obvious.

The following lemma will be used in proving some of the theorems of this section.

Lemma 4.1:

For a r.v X, define $Z=aX$, where $a>0$ is a constant, the following equality holds

$$IH_{(w,Z)}^{(\alpha,\beta)}\left(f;t_0,t_1\right)=aIH_{(w,X)}^{(\alpha,\beta)}\left(f;\frac{t_0}{a},\frac{t_1}{a}\right)+\frac{\alpha\left(a-1\right)}{\beta\left(\beta-\alpha\right)}.$$

Proof:

$$IH_{(w,Z)}^{(\alpha,\beta)}\left(f;t_0,t_1\right)=\frac{\alpha}{\beta\left(\beta-\alpha\right)}\left[\int_{t_0}^{t_1}\left(\frac{z\left(f\left(\frac{z}{a}\right)\right)}{a\left(\Pr\left(t_0<z<t_1\right)\right)}\right)^{\alpha-\beta+1}dz-1\right].$$

Setting $Z=aX$, a strictly increasing function of X, we have

$$IH_{(w,Z)}^{(\alpha,\beta)}\left(f;t_0,t_1\right)=\frac{\alpha}{\beta\left(\beta-\alpha\right)}\left[a\int_{\frac{t_0}{a}}^{\frac{t_1}{a}}\left(x\frac{f\left(x\right)}{F\left(\frac{t_1}{a}\right)-F\left(\frac{t_0}{a}\right)}\right)^{\alpha-\beta+1}dx-1\right]$$

$$=aIH_{(w,X)}^{(\alpha,\beta)}\left(f;\frac{t_0}{a},\frac{t_1}{a}\right)+\frac{\alpha\left(a-1\right)}{\beta\left(\beta-\alpha\right)}.$$

Theorem 4.3:

For the r.v $X\in$DWGIE, the following inequality holds

$$h_0(t_0,t_1) > \left[\frac{p\left(qIH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)+1\right)}{t_0^{\alpha-\beta+1}}\right]^{\frac{1}{\alpha-\beta}},$$

where,

$$p = \alpha-\beta+1 \text{ and } q = \frac{\beta(\beta-\alpha)}{\alpha}.$$

Proof:

From (19), we have

$$\frac{\beta(\beta-\alpha)}{\alpha}\frac{\partial}{\partial t_0}IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1) = (\alpha-\beta+1)h_0(t_0,t_1)\left(\frac{\beta(\beta-\alpha)}{\alpha}IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)+1\right) - t_0^{\alpha-\beta+1}h_0^{\alpha-\beta+1}(t_0,t_1).$$

Since, for any fixed t_1, $IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)$ is decreasing with respect to t_0, and $\beta > \alpha$, therefore,

$$h_0(t_0,t_1)\left[(\alpha-\beta+1)\left(\frac{\beta(\beta-\alpha)}{\alpha}IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)+1\right) - t_0^{\alpha-\beta+1}h_0^{\alpha-\beta}(t_0,t_1)\right] \le 0,$$

which leads to

$$h_0(t_0,t_1) > \left[p\left(qIH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)+1\right)\right]^{\frac{1}{\alpha-\beta}},$$

where, $p = \alpha-\beta+1$ and $q = \frac{\beta(\beta-\alpha)}{\alpha}$.

Theorem 4.4:

Let $X \in$ IWGIE, define $Z = aX$, where $a > 0$ is a constant, then $Z \in$ IWGIE.

Proof:

Since, $X \in$ IWGIE,

Therefore,

$$\frac{\partial}{\partial t_1}IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1) \ge 0.$$

Thus, by applying Lemma 1, it is obvious that $Z \in$ IWGIE.

Theorem 4.5:

For the r.v X having p.d.f $f(x)$ and c.d.f $F(x)$, then for $\beta > \alpha$, we obtain a lower bound of $IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)$ as follows

$$IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1) \geq \frac{\alpha}{\beta(\beta-\alpha)}\left[\exp\left(\begin{array}{l}(\alpha-\beta+1)\int_{t_0}^{t_1}\frac{f(x)\log x}{F(t_1)-F(t_0)}dx \\ \qquad +(\beta-\alpha)IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)\end{array}\right)-1\right]. \tag{23}$$

Proof:

From log-sum inequality, we have

$$\int_{t_0}^{t_1} f(x)\log\frac{f(x)}{\left(x\frac{f(x)}{F(t_1)-F(t_0)}\right)^{\alpha-\beta+1}}dx \geq \int_{t_0}^{t_1} f(x)dx\log\frac{\int_{t_0}^{t_1} f(x)dx}{\int_{t_0}^{t_1}\left(x\frac{f(x)}{F(t_1)-F(t_0)}\right)^{\alpha-\beta+1}dx}$$

$$=\left(F(t_1)-F(t_0)\right)\left[\log\left(F(t_1)-F(t_0)\right)-\log\left(\frac{\beta(\beta-\alpha)}{\alpha}IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)+1\right)\right]. \tag{24}$$

The L.H.S of (24) can be rewritten as

$$(\beta-\alpha)\int_{t_0}^{t_1} f(x)\log f(x)dx-(\alpha-\beta+1)$$
$$\left(\int_{t_0}^{t_1} f(x)\log x dx+\int_{t_0}^{t_1} f(x)\log\left(F(t_1)-F(t_0)\right)dx\right). \tag{25}$$

Using (25) in (24), we get (23).

Finally, for uniform distribution, we derive the general expressions of all the entropies that are mentioned in this article and are shown in Table 3.

Table 3
Different Types of Entropies with respect to Uniform Distribution

$f(x)$	$\frac{1}{b},\ 0<x<b$	Monotonicity
$H_X(f)$	$\log b$	–
$H_{(w,X)}(f)$	$\frac{1}{2}b\log b$	–
$H_X(f;t)$	$\log(b-t)$	Decreasing in t
$H_{(w,X)}(f;t)$	$\frac{1}{2}(b+t)\log(b-t)$	Decreasing in t
$IH_X(f;t_0,t_1)$	$\log(t_1-t_0)$	Decreasing in t_0 for $(t_1,t_2)\in D$
$IH_{(w,X)}(f;t_0,t_1)$	$\frac{1}{2}(t_1+t_0)\log(t_1-t_0)$	Decreasing in t_0 for $(t_0,t_1)\in D$
$H_X^{(\alpha,\beta)}(f)$	$\frac{\alpha}{\beta(\beta-\alpha)}\left(b^{\beta-\alpha}-1\right)$	–
$IH_{(\alpha,\beta)}(X;t_1,t_2)$	$\frac{\alpha}{\beta(\beta-\alpha)}\left((t_1-t_0)^{\beta-\alpha}-1\right)$	Decreasing in t_0 for $(t_0,t_1)\in D$
$H_{(\alpha,\beta)}^{w}(X)$	$\frac{\alpha}{\beta(\beta-\alpha)}\left(\frac{b}{\alpha-\beta+2}-1\right)$	–
$IH_{(w,X)}^{(\alpha,\beta)}(X;t_1,t_2)$	$\frac{\alpha}{\beta(\beta-\alpha)}\left(\frac{t_1^{\alpha-\beta+2}-t_0^{\alpha-\beta+2}}{(\alpha-\beta+2)(t_1-t_0)^{\alpha-\beta+1}}-1\right)$	Decreasing in t_0 for $(t_0,t_1)\in D$

It is clear from the Fig. 2 that all the interval entropies in Table. 2 are increasing with respect to t_1 for any fixed t_0.

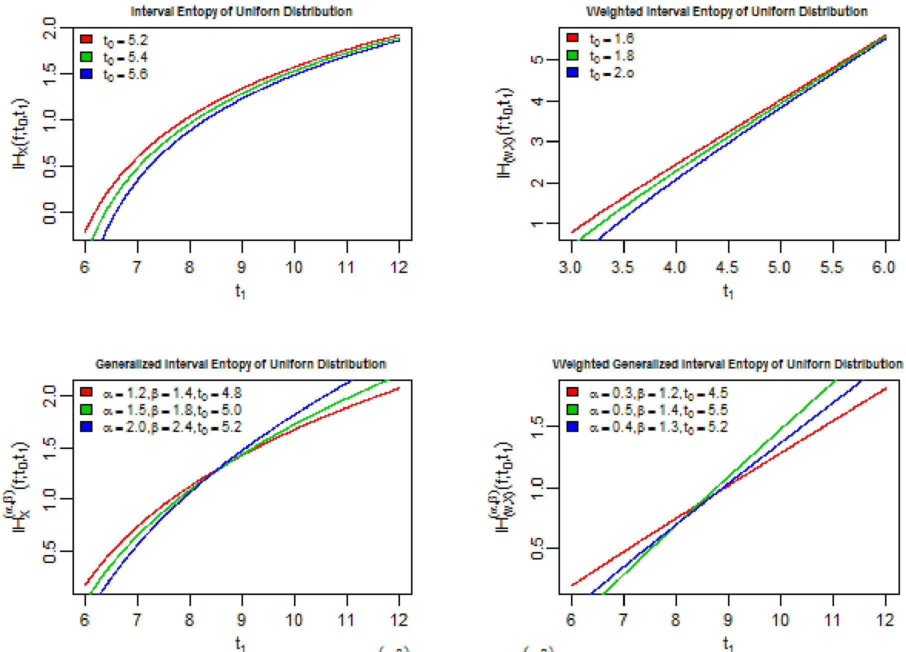

Fig.2. $IH_X(f;t_0,t_1)$, $IH_{(w,X)}(f;t_0,t_1)$, $IH_X^{(\alpha,\beta)}(f;t_0,t_1)$ and $IH_{(w,X)}^{(\alpha,\beta)}(f;t_0,t_1)$ **plots of Uniform Distribution with respect to** t_1 **for any fixed** t_0.

5. CONCLUSION

In this article, we have studied a new length biased shift-dependent interval uncertainty measure. A numerical comparison which differentiates a generalized entropy from its weighted version has been presented. Also, we have focused on several important characterization results related to the proposed measure. Further, some momentous properties of the measure have been presented. Finally, for uniform distribution, we have inferred the expressions of all the entropies that are used in this particular article.

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