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TWO-PARAMETER STOCHASTIC RESTRICTED PRINCIPAL COMPONENT ESTIMATOR IN LINEAR REGRESSION MODEL

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ABSTRACT

In this paper, an alternative estimator has been developed for the estimation of unknown regression coefficient vector in linear regression model to combat the problem of multicollinearity when additional stochastic restrictions are available. The proposed estimator is a generalization of the mixed regression (MR) estimator (Theil and Goldberger (1961)) and the principal component two parameter (PCTP) estimator (Huang and Yang (2015)), named as two parameter stochastic restricted principal component (TPSRPC) estimator. Necessary and sufficient conditions for superiority of the TPSRPC estimator over some other competing stochastic restricted estimators have been derived under the MSE matrix and scalar mean squared error (MSE) criteria. Further, tests to verify the conditions under MSE matrix have been derived. A Monte Carlo simulation and a numerical example have been given to evaluate the performance of the estimators.

KEYWORDS

Multicollinearity, Principal Component Two Parameter estimator, Stochastic linear restrictions, Mean squared error.

1. INTRODUCTION

In linear regression model, near linear dependency among the explanatory variables causes the problem of multicollinearity, which reduces the efficiency of the ordinary least squares (OLS) estimator and gives wide confidence intervals leading to wrong conclusions. Some biased estimators have been introduced to overcome the problem of multicollinearity, such as the ordinary ridge regression (ORR) estimator by Hoerl and Kennard (1970), principal component regression estimator by Farebrother (1972), the $r-k$ class estimator by Baye and Parker (1984) and principal component two parameter estimator by Chang and Yang (2012), etc. Moreover, when the error terms are found to be autocorrelated along with the problem of multicollinearity, Trenkler (1984) generalized the ORR estimator and introduced ridge regression (RR) estimator. On the same lines, other estimators were also generalized to deal with such situations, for instance, Şiray et al. (2014) and Huang and Yang (2015) generalized the $r-k$ class and the principal component two parameter estimators in the presence of autocorrelated errors, respectively.

An another efficacious tool to combat the problem of multicollinearity is to make use of some prior information, either exact or stochastic or both. However, the uncertainty in specifying the exact restrictions makes it difficult to use in practice. In such cases, stochastic restrictions may serve the solution as they can be obtained from some previous studies or from some prior data which inherit the information on the parameters in the stochastic form. Theil and Goldberger (1961), Durbin(1953), Theil (1963) and Bayhan and Bayhan (1998) suggested mixed regression (MR) estimator for the estimation of the regression coefficient vector by merging stochastic linear restrictions into the sample model.

Ozkale (2009) proposed stochastic restricted ridge regression (SRRR) estimator by grafting the techniques of the MR and RR estimators. They compared the performance of the SRRR estimator with the MR estimator in terms of variance and MSE matrices. He and Wu (2014) developed stochastic restricted PCR estimator by combining MR and PCR estimators and compared their properties under MSE matrix criterion. Chandra and Sarkar (2016) proposed stochastic restricted $r-k$ class (SRrk) estimator and compared this with the MR, RR and the SRRR estimators. They compared the performance of the proposed estimator with some competing estimators in MSE matrix sense and developed tests to verify the conditions.

In this paper, we introduce a new two parameter stochastic restricted principal component (TPSRPC) estimator by combining the approaches of the MR estimator and the principal component two parameter estimator (Huang and Yang (2015)). The TPSRPC estimator is a generalization of the MR, SRRR, SRPCR and SRrk estimators. The performance of the TPSRPC estimator has been compared with the estimators obtained as its special case in terms of the MSE matrix and scalar MSE[1]. Further, following Sarkar (1996), Sarkar and Chandra (2015) and Chandra and Sarkar (2016), tests for verifying the superiority conditions under MSE matrix have been developed. To compare the performance of the estimators, a Monte Carlo simulation and numerical example have also been provided.

Further, the paper is organized as follows: The new estimator is introduced in Section 2. In Section 3, properties of the new estimator have been studied and compared with other competing estimators by means of MSE matrix and scalar MSE criteria. Section 4 proposes tests to verify the conditions of superiority of the TPSRPC estimator over the others in MSE matrix sense. Section 5 discusses the selection of the biasing parameters for which scalar MSE of the estimator is minimum. Further, a numerical example and a Monte Carlo simulation study have been presented in Sections 6 and 7, respectively. The paper is concluded in Section 8.

2. MODEL SPECIFICATIONS AND THE ESTIMATORS

Let us consider the linear regression model

$$y = X\beta + u, \tag{1}$$

where y is an $n \times 1$ vector of observations on dependent variable, X is an $n \times p$ full column rank matrix of observations on p explanatory variables, β is a $p \times 1$ vector of unknown parameters and u is an $n \times 1$ vector of disturbance terms with mean vector 0

[1] The notation MSE is used for scalar MSE in all throughout the paper

and covariance matrix $\sigma^2\Omega$, where Ω is assumed to be a known symmetric positive definite $n \times n$ matrix.

The generalized least squares estimator for β in (1) is given by

$$\hat{\beta}_G = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y \tag{2}$$

and is an unbiased estimator with minimum variance. However, the collinearity among the explanatory variables in the model causes ill-conditioning of $X'\Omega^{-1}X$ matrix, which in turn inflates the variance of the estimator.

To overcome this problem, Trenkler (1984) obtained ridge regression (RR) estimator of β following the technique of Hoerl and Kennard (1970), which is expressed as

$$\hat{\beta}_G(k) = (X'\Omega^{-1}X + kI)^{-1}X'\Omega^{-1}y, k > 0. \tag{3}$$

Let us assume that in addition to the sample information in (1), we have some prior information on the parameters in the form of stochastic linear restrictions as follows

$$h = H\beta + v, \tag{4}$$

where, h is an $m \times 1$ random vector, H is an $m \times p$ matrix of known coefficients with full row rank m, and v is an $m \times 1$ random vector with mean vector 0 and covariance matrix $\sigma^2 W$. Further, it is assumed that u and v are independently distributed. Moreover, the usual methods of finding stochastic restrictions in the form of (4) are: to use some old dataset (see, for instance Chandra and Sarkar(2016), Ozkale(2009)), or from linear restrictions available from some theory which possesses some uncertainty (see, for example Li and Yang(2010), Yang and Cui(2011)).

Theil and Goldberger (1961) compounded the sample information (1) with prior information in (4) as follows

$$\begin{pmatrix} y \\ h \end{pmatrix} = \begin{pmatrix} X \\ H \end{pmatrix} \beta + \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\text{m}y_M = X_M\beta + u_M, \tag{5}$$

where $E(u_M) = 0$ and $Cov(u_M) = \sigma^2\Omega_M = \sigma^2 \begin{pmatrix} \Omega & 0 \\ 0 & W \end{pmatrix}$. Since Ω_M is a positive definite matrix, there exists an orthogonal matrix P such that $PP' = \Omega_M^{-1}$. On pre-multiplying model in (5) by P, we have

$$Py_M = PX_M\beta + Pu_M$$

$$y_M^* = X_M^*\beta + u_M^*, \tag{6}$$

where $y_M^* = Py_M$, $X_M^* = PX_M$ and $u_M^* = Pu_M$ is such that $E(u_M^*) = 0$ and $Cov(u_M^*) = \sigma^2 I$.

Theil and Goldberger (1961) applied OLS technique to the model (6), and proposed mixed regression (MR) estimator for β in model (1), which is given as

$$\hat{\beta}_M = (X'\Omega^{-1}X + H'W^{-1}H)^{-1}(X'\Omega^{-1}y + H'W^{-1}h). \tag{7}$$

Ozkale (2009) applied the ridge regression technique to model (6) and obtained stochastic restricted ridge regression (SRRR) estimator, which is expressed as

$$\hat{\beta}_{SR}(k) = (X'\Omega^{-1}X + H'W^{-1}H + kI)^{-1}(X'\Omega^{-1}y + H'W^{-1}h), k > 0. \quad (8)$$

The authors examined the performance of the SRRR estimator with the MR estimator using variance and MSE matrix as comparison criteria.

Further, in order to define some estimators, let us assume that $T = (t_1, t_2, \dots, t_p)$ is a $p \times p$ matrix of orthonormal eigen vectors of $X_M^*{}'X_M^*$ matrix such that $T'X_M^*{}'X_M^*T = T'(X'\Omega^{-1}X + H'W^{-1}H)T = G = diag(g_1, g_2, \dots, g_p)$ is a $p \times p$ diagonal matrix of eigen values of $X'\Omega^{-1}X + H'W^{-1}H$ matrix such that $g_1 \geq g_2 \geq \cdots \geq g_p$. Now, let $T_r = (t_1, t_2, \dots, t_r)$ be $p \times r$ matrix after deleting last $p - r$ columns from T matrix, where $r \leq p$. Thus, $T_{r\prime}(X'\Omega^{-1}X + H'W^{-1}H)T_r = G_r = diag(g_1, g_2, \dots, g_r)$ and $T'_{p-r}\left(X'^{\Omega^{-1}}X + H'^{W^{-1}}H\right)T_{p-r} = G_{p-r} = diag(g_{r+1}, g_{r+2}, \dots, g_p)$. Also, $T'T = T_r'T_r + T_{p-r}{}'T_{p-r}$.

Chandra and Sarkar 2016) proposed two stochastic restricted estimators, one is stochastic restricted PCR estimator which is obtained by combining the techniques of the MR and PCR estimators. The estimator is expressed as

$$\hat{\beta}_{SR}(r) = T_r(T_r'X'\Omega^{-1}XT_r + T_r'H'W^{-1}HT_r)^{-1}(T_r'X'\Omega^{-1}y + T_r'H'W^{-1}h). \quad (9)$$

This estimator is a general form of the stochastic restricted PCR estimator introduced by Wu and Yang(2013) when the errors are assumed to be non- autocorrelated. The second estimator is stochastic restricted $r - k$ (SRrk) class estimator, which is an amalgamation of the ideas underlying the MR estimator and the $r - k$ class estimator (Baye and Parker(1984)) and the form of the estimator is given as

$$\hat{\beta}_{SR}(r,k) = T_r(T_r'X'\Omega^{-1}XT_r + T_r'H'W^{-1}HT_r + kI_r)^{-1} (T_r'X'\Omega^{-1}y + T_r'H'W^{-1}h), k > 0 \quad (10)$$

This estimator gives the MR, SRRR and SRPCR estimators as its special case for certain values of r and k. They evaluated the performance of the SRrk estimator with the MR, SRRR and the RR estimators by means of MSE matrix criterion and also proposed tests for verifying the conditions.

Following Huang and Yang(2015), the PCTP estimator of β in model (6) is given as

$$\hat{\beta}_{SR}(r,k,d) = T_r(T_r'X_M^*{}'X_M^*T_r + I_r)^{-1}(T_r'X_M^*{}'X_M^*T_r + dI_r) \times (T_r'X_M^*{}'X_M^*T_r + kI_r)^{-1}T_r'X_M^*{}'y_M^*$$

After substituting the values of X_M^* and y_M^*, we obtain stochastic restricted PCTP (TPSRPC) estimator, given as

$$\hat{\beta}_{SR}(r,k,d) = T_r(T_r'(X'\Omega^{-1}X + H'W^{-1}H)T_r + I_r)^{-1}(T_r'(X'\Omega^{-1}X + H'W^{-1}H)T_r + dI_r)$$

$$(T_r'(X'\Omega^{-1}X + H'W^{-1}H)T_r + kI_r)^{-1}(T_r'X'\Omega^{-1}y + T_r'H'W^{-1}h) = T_rG_{r1}^{-1}G_{rd}G_{rk}^{-1}T_r'(X'\Omega^{-1}y + H'W^{-1}h), k > 0, 0 < d < 1, \quad (11)$$

where $G_{r1} = G_r + I_r$, $G_{rd} = G_r + dI_r$ and $G_{rk} = G_r + kI_r$. This estimator is a general class of estimators which includes the MR, SRRR, SRPCR estimators as its special case. For example, when $(r, k, d) = (p, 0, 1)$, the TPSRPC estimator reduces to the MR estimator, for $(r, k, d) = (p, k, 1)$ it gives the SRRR estimator, for $(r, k, d) = (r, 0, 1)$ we get the SRPCR estimator, and a substitution of $d = 1$ in the TPSRPC estimator results in the SRrk estimator.

Further, we will study the properties of the proposed estimator and compare its performance with the other competing estimators under the MSE and MSE matrix criteria in the next section.

3. COMPARISON OF THE ESTIMATORS

Let $M(\tilde{\beta})$ stands for the MSE matrix of an estimator $\tilde{\beta}$ of β, then $M(\tilde{\beta})$ is defined as

$$M(\tilde{\beta}) = Cov(\tilde{\beta}) + Bias(\tilde{\beta})Bias(\tilde{\beta})', \tag{12}$$

where $Cov(\tilde{\beta})$ is covariance matrix and $Bias(\tilde{\beta}) = E(\tilde{\beta}) - \beta$ is the bias vector of the estimator $\tilde{\beta}$ of β, respectively.

From (11), the bias and covariance matrix of the TPSRPC estimator are given as

$$Bias\left(\hat{\beta}_{SR}(r,k,d)\right) = \left(T_r G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r T_r' - I_p\right)\beta \tag{13}$$

and

$$Cov\left(\hat{\beta}_{SR}(r,k,d)\right) = \sigma^2 T_r G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r G_{rk}^{-1} G_{rd} G_{r1}^{-1} T_r', \tag{14}$$

respectively.

Thus, on substituting the expressions in (13) and (14) into (12), the simplified form of the MSE matrix of the TPSRPC estimator is obtained as

$$\begin{aligned} M\left(\hat{\beta}_{SR}(r,k,d)\right) = \sigma^2 T_r G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r G_{rk}^{-1} G_{rd} G_{r1}^{-1} T_r' \ (15) \\ +\left(T_r G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r T_r' - I_p\right)\beta\beta'^{\left(T_r G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r T_r' - I_p\right)}. \end{aligned} \tag{15}$$

The MSE matrix of the special cases of the proposed estimator can be obtained by substituting suitable values of r, k and d in (15), such as

$$M(\hat{\beta}_M) = \sigma^2 T G^{-1} T', \tag{16}$$

$$M\left(\hat{\beta}_{SR}(k)\right) = \sigma^2 T G_k^{-1} G G_k^{-1} T' + k^2 T G_k^{-1} T' \beta\beta' T G_k^{-1} T', \tag{17}$$

$$M\left(\hat{\beta}_{SR}(r)\right) = \sigma^2 T_r G_r^{-1} T_r' + \left(T_r T_r' - I_p\right)\beta\beta'\left(T_r T_r' - I_p\right), \tag{18}$$

$$M(\hat{\beta}_{SR}(r,k)) = \sigma^2 T_r G_{rk}^{-1} G_r G_{rk}^{-1} T_r' + (T_r G_{rk}^{-1} G_r T_r' - I_p)\beta\beta'(T_r G_{rk}^{-1} G_r T_r' - I_p). \tag{19}$$

Furthermore, MSE of an estimator, though being weaker criterion than the MSE matrix, is a widely accepted comparison criterion. The MSE of an estimator can be obtained using MSE matrix as follows:

$$m(\tilde{\beta}) = tr(M(\tilde{\beta})) = tr(Cov(\tilde{\beta})) + Bias(\tilde{\beta})' Bias(\tilde{\beta}). \tag{20}$$

Further, using the expressions in (20) and (15), the MSE of the TPSRPC estimator is given as

$$m\left(\hat{\beta}_{SR}(r,k,d)\right) = tr\left(M\left(\hat{\beta}_{SR}(r,k,d)\right)\right)$$
$$= \sum_{i=1}^{r} \frac{g_i(g_i+d)^2\sigma^2 + (k+(1+k-d)g_i)^2\alpha_i^2}{(g_i+1)^2(g_i+k)^2} + \sum_{i=r+1}^{p} \alpha_i^2, \quad (21)$$

where α_i is the i^{th} element of $\alpha = T'\beta$. And MSEs of special cases of the estimators are obtained as

$$m(\hat{\beta}_M) = \sum_{i=1}^{p} \sigma^2/g_i, \quad (22)$$

$$m(\hat{\beta}_{SR}(k)) = \sum_{i=1}^{p} (g_i\sigma^2 + k^2\alpha_i^2)/(g_i+k)^2, \quad (23)$$

$$m(\hat{\beta}_{SR}(r)) = \sum_{i=1}^{r} \sigma^2/g_i + \sum_{i=r+1}^{p} \alpha_i^2, \quad (24)$$

$$m(\hat{\beta}_{SR}(r,k)) = \sum_{i=1}^{r} (g_i\sigma^2 + k^2\alpha_i^2)/(g_i+k)^2 + \sum_{i=r+1}^{p} \alpha_i^2. \quad (25)$$

Further, we will compare the performance of the TPSRPC estimator with the MR, SRRR, SRPCR and the SRrk estimators under the MSE matrix and MSE criteria. To facilitate the comparison between the estimators under the MSE matrix criterion, following lemmas are used:

Lemma 1 (Farebrother (1976))

Let A be a positive definite matrix, namely $A > 0$ and α be some vector, then $A - \alpha\alpha' \geq 0$ if and only if $\alpha' A^{-1}\alpha \leq 1$.

Lemma 2 (Rao and Totenburg (1995))

Assume that $\hat{\beta}_j = A_j y, j = 1,2$ be two competing linear estimators of β. Suppose that $D = Cov(\hat{\beta}_1) - Cov(\hat{\beta}_2) > 0$, where $Cov(\hat{\beta}_j), j = 1,2$ denotes the covariance matrix of $\hat{\beta}_j$. Then $\Delta(\hat{\beta}_1, \hat{\beta}_2) = M(\hat{\beta}_1) - M(\hat{\beta}_2) \geq 0$ if and only if $d_{2\prime}(D + d_1 d_{1\prime})^{-1} d_2 \leq 1$, where d_j denote the bias vector of $\hat{\beta}_j$.

Lemma 3 (Baksalary and Trenkler (1991))

Let $C_{n\times p}$ be the set of $n \times p$ complex matrices and $H_{n\times n}$ be the subset of $C_{n\times p}$ consisting of Hermitian matrices. Further, given $L \in C_{n\times p}$, L^*, $R(L)$ and $\kappa(L)$ stand for the conjugate transpose, the range and the set of all generalized inverses, respectively of L. Now, let $D \in H_{n\times n}$, a_1 and $a_2 \in C_{n\times 1}$ be linearly independent, $f_{ij} = a_i^* D^- a_j$, $i,j = 1,2$ for $D^- \in \kappa(L)$, and if $a_1 \notin R(D)$, let $s = [a_1^*(I - DD^-)^*(I - DD^-)a_2]/[a_1^*(I - DD^-)^*(I - DD^-)a_1]$. Then $D + a_1 a_1^* - a_2 a_2^* \geq 0$ if and only if one of the following sets of conditions holds:

i) $D \geq 0$, $a_i \in R(D)$, $i = 1,2$, $(f_{11} + 1)(f_{22} - 1) \leq |f_{12}|^2$;
ii) $D > 0$, $a_1 \notin R(D)$, $a_2 \in R(D: a_1)$, $(a_2 - sa_1)^* D^- (a_2 - sa_1) \leq 1 - |s|^2$;
iii) $D = U\Delta U^* - \lambda vv^*$, $a_i \in R(D)$, $i = 1,2$, $v^* a_1 \neq 0$, $f_{11} + 1 \leq 0$, $f_{22} - 1 \leq 0$, $(f_{11} + 1)(f_{22} - 1) \geq |f_{12}|^2$;

where $(U: v)$ is a subunitary matrix (U possibly absent), Δ is a positive definite diagonal matrix (occuring when U is present) and λ is a positive scalar. Further, the condition (a), (b) and (c) are all independent of the choice of $D^- \in \kappa(L)$.

3.1 Comparison of the TPSRPC Estimator with the MR Estimator

Theorem 3.1

A necessary and sufficient condition for the superiority of the TPSRPC estimator over the MR estimator in MSE matrix sense is given by

$$\beta'[T_r((1 + k - d)G_r + kI_r)G_r(2G_r^2 + (1 + k + d)G_r + kI_r)^{-1}T_r' + T_{p-r}G_{p-r}T_{p-r}{}']\beta \leq \sigma^2.$$

Proof.

From (15) and (16), the difference of the MSE matrices of the MR and TPSRPC estimators, say Δ_1, can be obtained as

$$\begin{aligned}\Delta_1 &= M(\hat{\beta}_M) - M\left(\hat{\beta}_{SR}(r,k,d)\right)\\ &= \sigma^2[TG^{-1}T' - T_rG_{r1}^{-1}G_{rd}G_{rk}^{-1}G_rG_{rk}^{-1}G_{rd}G_{r1}^{-1}T_r']\\ &\quad -(T_rG_{r1}^{-1}G_{rd}G_{rk}^{-1}G_rT_r{}' - I_p)\beta\beta'(T_rG_{r1}^{-1}G_{rd}G_{rk}^{-1}G_rT_r{}' - I_p)\\ &= \sigma^2\left[T_rG_{r1}^{-1}G_{rk}^{-1}G_r^{-1}(G_{r1}^2G_{rk}^2 - G_{rd}^2G_r^2)G_{r1}^{-1}G_{rk}^{-1}T_r{}' + T_{p-r}{}'G_{p-r}^{-1}T_{p-r}{}'\right]\\ &\quad -(T_rG_{r1}^{-1}G_{rd}G_{rk}^{-1}G_rT_r{}' - I_p)\beta\beta'(T_rG_{r1}^{-1}G_{rd}G_{rk}^{-1}G_rT_r{}' - I_p)\\ &= \sigma^2TG^{-1}M_1T' - a_1a_1{}',\end{aligned} \tag{26}$$

where

$$M_1 = \begin{pmatrix} G_{r1}^{-1}G_{rk}^{-1}(2G_r^2 + (1 + k + d)G_r + kI_r) \times ((1 + k - d)G_r + kI_r)G_{r1}^{-1}G_{rk}^{-1} & 0 \\ 0 & I_{p-r} \end{pmatrix}$$

and

$$a_1 = (T_rG_{r1}^{-1}G_{rd}G_{rk}^{-1}G_rT_r' - I_p)\beta = -TQT'\beta \text{ with}$$
$$Q = \begin{pmatrix} G_{r1}^{-1}G_{rk}^{-1}((1 + k - d)G_r + kI_r) & 0 \\ 0 & I_{p-r} \end{pmatrix}.$$

It can be clearly observed that M_1 is a positive definite matrix for all values of $k > 0$ and $0 < d < 1$. Thus, *Lemma* 1 can be applied to examine the positive definiteness of Δ_1. Therefore, the necessary and sufficient condition is obtained as

$$a_1{,}TM_1^{-1}GT'a_1 \leq \sigma^2. \tag{27}$$

The condition can be further simplified as follows

$$\beta'TQ'M_1^{-1}GQT'\beta = \beta'TG^*T'\beta \leq \sigma^2, \tag{28}$$

where

$$G^* = Q'M_1^{-1}GQ = \begin{pmatrix} ((1 + k - d)G_r + kI_r)G_r \times (2G_r^2 + (1 + k + d)G_r + kI_r)^{-1} & 0 \\ 0 & G_{p-r} \end{pmatrix}.$$

Moreover, a substitution of $d = 1$ in Theorem 3.1, gives the necessary and sufficient condition of superiority of the SRrk estimator over the MR estimator under the MSE matrix sense obtained by Chandra and Sarkar (2016) as follows:

$$\beta'\left[T_r\left(\frac{2}{k}I_r + G_r^{-1}\right)^{-1} T_r' + T_{p-r}G_{p-r}T_{p-r}'\right]\beta \leq \sigma^2.$$

Theorem 3.2

If $\sigma^2 - g_i\alpha_i^2 > 0$ for all $i = r+1, r+2, \dots, p$ then the TPSRPC estimator dominates the MR estimator in the MSE sense when $(2g_i + (1+k+d))\sigma^2 - (k + (1+k-d)g_i)\alpha_i^2 \geq 0$ for all $i = 1,2,\dots,r$.

Proof.

Further, from (21) and (22), the difference of the MSEs of the MR and the TPSRPC estimators is obtained as follows

$$m(\hat{\beta}_M) - m(\hat{\beta}_{SR}(r,k,d)) = \sum_{i=1}^{r} \frac{(k + (k+1-d)g_i)[(2g_i + (1+k+d))\sigma^2 - (k + (1+k-d)g_i)\alpha_i^2]}{(g_i+1)^2(g_i+k)^2} + \sum_{i=r+1}^{p} \frac{\sigma^2 - g_i\alpha_i^2}{g_i}. \quad (29)$$

It is clear from the above expression that the difference $m(\hat{\beta}_M) - m(\hat{\beta}_{SR}(r,k,d))$ is positive when both first and second summations are positive.

3.2 Comparison of the TPSRPC Estimator with the SRRR Estimator

Theorem 3.3

A necessary and sufficient condition for superiority of the TPSRPC estimator over the SRRR estimator in MSE matrix sense is given by

$$\beta' T Q' G_k^{-1}(\sigma^2 G M_2 + k^2 T'\beta\beta' T)^{-1} G_k^{-1} Q T'\beta \leq 1. \quad (30)$$

Proof.

Using expressions in (15) and (17), the difference in the MSE matrices, say Δ_2, can be obtained as

$$\begin{aligned}\Delta_2 &= M(\hat{\beta}_{SR}(k)) - M(\hat{\beta}_{SR}(r,k,d)) \\ &= \sigma^2 T G_k^{-1} G G_k^{-1} T' - \sigma^2 T_r G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r G_{rk}^{-1} G_{rd} G_{r1}^{-1} T_r' \\ &\quad + k^2 T G_k^{-1} T'\beta\beta' T G_k^{-1} T' - (T_r G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r T_r'' \\ &\quad - I_p)\beta\beta'(T_r G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r T_r' - I_p) \\ &= D_2 + a_1 a_1' - a_2 a_2',\end{aligned} \quad (31)$$

where

$D_2 = \sigma^2[T G_k^{-1} G G_k^{-1} T' - T_r G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r G_{rk}^{-1} G_{rd} G_{r1}^{-1} T_r'], a_1 = kTG_k^{-1}T'\beta$ and $a_2 = (T_r G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r T_r' - I_p)\beta = -TQT'\beta$.

D_2 can be further simplified as

$$\begin{aligned} D_2 &= \sigma^2 T_r G_{rk}^{-1} G_r G_{r1}^{-1}(G_{r1}^2 - G_{rd}^2) G_{r1}^{-1} G_{rk}^{-1} T_r' \\ &\quad +\sigma^2 T_{p-r} G_{(p-r)k}^{-1} G_{p-r} G_{(p-r)k}^{-1} T_{p-r}{}' \\ &= \sigma^2 (1-d) T_r G_{rk}^{-1} G_r G_{r1}^{-1}(2G_r + (1+d)I_r) G_{r1}^{-1} G_{rk}^{-1} T_r' \\ &\quad + \sigma^2 T_{p-r} G_{(p-r)k}^{-1} G_{p-r} G_{(p-r)k}^{-1} T_{p-r}{}' \\ &= \sigma^2 T G_k^{-1} G M_2 G_k^{-1} T', \end{aligned} \tag{32}$$

where

$$M_2 = \begin{pmatrix} (1-d) G_{r1}^{-1}(2G_r + (1+d)I_r) G_{r1}^{-1} & 0 \\ 0 & I_{p-r} \end{pmatrix}.$$

From (3.20), it can be noticed that D_2 is a positive definite matrix for all values of $k > 0$ and $0 < d < 1$. Further, from *Lemma* 2, $\Delta_2 \geq 0$ if and only if

$$a_{2}{}'(D_2 + a_1 a_1{}')^{-1} a_2 \leq 1. \tag{33}$$

On substituting the values of a_1, a_2 and D_2, we get

$$\beta' T Q T' (\sigma^2 T G_k^{-1} G M_2 G_k^{-1} T' + k^2 T G_k^{-1} T' \beta \beta' T G_k^{-1} T' \beta) T Q T' \beta \leq 1. \tag{34}$$

A further simplification gives the condition for the dominance of the TPSRPC estimator over the SRRR estimator.

Theorem 3.4

The TPSRPC estimator outperforms the SRRR estimator in MSE sense if $k < \frac{\sigma^2 - g_i \alpha_i^2}{2\alpha_i^2}$ *for all* $i = r+1, r+2, \ldots, p$ *and* $(2g_i + 1 + d)\sigma^2 - (2k + (2k + 1 - d) g_i)\alpha_i^2 > 0$ *for all* $i = 1, 2, \ldots, r$.

Proof.

Using (21) and (23), the difference of the MSEs of the TPSRPC estimator and the SRRR estimator is given by

$$\begin{aligned} & m(\hat{\beta}_{SR}(k)) - m(\hat{\beta}_{SR}(r,k,d)) \\ &\quad = \sum_{i=1}^{r} \frac{(1-d) g_i \{(2g_i + 1 + d)\sigma^2 - (2k + (2k+1-d) g_i)\alpha_i^2\}}{(g_i + 1)^2 (g_i + k)^2} \\ &\qquad + \sum_{i=r+1}^{p} \frac{g_i(\sigma^2 - g_i \alpha_i^2) - 2k g_i \alpha_i^2}{(g_i + k)^2}. \end{aligned} \tag{35}$$

Clearly, $m(\hat{\beta}_{SR}(k)) - m(\hat{\beta}_{SR}(r,k,d))$ is positive when first and second summations take positive values. Now, the second summation is positive if $g_i(\sigma^2 - g_i \alpha_i^2) - 2k g_i \alpha_i^2 > 0$ for all $i = r+1, r+2, \ldots, p$, which holds when

$$k < \frac{\sigma^2 - g_i \alpha_i^2}{2\alpha_i^2} \text{ forall } i = r+1, r+2, \ldots, p.$$

Further, the first summation will be positive if $(1-d) g_i \{(2g_i + 1 + d)\sigma^2 - (2k + (2k + 1 - d) g_i)\alpha_i^2\} > 0$ for all $i = 1, 2, \ldots, r$.

3.3 Comparison of the TPSRPC Estimator with the SRPCR Estimator

Theorem 3.5

A necessary and sufficient condition for superiority of the TPSRPC estimator over the SRPCR estimator under the MSE matrix criterion is $T_r{}'\beta = 0$.

Proof.

From (15) and (18), the difference of the MSE matrices of the TPSRPC estimator and the SRPCR estimator, say Δ_3, is obtained as

$$\begin{aligned}\Delta_3 &= M(\hat{\beta}_{SR}(r)) - M(\hat{\beta}_{SR}(r,k,d))\\ &= \sigma^2 T_r G_r^{-1} T_r' + \left(T_r T_r' - I_p\right)\beta\beta'\left(T_r T_r' - I_p\right)\\ &\quad -\sigma^2 T_r G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r G_{rk}^{-1} G_{rd} G_{r1}^{-1} T_r{}'\\ &\quad -\left(T_r G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r T_r{}' - I_p\right)\beta\beta'\left(T_r G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r T_r{}' - I_p\right)\\ &= D_3 + a_1 a_1{}' - a_2 a_2{}', \qquad (36)\end{aligned}$$

where

$$\begin{aligned}D_3 &= \sigma^2 T_r (G_r^{-1} - G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r G_{rk}^{-1} G_{rd} G_{r1}^{-1}) T_r', \; a_1 = \left(T_r T_r' - I_p\right)\beta\\ &= -T_{p-r} T_{p-r}{}'\beta \text{ and } a_2 = \left(T_r G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r T_r{}' - I_p\right)\beta = -TQT'\beta.\end{aligned}$$

Now, we simplify D_3 as

$$\begin{aligned}D_3 &= \sigma^2 T_r (G_r^{-1} - G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r G_{rk}^{-1} G_{rd} G_{r1}^{-1}) T_r{}'\\ &= \sigma^2 T_r G_{r1}^{-1} G_{rk}^{-1} B G_{rk}^{-1} G_{r1}^{-1} T_r{}', \qquad (37)\end{aligned}$$

where $B = 2(k+1-d)G_r^2 + (4k + k^2 + 1 - d^2)G_r + 2k(1+k) + k^2 G_r^{-1}$ is positive definite for all values of $k > 0$ and $0 < d < 1$, thus D_3 is also a positive definite matrix.

Further, by using the method of Moore-Penrose to find the generalized inverse of a matrix, we have the generalized inverse D_3^- of D_3 given as

$$D_3^- = \sigma^{-2} T_r G_{r1} G_{rk} B^{-1} G_{rk} G_{r1} T_r'. \qquad (38)$$

It is easy to verify that $D_3 D_3^- = T_r T_r'$ and since $a_1 = -T_{p-r} T_{p-r}{}'\beta \neq D_3 D_3^- a_1$, $a_1 \notin R(D_3)$. Hence, part (a) of *Lemma* 3 is not applicable here. Further, $a_2 = a_1 + D_3 \eta_1$ with $\eta_1 = -\sigma^2 T_r G_{r1} G_{rk} B^{-1}((k+1-d)G_r + kI_r)T_r{}'\beta$, thus $a_2 \in R(D_1 : a_1)$, hence part (b) of the *Lemma* 3 can be applied here. Furthermore, it is easy to check that in our case $s = 1$ from the definition of s. Using *Lemma* 3, $\Delta_3 \geq 0$, that is the TPSRPC estimator dominates the SRPCR estimator, if and only if

$$(a_2 - a_1)'^{D_3^-}(a_2 - a_1) = \eta_1^{\prime D_3 \eta_1} \leq 0. \qquad (39)$$

Further,

$$\begin{aligned}\eta_1{}' D_3 \eta_1 &= \sigma^{-2}\beta' T_r ((k+1-d)G_r + kI_r) B^{-1}\\ &\qquad ((k+1-d)G_r + kI_r) T_r'\beta\\ &= \beta' W_1 W_1 \beta \leq 0 \qquad (40)\end{aligned}$$

where $W_1 = \sigma^{-1} B^{-\frac{1}{2}}((k+1-d)G_r + kI_r)'$.

Since $\beta' W_1 B^{-1} W_1 \beta$ is a positive definite matrix, the above condition holds when $W_1\beta = \sigma^{-1} B^{-1/2}((k+1-d)G_r + kI_r)T_r'\beta = 0$, i.e. $T_r'\beta = 0$.

Theorem 3.6

The TPSRPC estimator dominates the SRPCRE in the MSE sense when $(2g_i + (1 + k + d))\sigma^2 - (k + (1 + k - d)g_i)\alpha_i^2 \geq 0$ *for all* $i = 1,2,\dots,r$.

Proof.

From (21) and (24), the difference in the MSEs of the SRPCR and TPSRPC estimators is obtained as

$$m\left(\hat{\beta}_{SR}(r)\right) - m\left(\hat{\beta}_{SR}(r,k,d)\right) = \sum_{i=1}^{r} \frac{(k+(k+1-d)g_i)\{(2g_i+(1+k+d))\sigma^2-(k+(1+k-d)g_i)\alpha_i^2\}}{(g_i+1)^2(g_i+k)^2} \quad (41)$$

Evidently, the difference is positive when the term $(2g_i + (1 + k + d))\sigma^2 - (k + (1 + k - d)g_i)\alpha_i^2$ is positive for all $i = 1,2,\dots,r$.

3.4 Comparison of the TPSRPC Estimator with the SRrk Class Estimator

Theorem 3.7

A necessary and sufficient condition for the superiority of the TPSRPC estimator over the SRrk class estimator in the MSE matrix sense is $T_r'\beta = 0$.

Proof.

From (15) and (19), the difference in the MSE matrices of the SRrk and TPSRPC estimators, say Δ_4, is obtained as

$$\begin{aligned}\Delta_4 &= M(\hat{\beta}_{SR}(r,k)) - M(\hat{\beta}_{SR}(r,k,d)) \\ &= \sigma^2 T_r G_{rk}^{-1} G_r G_{rk}^{-1} T_r' - \sigma^2 T_r G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r G_{rk}^{-1} G_{rd} G_{r1}^{-1} T_r' \\ &\quad + (T_r G_{rk}^{-1} G_r T_r' - I_p)\beta\beta'(T_r G_{rk}^{-1} G_r T_r' - I_p) \\ &\quad - \left(T_r G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r T_r' - I_p\right)\beta\beta'\left(T_r G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r T_r' - I_p\right) \\ &= D_4 + a_1 a_{1'} - a_2 a_2', \end{aligned} \quad (42)$$

where $D_4 = \sigma^2 T_r(G_{rk}^{-1} G_r G_{rk}^{-1} - G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r G_{rk}^{-1} G_{rd} G_{r1}^{-1})T_r'$, $a_1 = (T_r G_{rk}^{-1} G_r T_r' - I_p)\beta$ and $a_2 = (T_r G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r T_r' - I_p)\beta$.

Further, D_4 can be simplified as

$$D_4 = \sigma^2(1-d) T_r G_{rk}^{-1} G_{r1}^{-1} (2G_r^2 + (1+d)G_r) G_{r1}^{-1} G_{rk}^{-1} T_{r\prime}. \quad (43)$$

And the generalized inverse D_4^- of D_4 is obtained as

$$D_4^- = \frac{\sigma^{-2}}{(1-d)} T_r G_{rk} G_{r1} (2G_r^2 + (1+d)G_r)^{-1} G_{r1} G_{rk} T_r'. \quad (44)$$

It is simple to obtain that $D_4 D_4^- = T_r T_r'$ and $D_4 D_4^- a_1 \neq a_1$, that is $a_1 \notin R(D_4)$, thus part (a) of *Lemma* 3 can not be applied here. Further, $a_2 = a_1 + D_4\eta_2$, where $\eta_2 = -\sigma^{-2} T_r G_{rk} G_{r1} (2G_r^2 + (1+d)G_r)^{-1} G_r T_r' \beta$ which implies that $a_2 \in R(D_4 : a_1)$. Furthermore, it is easy to verify from the definition of s that in this case $s = 1$. By using part (b) of *Lemma* 3, $\Delta_4 \geq 0$, that is the TPSRPC estimator performs better than the SRrk class estimator if and only if

$$(a_2 - a_1)' D_4^- (a_2 - a_1) = \eta_2' D_4 \eta_2 \leq 0. \quad (45)$$

On further simplification, we have

$$\eta_2' D_4 \eta_2 = \sigma^{-2}(1-d)^2 \beta' T_r (2(1-d) + (1-d^2) G_r^{-1})^{-1} T_r' \beta \\ = \beta' W_2' W_2 \beta \leq 0, \tag{46}$$

where $W_2 = \sigma^{-1}(1-d)(2(1-d) + (1-d^2)G_r^{-1})^{-\frac{1}{2}} T_r'$.

However, $\beta' W_2' W_2 \beta$ being a quadratic form, (46) holds only if $W_2\beta = \sigma^{-1}(1-d)(2(1-d) + (1-d^2)G_r^{-1})^{-1/2} T_r'\beta = 0$ which in turn holds when $T_r'\beta = 0$.

Theorem 3.8

The TPSRPC estimator dominates the SRrk estimator in the MSE sense when $(2g_i + 1 + d)\sigma^2 - ((2k + 1 - d)g_i + 2k)\alpha_i^2 \geq 0$ *for all* $i = 1,2,\dots,r$.

Proof.

The difference of the MSEs of the TPSRPC estimator and the SRrk estimator using (21) and (25) is given as

$$m\left(\hat{\beta}_{SR}(r,k)\right) - m\left(\hat{\beta}_{SR}(r,k,d)\right) \\ = \sum_{i=1}^{r} \frac{(1-d)g_i\{(2g_i + 1 + d)\sigma^2 - (2k + (2k + 1 - d)g_i)\alpha_i^2\}}{(g_i + 1)^2 (g_i + k)^2}. \tag{47}$$

Since $0 < d < 1$, $1 - d$ is always positive, hence the difference is positive when the term $(2g_i + 1 + d)\sigma^2 - (2k + (2k + 1 - d)g_i)\alpha_i^2$ is positive for $i = 1,2,\dots,r$.

4. TESTS FOR VERIFYING CONDITIONS UNDER MSE MATRIX

The superiority conditions of the TPSRPC estimator over the others which are stated in Theorems 3.1, 3.3, 3.5 and 3.7 in MSE matrix sense involve unknown parameters β and σ^2 and hence can not be verified in practice. Following Sarkar(1996), Sarkar and Chandra (2015), Chandra and Sarkar(2016), tests are proposed to verify the conditions under the MSE matrix criterion. However, since the condition stated in Theorem 3.3 for the superiority of the TPSRPC estimator over the SRRR estimator is quite complex, the tests are developed for the rest three conditions. Moreover, the superiority conditions for the TPSRPC estimator over the SRPCR and SRrk estimators stated in Theorems 3.5 and 3.7 are same, hence one test would suffice to test the superiority of the TPSRPC estimator over the SRPCR and SRrk estimators. Thus, in all, we would develop two tests to verify the conditions of superiority of the TPSRPC estimator over the MR estimator, and over the SRPCR and SRrk estimators.

4.1 Testing between the TPSRPC Estimator and the MR Estimator

After a careful observation of the superiority condition for the TPSRPC estimator over the MR estimator under MSE matrix criterion which is stated in Theorem 3.1, it is found that it is difficult to propose test to verify this condition. Further, since $G - G^*$ is n.n.d. matrix, it is apparent that $\beta' T G T' \beta \leq \sigma^2$ is sufficient to hold $\beta' T G^* T' \beta \leq \sigma^2$. Hence, following Chandra and Sarkar(2016), we develop a test for testing the sufficient condition.

To test the sufficient condition $\beta' TGT' \beta \leq \sigma^2$, the null and alternative hypotheses are given as follows:

$$H_{0a}: \frac{\beta' TGT' \beta}{\sigma^2} \leq 1 \text{ and } H_{1a}: \frac{\beta' TGT' \beta}{\sigma^2} > 1. \quad (48)$$

Under the assumption of normality of disturbance terms, $G^{1/2}T'\hat{\beta}_M$ follows normal distribution with mean vector $G^{1/2}T'\beta$ and covariance matrix $\sigma^2 I_n$. Further, the test statistic is obtained as

$$F_1 = \frac{\beta' TGT' \beta / p}{e'e/(n-p)} \sim F_{(p,n-p)}(\lambda), \quad (49)$$

where, $e'e/(n-p)$ is the MR estimator of σ^2. The statistic F_1 follows non-central F-distribution with degrees of freedom p and $n-p$ with non-centrality parameter $\lambda = \beta' TGT' \beta / \sigma^2$.

Further, a non-central F-distribution, say, $F_{(p,n-p)}(\lambda)$ with non-centrality parameter λ can be approximated to $(1 + \lambda/p)F_{(p^*,n-p)}$, with $p^* = p + \frac{\lambda^2}{(p+2\lambda)}$, which is greater than p and approximated to the nearest integer value (see Johnson and Kotz (1970) for details). Thus, an equivalent test statistic to F_1 in (49) can be restated in terms of central F-distribution, which is as follows:

$$F_1^* = \frac{F_1}{1 + \frac{\lambda}{p}} \sim F_{(p^*,n-p)} \quad (50)$$

Further, for a test of size α under $\lambda = 1$,

$$P_{\lambda=1}(F_{(p^*,n-p)}(\lambda) > C_r) = \alpha,$$

$P_\lambda(F_{(p^*,n-p)}(\lambda) > C_r) < \alpha$ for all $\lambda < 1$ where C_r is the critical value at $\alpha\%$ level of significance. Hence, we carry out the test with the value of the non-centrality parameter $\lambda = 1$. (see Sarkar and Chandra (2015) and Chandra and Sarkar (2016), for details).

4.2 Testing between the TPSRPC Estimator and the SRPCR and SRrk Estimators

The null and alternative hypotheses for testing the conditions stated in Theorems 3.5 and 3.7 are given as:

$$H_{0b}: T_r{}'\beta = 0 \text{ and } H_{1b}: T_r{}'\beta \neq 0. \quad (51)$$

Under the assumption of normality of disturbance terms, $\hat{\beta}_{SR}(r,k,d)$ is normally distributed with mean $T_r G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r T_r{}'\beta$ and covariance matrix $\sigma^2 T_r G_{r1}^{-1} G_{rd} G_{rk}^{-1} G_r G_{rk}^{-1} G_{rd} G_{r1}^{-1} T_r{}'$. Hence, $T_r{}'\hat{\beta}_{SR}(r,k,d)$ is an unbiased estimator of $T_r{}'\beta$ under the null hypothesis. Thus, the test statistic can be defined as

$$F_2 = \frac{\hat{\beta}_{SR}(r,k,d)' T_r G_{r1} G_{rd}^{-1} G_{rk} G_r^{-1} G_{rk} G_{rd}^{-1} G_{r1} T_r{}' \hat{\beta}_{SR}(r,k,d)/r}{e'e/(n-p)}, \quad (52)$$

which follows F-distribution with degrees of freedom r and $n-p$.

5. SELECTION OF k AND d

In this section, we will discuss the selection of the biasing parameters k and d in the TPSRPC estimator by means of minimizing the MSE. For that purpose, using the expression in (21), the first partial derivatives of the MSE of the TPSRPC estimator with respect to k and d are obtained as

$$\frac{\partial m\left(\hat{\beta}_{SR}(r,k,d)\right)}{\partial k} = \sum_{i=1}^{r} \frac{-2\sigma^2 g_i(g_i+d)^2 + 2g_i(g_i+d)((1-d)g_i + k(g_i+1))\alpha_i^2}{(g_i+1)^2(g_i+k)^3} \quad (53)$$

and

$$\frac{\partial m\left(\hat{\beta}_{SR}(r,k,d)\right)}{\partial d} = \sum_{i=1}^{r} \frac{g_i\left[\sigma^2(g_i+d) - \left((1-d)g_i + k(g_i+1)\right)\alpha_i^2\right]}{(g_i+1)^2(g_i+k)^2}, \quad (54)$$

respectively.

Further, the optimum value of k for a fixed value of d can be found as a solution to $\frac{\partial m(\hat{\beta}_{SR}(r,k,d))}{\partial k} = 0$, which is obtained to be

$$k = \frac{g_i(\alpha_i^2 - \sigma^2) - d(g_i\alpha_i^2 - \sigma^2)}{(g_i+1)\alpha_i^2} \text{ forall } i = 1,2,\dots,r. \quad (55)$$

Since the value of k varies for different values of i, [Kibria(2003) suggested to use arithmetic and geometric mean of the values obtained in (55) as optimum value. Following the same ideology, we suggest the optimum values of k for a fixed d as follows

$$k_{AM} = \frac{1}{r}\sum_{i=1}^{r} \frac{g_i(\alpha_i^2 - \sigma^2) - d(g_i\alpha_i^2 - \sigma^2)}{(g_i+1)\alpha_i^2}, \quad (56)$$

$$k_{GM} = \left(\prod_{i=1}^{r} \frac{g_i(\alpha_i^2-\sigma^2) - d(g_i\alpha_i^2-\sigma^2)}{(g_i+1)\alpha_i^2}\right)^{1/r}. \quad (57)$$

Here it is important to note that for the positivity of k_{AM} and existence of k_{GM}, the value of k in (55) should be positive for all $i = 1,2,\dots,r$, hence we have the following conditions. If $\alpha_i^2 - \sigma^2$ and $g_i\alpha_i^2 - \sigma^2$, both are either positive or negative, for all $i = 1,2,\dots,r$ then k is positive for $d < \min_{i\in N_r}\left\{\frac{g_i(\alpha_i^2-\sigma^2)}{(g_i\alpha_i^2-\sigma^2)}\right\}$.

If $\alpha_i^2 - \sigma^2 > 0$ and $g_i\alpha_i^2 - \sigma^2 < 0$ for all $i = 1,2,\dots,r$, then k is positive for all values of $0 < d < 1$.

When $\alpha_i^2 - \sigma^2 < 0$ and $g_i\alpha_i^2 - \sigma^2 > 0$, for all $i = 1,2,\dots,r$ then there is no d for which k is positive. In this case we have to chose a value of d such that k_{AM} is positive and k_{GM} exists.

Furthermore, the optimum value of d for a fixed k, is obtained as a solution of $\frac{\partial m(\hat{\beta}_{SR}(r,k,d))}{\partial d} = 0$, which on using expression in (54) is found to be

$$d_{opt} = \frac{\sum_{i=1}^{r} \frac{g_i(k(g_i+1)\alpha_i^2 + g_i(\alpha_i^2 - \sigma^2))}{(g_i+1)^2(g_i+k)^2}}{\sum_{i=1}^{r} \frac{g_i(g_i\alpha_i^2+\sigma^2)}{(g_i+1)^2(g_i+k)^2}} \tag{58}$$

For d_{opt} to be positive, $g_i(k(g_i+1)\alpha_i^2 + g_i(\alpha_i^2 - \sigma^2))$ should be positive for all $i = 1,2,\ldots,r$. Hence, if $\sigma^2 < \alpha_i^2$, $i = 1,2,\ldots,r$, d_{opt} is positive for all $k > 0$, however if $\sigma^2 > \alpha_i^2$, $i = 1,2,\ldots,r$, d_{opt} will be positive for $k > \max_{i=1,2,\ldots,r}\left\{\frac{g_i(\sigma^2-\alpha_i^2)}{(g_i+1)\alpha_i^2}\right\}$. Further, d_{opt} will be less than 1 when $g_i(k(g_i+1)\alpha_i^2 + g_i(\alpha_i^2 - \sigma^2)) < g_i(g_i\alpha_i^2 + \sigma^2)$ for all $i = 1,2,\ldots,r$, which will hold for $k < \frac{\sigma^2}{\alpha_{\max}^2}$.

For operational purposes, the unknown parameters α_i and σ^2 can be replaced by their respective unbiased estimators. Note that on substituting $r = p$, $\Omega = I$ and $H = 0$ in the expressions of optimum values of k and d, we will find the optimum values of k and d for two-parameter estimator which was obtained by Yang and Chang (2010).

6. NUMERICAL EXAMPLE

In order to illustrate our theoretical results, we now consider in this section the data set on Total National Research and Development Expenditures as a Percent of Gross National Product, originally due to Gruber (1998) and also analyzed by Zhong and Yang (2007). It represents the relationship between the dependent variable Y, the percents spent by the U.S., and the four other independent variables X_1, X_2, X_3, and X_4. The variables X_1, X_2, X_3 and X_4, respectively represent the percent spent by France, West Germany, Japan and the former Soviet Union.

For this data set, the OLS estimator of $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)'$ is given as $\hat{\beta}$ = (0.6455, 0.0896, 0.1436, 0.1526)$'$. The Durbin-Watson statistic for the residuals is found to be 2.633842 with the p-value of 0.576, which concludes that there is no autocorrelation, hence we take $\Omega = I$. We obtain the eigenvalues of $X'X$ matrix as λ_1 = 302.9626, λ_2 = 0.7283, λ_3= 0.0446, and λ_4 = 0.0345 and the condition index ($\sqrt{\lambda_{\max}/\lambda_{\min}} = \sqrt{\lambda_1/\lambda_4}$) is approximately 93.682, which indicates an ill-conditioned design matrix. Further, the estimated value of σ^2 is obtained to be 0.001530. Following Yang and Cui(2011), let us consider the following stochastic restrictions

$$h = H\beta + v, H = (1\ -2\ -2\ -2), v \sim N(0, \sigma^2).$$

The eigen values of the matrix $X'X + H'H$ are found to be g_1 =310.7975, g_2 =5.2094, g_3 =0.7274 and g_4 =0.0355, hence the condition index $\sqrt{g_1/g_4}$ is obtained as 93.459, which shows a severe multicollinearity in the data set. Furthermore, as the eigen values g_1, g_2, g_3 and g_4 represent the variation explained by the four principal components (PCs) obtained from variables in X_M matrix in model (5). Thus, it can be seen that the proportion of variation explained by the PCs are 98.1145, 1.6445, 0.2296 and 0.0112. Following Johnson and Wichern (2007), the value of r is chosen to be 3, which accounts for 99.98% of variation in X_M matrix and the values of α_i^2 are obtained to be $0.1901290, 0.2415385, 0.03686222$ for $i = 1,2,3$. Now, since for this data, $\sigma^2 < \alpha_i^2$ for $i = 1,2,3$, as discussed in Section 5, we choose the value of k as 0.003 which is less than $\sigma^2/\alpha_{\max}^2 = 0.0063$ and using the expression in (58), the optimum value of d for the value of k is obtained to be 0.9403.

The MSEs of the estimators are estimated using the expressions in (21) to (25). The results obtained are presented in Table 1.

Table 1
Estimated Coefficients and MSEs for the Data Set on Total National Research and Development Expenditures

	MR	**SRRR**	**SRPCR**	**SRrk**	**TPSRPC**
$\boldsymbol{\beta_1}$	0.646243	0.64574	0.645987	0.645504	0.639355
$\boldsymbol{\beta_2}$	0.089375	0.089069	0.086376	0.086304	0.086414
$\boldsymbol{\beta_3}$	0.143229	0.142983	0.145243	0.144841	0.142253
$\boldsymbol{\beta_4}$	0.152543	0.153158	0.153401	0.15395	0.158572
$\widehat{\boldsymbol{MSE}}$	**0.045418**	**0.038972**	**0.002416**	**0.002399**	**0.002331**

From the MSE values in Table 1, we can observe that the TPSRPC estimator has a very significant improvement over the MR and SRRR estimators in the MSE sense. However, the SRPCR, SRrk and the TPSRPC estimators exhibit almost equal MSEs for this data set and the difference can be only noticed after 4^{th} or 5^{th} decimal places. If the difference after 5^{th} decimal place is accounted, it can be said that the TPSRPC estimator gives minimum MSE among all the estimators.

Further, the value of F_1^* in (50) to test H_{0a} against H_{1a} is obtained to be 7889.326 with critical value 9.148301 at 1% level of significance. Hence, the null hypothesis for dominance of TPSRPC estimator over the MR estimator is rejected. Furthermore, the value of F_2 in (52) to test H_{0b} against H_{1b} is found to be 13148.88 with critical value 9.779538 at 1% level of significance. Clearly, the null hypothesis H_{0b} for superiority of the SRTPCTP estimator over the SRPCR and the SRrk estimators is also rejected. Thus, it can be concluded from the test results that the TPSRPC estimator does not perform better than the MR, SRPCR and the SRrk estimators in MSE matrix sense.

7. MONTE CARLO SIMULATION

In this section, we will evaluate the performance of the estimators through Monte-Carlo simulation. Following McDonald and Galarneau (1975) and Gibbons (1981), the observations in X matrix have been generated as follows:

$$x_{ij} = (1-\rho^2)^{1/2} z_{ij} + \rho z_{ip+1}, i = 1,2,\ldots,n, j = 1,2,\ldots,p, \tag{59}$$

where z_{ij} are generated from standard normal pseudo-random numbers and x_{ij}'s are generated such that the correlation between any pair of variables in X-matrix is ρ^2. In this study, we consider the values of ρ to be 0.95, 0.99 and 0.999. Following McDonald and Galarneau(1975), Gibbons(1981), Kibria(2003) and others, β has been chosen as the normalized eigenvector corresponding to the largest eigenvalue of the $X'X$ matrix. The dependent variable y is obtained by

$$y = X\beta + u. \tag{60}$$

Further, the H matrix has been constructed by m independent row vectors of standard normal pseudo-random numbers and h is obtained as

$$h = H\beta + v. \tag{61}$$

Following Firinguetti (1989), Judge et al.(1985) and Chandra and Sarkar (2016), the error vectors u and v in (60) and (61), respectively are generated independently from AR(1) process as follows:

$$u_i = \phi u_{i-1} + e_{1i}, v_j = \phi v_{i-1} + e_{2j}, i = 1,2,\dots,n, j = 1,2,\dots,m, \tag{62}$$

where e_{1i} and e_{2j} are independent normal pseudo-random numbers with mean 0 and variance σ_e^2 and ϕ is autoregressive coefficient such that $|\phi| < 1$. Further, the covariance matrix Ω for AR(1) errors is given by

$$\Omega = (\omega_{ij})_{n\times n}, \omega_{ij} = \sigma^2\phi^{|i-j|}, \tag{63}$$

where $\sigma^2 = \frac{\sigma_e^2}{1-\phi^2}$. And the covariance matrix W can be obtained by taking first m rows and columns of Ω matrix.

The value of r is chosen so that the first r components account for at least 90% of the variation in variables in X_M matrix. In this simulation we chose $n = 50$, $p = 5$, σ_e^2 =0.5, 1, 10, m =3, 5 and ϕ =0, 0.6, 0.9. Since the TPSRPC estimator gives other estimators as its special cases when k and/or d approach to either 0 or 1, the values of k and d are chosen to be k =0.01,0.5,0.99 and d =0.01,0.5,0.99. The experiment is repeated 2000 times by generating errors in every iteration and estimated MSE (EMSE) is calculated by the following formula:

$$EMSE(\hat{\beta}) = \frac{1}{2000}\sum_{i=1}^{2000} (\hat{\beta}_{(i)} - \beta)'(\hat{\beta}_{(i)} - \beta), \tag{64}$$

where $\hat{\beta}_{(i)}$ is the estimated value of β in i^{th} iteration. The degree of multicollinearity is measured by condition index (CI), which is calculated as $CI = \sqrt{\frac{g_1}{g_5}}$ with g_1 and g_5 as maximum and minimum eigen values of $X'\Omega^{-1}X + H'W^{-1}H$ matrix. The EMSE values along with the CI are presented[2] in Tables 2 to 4.

[2] For brevity, we have only presented results for $\sigma_e^2 = 10$ as the conclusions for σ_e^2 =0.5 and 1 are similar to that for $\sigma_e^2 = 10$.

Table 2: Estimated Mean Squared Error when $k = 0.01$

d		$m = 3$			$m = 5$		
		$\rho = 0.95$	$\rho = 0.99$	$\rho = 0.995$	$\rho = 0.95$	$\rho = 0.99$	$\rho = 0.995$
				$\phi = 0$			
0.01	*CI*	*8.40287*	*18.98482*	*26.92654*	*7.81336*	*15.39877*	*19.88059*
	MR	1.11012	2.72763	4.53618	1.05993	2.22707	3.14389
	SRRR	1.08576	2.32181	3.21744	1.04066	2.00713	2.61095
	SRPCR	0.60882	0.65754	0.66850	0.63078	0.70188	0.71724
	SRrk	0.60812	0.65534	0.66588	0.62902	0.69623	0.71051
	TPSRPC	**0.58504**	**0.59057**	**0.59137**	**0.57951**	**0.58118**	**0.58128**
0.5	MR	1.11012	2.72763	4.53618	1.05993	2.22707	3.14389
	SRRR	1.08576	2.32181	3.21744	1.04066	2.00713	2.61095
	SRPCR	0.60882	0.65754	0.66850	0.63078	0.70188	0.71724
	SRrk	0.60812	0.65534	0.66588	0.62902	0.69623	0.71051
	TPSRPC	**0.59384**	**0.61546**	**0.61987**	**0.59817**	**0.62236**	**0.62706**
0.99	MR	1.11012	2.72763	4.53618	1.05993	2.22707	3.14389
	SRRR	1.08576	2.32181	3.21744	1.04066	2.00713	2.61095
	SRPCR	0.60882	0.65754	0.66850	0.63078	0.70188	0.71724
	SRrk	0.60812	0.65534	0.66588	0.62902	0.69623	0.71051
	TPSRPC	**0.60778**	**0.65440**	**0.66480**	**0.62828**	**0.69444**	**0.70848**
ϕ =0.6							
0.01	*CI*	*9.92770*	*22.25548*	*31.50169*	*8.93203*	*17.13814*	*22.57750*
	MR	0.97185	2.05826	3.22448	0.93544	1.76019	2.43317
	SRRR	0.95898	1.84177	2.51427	0.92550	1.64962	2.15242
	SRPCR	0.64594	0.69321	0.70329	0.64642	0.74222	0.76437
	SRrk	0.64556	0.69206	0.70193	0.64548	0.73748	0.75843
	TPSRPC	**0.62700**	**0.64011**	**0.64235**	**0.61173**	**0.61942**	**0.62012**
0.5	MR	0.97185	2.05826	3.22448	0.93544	1.76019	2.43317
	SRRR	0.95898	1.84177	2.51427	0.92550	1.64962	2.15242
	SRPCR	0.64594	0.69321	0.70329	0.64642	0.74222	0.76437
	SRrk	0.64556	0.69206	0.70193	0.64548	0.73748	0.75843
	TPSRPC	**0.63495**	**0.66220**	**0.66760**	**0.62529**	**0.66404**	**0.67172**
0.99	MR	0.97185	2.05826	3.22448	0.93544	1.76019	2.43317
	SRRR	0.95898	1.84177	2.51427	0.92550	1.64962	2.15242
	SRPCR	0.64594	0.69321	0.70329	0.64642	0.74222	0.76437
	SRrk	0.64556	0.69206	0.70193	0.64548	0.73748	0.75843
	TPSRPC	**0.64532**	**0.69139**	**0.70116**	**0.64501**	**0.73574**	**0.75636**
ϕ =0.9							
0.01	*CI*	*10.17962*	*22.74401*	*32.16925*	*9.18977*	*17.84168*	*23.96531*
	MR	0.89938	1.49216	2.08111	0.88434	1.44065	1.93233
	SRRR	0.89418	1.40795	1.80891	0.87972	1.38720	1.78648
	SRPCR	0.73260	0.77855	0.78851	0.71287	0.81497	0.84152
	SRrk	0.73235	0.77772	0.78753	0.71242	0.81161	0.83709
	TPSRPC	**0.71768**	**0.73405**	**0.73701**	**0.69375**	**0.70882**	**0.71079**
0.5	MR	0.89938	1.49216	2.08111	0.88434	1.44065	1.93233
	SRRR	0.89418	1.40795	1.80891	0.87972	1.38720	1.78648
	SRPCR	0.73260	0.77855	0.78851	0.71287	0.81497	0.84152
	SRrk	0.73235	0.77772	0.78753	0.71242	0.81161	0.83709
	TPSRPC	**0.72423**	**0.75331**	**0.75922**	**0.70154**	**0.74951**	**0.76014**
0.99	MR	0.89938	1.49216	2.08111	0.88434	1.44065	1.93233
	SRRR	0.89418	1.40795	1.80891	0.87972	1.38720	1.78648
	SRPCR	0.73260	0.77855	0.78851	0.71287	0.81497	0.84152
	SRrk	0.73235	0.77772	0.78753	0.71242	0.81161	0.83709
	TPSRPC	**0.73217**	**0.77719**	**0.78691**	**0.71218**	**0.81016**	**0.83529**

Table 3: Estimated Mean Squared Error when $k = 0.5$

d		$m = 3$			$m = 5$		
		$\rho = 0.95$	$\rho = 0.99$	$\rho = 0.995$	$\rho = 0.95$	$\rho = 0.99$	$\rho = 0.995$
				$\phi = 0$			
0.01	*CI*	*8.40287*	*18.98482*	*26.92654*	*7.81336*	*15.39877*	*19.88059*
	MR	1.11012	2.72763	4.53618	1.05993	2.22707	3.14389
	SRRR	0.69754	0.67666	0.66440	0.69376	0.67136	0.65740
	SRPCR	0.60882	0.65754	0.66850	0.63078	0.70188	0.71724
	SRrk	0.58922	0.60280	0.60511	0.58787	0.59562	0.59651
	TPSRPC	**0.58379**	**0.58450**	**0.58449**	**0.57614**	**0.57537**	**0.57526**
0.5	MR	1.11012	2.72763	4.53618	1.05993	2.22707	3.14389
	SRRR	0.69754	0.67666	0.66440	0.69376	0.67136	0.65740
	SRPCR	0.60882	0.65754	0.66850	0.63078	0.70188	0.71724
	SRrk	0.58922	0.60280	0.60511	0.58787	0.59562	0.59651
	TPSRPC	**0.58504**	**0.59057**	**0.59137**	**0.57951**	**0.58118**	**0.58128**
0.99	MR	1.11012	2.72763	4.53618	1.05993	2.22707	3.14389
	SRRR	0.69754	0.67666	0.66440	0.69376	0.67136	0.65740
	SRPCR	0.60882	0.65754	0.66850	0.63078	0.70188	0.71724
	SRrk	0.58922	0.60280	0.60511	0.58787	0.59562	0.59651
	TPSRPC	**0.58911**	**0.60250**	**0.60477**	**0.58765**	**0.59525**	**0.59612**
				$\phi = 0.6$			
0.01	*CI*	*9.92770*	*22.25548*	*31.50169*	*8.93203*	*17.13814*	*22.57750*
	MR	0.97185	2.05826	3.22448	0.93544	1.76019	2.43317
	SRRR	0.72111	0.72953	0.72369	0.71217	0.72363	0.71711
	SRPCR	0.64594	0.69321	0.70329	0.64642	0.74222	0.76437
	SRrk	0.63276	0.65572	0.66000	0.61966	0.64003	0.64266
	TPSRPC	**0.62302**	**0.62897**	**0.62981**	**0.60707**	**0.60792**	**0.60779**
0.5	MR	0.97185	2.05826	3.22448	0.93544	1.76019	2.43317
	SRRR	0.72111	0.72953	0.72369	0.71217	0.72363	0.71711
	SRPCR	0.64594	0.69321	0.70329	0.64642	0.74222	0.76437
	SRrk	0.63276	0.65572	0.66000	0.61966	0.64003	0.64266
	TPSRPC	**0.62700**	**0.64011**	**0.64235**	**0.61173**	**0.61942**	**0.62012**
0.99	MR	0.97185	2.05826	3.22448	0.93544	1.76019	2.43317
	SRRR	0.72111	0.72953	0.72369	0.71217	0.72363	0.71711
	SRPCR	0.64594	0.69321	0.70329	0.64642	0.74222	0.76437
	SRrk	0.63276	0.65572	0.66000	0.61966	0.64003	0.64266
	TPSRPC	**0.63263**	**0.65537**	**0.65960**	**0.61947**	**0.63953**	**0.64211**
				$\phi = 0.9$			
0.01	*CI*	*10.17962*	*22.74401*	*32.16925*	*9.18977*	*17.84168*	*23.96531*
	MR	0.89938	1.49216	2.08111	0.88434	1.44065	1.93233
	SRRR	0.77982	0.81054	0.81273	0.76608	0.80703	0.81188
	SRPCR	0.73260	0.77855	0.78851	0.71287	0.81497	0.84152
	SRrk	0.72299	0.74914	0.75424	0.69891	0.73137	0.73683
	TPSRPC	**0.71351**	**0.72230**	**0.72368**	**0.69034**	**0.69450**	**0.69457**
0.5	MR	0.89938	1.49216	2.08111	0.88434	1.44065	1.93233
	SRRR	0.77982	0.81054	0.81273	0.76608	0.80703	0.81188
	SRPCR	0.73260	0.77855	0.78851	0.71287	0.81497	0.84152
	SRrk	0.72299	0.74914	0.75424	0.69891	0.73137	0.73683
	TPSRPC	**0.71768**	**0.73405**	**0.73701**	**0.69375**	**0.70882**	**0.71079**
0.99	MR	0.89938	1.49216	2.08111	0.88434	1.44065	1.93233
	SRRR	0.77982	0.81054	0.81273	0.76608	0.80703	0.81188
	SRPCR	0.73260	0.77855	0.78851	0.71287	0.81497	0.84152
	SRrk	0.72299	0.74914	0.75424	0.69891	0.73137	0.73683
	TPSRPC	**0.72287**	**0.74881**	**0.75386**	**0.69879**	**0.73085**	**0.73621**

Table 4: Estimated Mean Squared Error when $k = 0.99$

d		$m = 3$			$m = 5$		
		$\rho = 0.95$	$\rho = 0.99$	$\rho = 0.995$	$\rho = 0.95$	$\rho = 0.99$	$\rho = 0.995$
				$\phi = 0$			
0.01	*CI*	*8.40287*	*18.98482*	*26.92654*	*7.81336*	*15.39877*	*19.88059*
	MR	1.11012	2.72763	4.53618	1.05993	2.22707	3.14389
	SRRR	0.63296	0.61585	0.61124	0.62798	0.60675	0.60082
	SRPCR	0.60882	0.65754	0.66850	0.63078	0.70188	0.71724
	SRrk	**0.58505**	0.59067	0.59149	**0.57956**	0.58127	0.58138
	TPSRPC	0.58734	**0.58728**	**0.58723**	0.57988	**0.57965**	**0.57966**
0.5	MR	1.11012	2.72763	4.53618	1.05993	2.22707	3.14389
	SRRR	0.63296	0.61585	0.61124	0.62798	0.60675	0.60082
	SRPCR	0.60882	0.65754	0.66850	0.63078	0.70188	0.71724
	SRrk	**0.58505**	0.59067	0.59149	0.57956	0.58127	0.58138
	TPSRPC	0.58530	**0.58732**	**0.58755**	**0.57839**	**0.57853**	**0.57851**
0.99	MR	1.11012	2.72763	4.53618	1.05993	2.22707	3.14389
	SRRR	0.63296	0.61585	0.61124	0.62798	0.60675	0.60082
	SRPCR	0.60882	0.65754	0.66850	0.63078	0.70188	0.71724
	SRrk	0.58505	0.59067	0.59149	0.57956	0.58127	0.58138
	TPSRPC	**0.58504**	**0.59057**	**0.59137**	**0.57951**	**0.58118**	**0.58128**
				ϕ =0.6			
0.01	*CI*	*9.92770*	*22.25548*	*31.50169*	*8.93203*	*17.13814*	*22.57750*
	MR	0.97185	2.05826	3.22448	0.93544	1.76019	2.43317
	SRRR	0.66943	0.66699	0.66484	0.65825	0.65015	0.64607
	SRPCR	0.64594	0.69321	0.70329	0.64642	0.74222	0.76437
	SRrk	0.62707	0.64031	0.64258	0.61181	0.61962	0.62034
	TPSRPC	**0.62243**	**0.62565**	**0.62606**	**0.60762**	**0.60809**	**0.60808**
0.5	MR	0.97185	2.05826	3.22448	0.93544	1.76019	2.43317
	SRRR	0.66943	0.66699	0.66484	0.65825	0.65015	0.64607
	SRPCR	0.64594	0.69321	0.70329	0.64642	0.74222	0.76437
	SRrk	0.62707	0.64031	0.64258	0.61181	0.61962	0.62034
	TPSRPC	**0.62412**	**0.63153**	**0.63269**	**0.60876**	**0.61166**	**0.61183**
0.99	MR	0.97185	2.05826	3.22448	0.93544	1.76019	2.43317
	SRRR	0.66943	0.66699	0.66484	0.65825	0.65015	0.64607
	SRPCR	0.64594	0.69321	0.70329	0.64642	0.74222	0.76437
	SRrk	0.62707	0.64031	0.64258	0.61181	0.61962	0.62034
	TPSRPC	**0.62700**	**0.64011**	**0.64235**	**0.61173**	**0.61942**	**0.62012**
				ϕ =0.9			
0.01	*CI*	*10.17962*	*22.74401*	*32.16925*	*9.18977*	*17.84168*	*23.96531*
	MR	0.89938	1.49216	2.08111	0.88434	1.44065	1.93233
	SRRR	0.74736	0.75782	0.75860	0.73051	0.73896	0.73892
	SRPCR	0.73260	0.77855	0.78851	0.71287	0.81497	0.84152
	SRrk	0.71776	0.73427	0.73726	0.69381	0.70908	0.71108
	TPSRPC	**0.71164**	**0.71679**	**0.71752**	**0.69001**	**0.69184**	**0.69180**
0.5	MR	0.89938	1.49216	2.08111	0.88434	1.44065	1.93233
	SRRR	0.74736	0.75782	0.75860	0.73051	0.73896	0.73892
	SRPCR	0.73260	0.77855	0.78851	0.71287	0.81497	0.84152
	SRrk	0.71776	0.73427	0.73726	0.69381	0.70908	0.71108
	TPSRPC	**0.71426**	**0.72436**	**0.72605**	**0.69135**	**0.69831**	**0.69895**
0.99	MR	0.89938	1.49216	2.08111	0.88434	1.44065	1.93233
	SRRR	0.74736	0.75782	0.75860	0.73051	0.73896	0.73892
	SRPCR	0.73260	0.77855	0.78851	0.71287	0.81497	0.84152
	SRrk	0.71776	0.73427	0.73726	0.69381	0.70908	0.71108
	TPSRPC	**0.71768**	**0.73405**	**0.73701**	**0.69375**	**0.70882**	**0.71079**

7.1 MSE Comparisons

The simulation results in Tables 2 to 4 show that keeping other parameters fixed, as the value of ρ increases the EMSEs of all the estimators increase, which is likely as the increased level of multicollinearity reduces the performance of an estimator. However, the difference in the EMSE of the TPSRPC estimator over the other estimators increases as the collinearity increases. Thus, the superiority of the TPSRPC estimator over the other estimators will increase as the value of ρ increases. Further, as m increases i.e. for larger number of stochastic restrictions the EMSE of the estimators decreases for most of the values of other parameters.

The proposed estimator has lesser estimated MSE values than the MR, SRRR, SRPCR estimators for all values of ϕ, ρ, m, k and d. The TPSRPC estimator performs better than the SRrk estimator for almost all values of parameters considered here. However, in some cases, for instance, when $k = 0.99$, $\phi = 0$ and $d = 0.01$, the SRrk estimator performs better than the TPSRPC estimator. Moreover, for some values of the parameters, the SRrk estimator performs almost equally well to the TPSRPC estimator as we can see for the values $k = 0.01$, $\phi = 0.9$, $d = 0.99$; $k = 0.5$, $\phi = 0{,}0.9$, $d = 0.99$ and $k = 0.99$, $\phi = 0{,}0.6{,}0.9$, $d = 0.99$, the difference in the EMSEs of the TPSRPC and SRrk estimators can be noticed only after 3 rd and 4 th decimal places.

Further, the superiority of the TPSRPC estimator over the other estimators in simulation results can be easily justified by verifying the dominance conditions under scalar MSE criterion. For example, when $\rho = 0.95$, the values of $\sigma^2 - g_i\alpha_i^2$ for $i = 3{,}4{,}5$ (r is chosen to be 2 here) are 9.999892, 9.999955, 9.999992, which are all positive, further for $k = 0.01$, $d = 0.01$, the values of $(2g_i + (1 + k + d))\sigma^2 - (k + (1 + k - d)g_i)\alpha_i^2$ for $i = 1{,}2$ are obtained to be 570.30432, 47.39837, which are all positive, hence the condition for the dominance of the TPSRPC estimator over the MR estimator, stated in Theorem 3.2 is satisfied, which justifies the dominance of the TPSRPC estimator over the MR estimator in Table 2. Moreover, the value of $(2g_i + 1 + d)\sigma^2 - ((2k + 1 - d)g_i + 2k)\alpha_i^2$ for $i = 1{,}2$, when $k = 0.01$, $d = 0.01$, comes out to be 566.26610, 46.39836 which from Theorem 3.8, justifies the superiority of the TPSRPC estimator over the SRrk estimator in scalar MSE sense. Similarly, other conditions can also be verified easily.

Hence, taking all the above observations under consideration, it can be concluded that the TPSRPC estimator is superior to the other competing estimators considered in this study in the scalar MSE sense.

7.2 Testing the Conditions under MSE Matrix

The values of the test statistics F_1^* and F_2 for testing the conditions of superiority of the TPSRPC estimator over the MR estimator, and over the SRPCR and SRrk estimators stated in Theorems 3.1, 3.5 and 3.8, respectively have been calculated. The proportion of the times the null hypotheses H_{0a} and H_{0b}, which state the conditions that the TPSRPC estimator is superior than the other estimators are not rejected, are calculated. The proportions are denoted as P_{0a} and P_{0b}, respectively.

Table 5

The values of P_{0a}, for $\sigma_e^2 = 0.5$

ϕ	d	*m*=3				*m*=5			
		ρ=0.8	ρ =0.9	ρ =0.99	ρ =0.995	ρ =0.8	ρ =0.9	ρ =0.99	ρ =0.995
					***k*=0.01**				
0	0.01	21.70	15.70	11.55	11.30	2.40	0.40	0.15	0.15
	0.5	21.70	15.70	11.55	11.30	2.40	0.40	0.15	0.15
	0.99	21.70	15.70	11.55	11.30	2.40	0.40	0.15	0.15
0.6	0.01	11.10	8.70	5.15	5.10	0.00	0.00	0.00	0.00
	0.5	11.10	8.70	5.15	5.10	0.00	0.00	0.00	0.00
	0.99	11.10	8.70	5.15	5.10	0.00	0.00	0.00	0.00
0.9	0.01	32.20	25.80	20.80	20.80	0.00	0.00	0.00	0.00
	0.5	32.20	25.80	20.80	20.80	0.00	0.00	0.00	0.00
	0.99	32.20	25.80	20.80	20.80	0.00	0.00	0.00	0.00
					***k*=0.5**				
0	0.01	21.70	15.70	11.55	11.30	2.40	0.40	0.15	0.15
	0.5	21.70	15.70	11.55	11.30	2.40	0.40	0.15	0.15
	0.99	21.70	15.70	11.55	11.30	2.40	0.40	0.15	0.15
0.6	0.01	11.10	8.70	5.15	5.10	0.00	0.00	0.00	0.00
	0.5	11.10	8.70	5.15	5.10	0.00	0.00	0.00	0.00
	0.99	11.10	8.70	5.15	5.10	0.00	0.00	0.00	0.00
0.9	0.01	32.20	25.80	20.80	20.80	0.00	0.00	0.00	0.00
	0.5	32.20	25.80	20.80	20.80	0.00	0.00	0.00	0.00
	0.99	32.20	25.80	20.80	20.80	0.00	0.00	0.00	0.00
					***k*=0.99**				
0	0.01	21.70	15.70	11.55	11.30	2.40	0.40	0.15	0.15
	0.5	21.70	15.70	11.55	11.30	2.40	0.40	0.15	0.15
	0.99	21.70	15.70	11.55	11.30	2.40	0.40	0.15	0.15
0.6	0.01	11.10	8.70	5.15	5.10	0.00	0.00	0.00	0.00
	0.5	11.10	8.70	5.15	5.10	0.00	0.00	0.00	0.00
	0.99	11.10	8.70	5.15	5.10	0.00	0.00	0.00	0.00
0.9	0.01	32.20	25.80	20.80	20.80	0.00	0.00	0.00	0.00
	0.5	32.20	25.80	20.80	20.80	0.00	0.00	0.00	0.00
	0.99	32.20	25.80	20.80	20.80	0.00	0.00	0.00	0.00

Table 6

The Values of P_{0a}, for $\sigma_e^2 = 1$

ϕ	d	*m*=3				*m*=5			
		ρ=0.8	ρ =0.9	ρ =0.99	ρ =0.995	ρ =0.8	ρ =0.9	ρ =0.99	ρ =0.995
					***k*=0.01**				
0	0.01	86.20	85.15	81.95	81.75	75.75	40.60	36.65	36.95
	0.5	86.20	85.15	81.95	81.75	75.75	40.60	36.65	36.95
	0.99	86.20	85.15	81.95	81.75	75.75	40.60	36.65	36.95
0.6	0.01	87.35	83.40	79.90	79.70	35.60	11.40	9.05	9.00
	0.5	87.35	83.40	79.90	79.70	35.60	11.40	9.05	9.00
	0.99	87.35	83.40	79.90	79.70	35.60	11.40	9.05	9.00
0.9	0.01	95.50	94.60	92.60	92.70	32.45	10.25	8.45	8.40
	0.5	95.50	94.60	92.60	92.70	32.45	10.25	8.45	8.40
	0.99	95.50	94.60	92.60	92.70	32.45	10.25	8.45	8.40
					***k*=0.5**				
0	0.01	86.20	85.15	81.95	81.75	75.75	40.60	36.65	36.95
	0.5	86.20	85.15	81.95	81.75	75.75	40.60	36.65	36.95
	0.99	86.20	85.15	81.95	81.75	75.75	40.60	36.65	36.95
0.6	0.01	87.35	83.40	79.90	79.70	35.60	11.40	9.05	9.00
	0.5	87.35	83.40	79.90	79.70	35.60	11.40	9.05	9.00
	0.99	87.35	83.40	79.90	79.70	35.60	11.40	9.05	9.00
0.9	0.01	95.50	94.60	92.60	92.70	32.45	10.25	8.45	8.40
	0.5	95.50	94.60	92.60	92.70	32.45	10.25	8.45	8.40
	0.99	95.50	94.60	92.60	92.70	32.45	10.25	8.45	8.40
					***k*=0.99**				
0	0.01	86.20	85.15	81.95	81.75	75.75	40.60	36.65	36.95
	0.5	86.20	85.15	81.95	81.75	75.75	40.60	36.65	36.95
	0.99	86.20	85.15	81.95	81.75	75.75	40.60	36.65	36.95
0.6	0.01	87.35	83.40	79.90	79.70	35.60	11.40	9.05	9.00
	0.5	87.35	83.40	79.90	79.70	35.60	11.40	9.05	9.00
	0.99	87.35	83.40	79.90	79.70	35.60	11.40	9.05	9.00
0.9	0.01	95.50	94.60	92.60	92.70	32.45	10.25	8.45	8.40
	0.5	95.50	94.60	92.60	92.70	32.45	10.25	8.45	8.40
	0.99	95.50	94.60	92.60	92.70	32.45	10.25	8.45	8.40

Table 7
The Values of P_{0b}, for $\sigma_e^2 = 0.5$

ϕ	d	m=3				m=5			
		ρ=0.8	ρ =0.9	ρ =0.99	ρ =0.995	ρ =0.8	ρ =0.9	ρ =0.99	ρ =0.995
					k=0.01				
0	0.01	14.15	9.85	9.75	9.70	11.60	8.75	5.55	5.50
	0.5	14.15	9.85	9.75	9.70	11.60	8.75	5.55	5.50
	0.99	14.15	9.85	9.75	9.70	11.60	8.75	5.55	5.50
0.6	0.01	1.80	1.00	0.65	0.75	3.55	1.40	1.15	1.15
	0.5	1.80	1.00	0.65	0.75	3.55	1.40	1.15	1.15
	0.99	1.80	1.00	0.65	0.75	3.55	1.40	1.15	1.15
0.9	0.01	3.00	1.30	0.90	0.95	7.50	4.50	3.25	3.25
	0.5	3.00	1.30	0.90	0.95	7.50	4.50	3.25	3.25
	0.99	3.00	1.30	0.90	0.95	7.50	4.50	3.25	3.25
					k=0.5				
0	0.01	14.15	9.85	9.75	9.70	11.60	8.75	5.55	5.50
	0.5	14.15	9.85	9.75	9.70	11.60	8.75	5.55	5.50
	0.99	14.15	9.85	9.75	9.70	11.60	8.75	5.55	5.50
0.6	0.01	1.80	1.00	0.65	0.75	3.55	1.40	1.15	1.15
	0.5	1.80	1.00	0.65	0.75	3.55	1.40	1.15	1.15
	0.99	1.80	1.00	0.65	0.75	3.55	1.40	1.15	1.15
0.9	0.01	3.00	1.30	0.90	0.95	7.50	4.50	3.25	3.25
	0.5	3.00	1.30	0.90	0.95	7.50	4.50	3.25	3.25
	0.99	3.00	1.30	0.90	0.95	7.50	4.50	3.25	3.25
					k=0.99				
0	0.01	14.15	9.85	9.75	9.70	11.60	8.75	5.55	5.50
	0.5	14.15	9.85	9.75	9.70	11.60	8.75	5.55	5.50
	0.99	14.15	9.85	9.75	9.70	11.60	8.75	5.55	5.50
0.6	0.01	1.80	1.00	0.65	0.75	3.55	1.40	1.15	1.15
	0.5	1.80	1.00	0.65	0.75	3.55	1.40	1.15	1.15
	0.99	1.80	1.00	0.65	0.75	3.55	1.40	1.15	1.15
0.9	0.01	3.00	1.30	0.90	0.95	7.50	4.50	3.25	3.25
	0.5	3.00	1.30	0.90	0.95	7.50	4.50	3.25	3.25
	0.99	3.00	1.30	0.90	0.95	7.50	4.50	3.25	3.25

Table 8
The Values of P_{0b}, for $\sigma_e^2 = 1$

ϕ	d	*m=3* ρ=0.8	ρ =0.9	ρ =0.99	ρ =0.995	*m=5* ρ =0.8	ρ =0.9	ρ =0.99	ρ =0.995
					***k*=0.01**				
0	0.01	92.75	89.10	84.35	83.40	72.15	68.05	62.75	62.20
	0.5	92.75	89.10	84.35	83.40	72.15	68.05	62.75	62.20
	0.99	92.75	89.10	84.35	83.40	72.15	68.05	62.75	62.20
0.6	0.01	81.90	76.45	72.35	72.15	63.95	56.95	38.05	37.65
	0.5	81.90	76.45	72.35	72.15	63.95	56.95	38.05	37.65
	0.99	81.90	76.45	72.35	72.15	63.95	56.95	38.05	37.65
0.9	0.01	86.20	81.90	79.30	79.35	80.25	59.35	55.20	55.10
	0.5	86.20	81.90	79.30	79.35	80.25	59.35	55.20	55.10
	0.99	86.20	81.90	79.30	79.35	80.25	59.35	55.20	55.10
					***k*=0.5**				
0	0.01	92.75	89.10	84.35	83.40	72.15	68.05	62.75	62.20
	0.5	92.75	89.10	84.35	83.40	72.15	68.05	62.75	62.20
	0.99	92.75	89.10	84.35	83.40	72.15	68.05	62.75	62.20
0.6	0.01	81.90	76.45	72.35	72.15	63.95	56.95	38.05	37.65
	0.5	81.90	76.45	72.35	72.15	63.95	56.95	38.05	37.65
	0.99	81.90	76.45	72.35	72.15	63.95	56.95	38.05	37.65
0.9	0.01	86.20	81.90	79.30	79.35	80.25	59.35	55.20	55.10
	0.5	86.20	81.90	79.30	79.35	80.25	59.35	55.20	55.10
	0.99	86.20	81.90	79.30	79.35	80.25	59.35	55.20	55.10
					***k*=0.99**				
0	0.01	92.75	89.10	84.35	83.40	72.15	68.05	62.75	62.20
	0.5	92.75	89.10	84.35	83.40	72.15	68.05	62.75	62.20
	0.99	92.75	89.10	84.35	83.40	72.15	68.05	62.75	62.20
0.6	0.01	81.90	76.45	72.35	72.15	63.95	56.95	38.05	37.65
	0.5	81.90	76.45	72.35	72.15	63.95	56.95	38.05	37.65
	0.99	81.90	76.45	72.35	72.15	63.95	56.95	38.05	37.65
0.9	0.01	86.20	81.90	79.30	79.35	80.25	59.35	55.20	55.10
	0.5	86.20	81.90	79.30	79.35	80.25	59.35	55.20	55.10
	0.99	86.20	81.90	79.30	79.35	80.25	59.35	55.20	55.10

The proportions are reported in Tables 5 to 8 and it is observed that the values of P_{0a} and P_{0b} are very low for $\sigma_e^2 = 0.5$, i.e. the chances of the TPSRPC estimator being superior to the MR, SRPCR and SRrk estimators in MSE matrix sense are very less, almost zero in some cases. However, for $\sigma_e^2 = 1$ the case is reverse and the values of P_{0a} and P_{0b} are very high which means that chances that the TPSRPC estimator performs better than the MR, SRPCR and the SRrk estimators in MSE matrix sense are high for $\sigma_e^2 = 1$.

Further, for fixed values of ρ, ϕ and $k(d)$ the values of P_{0a} and P_{0b} are not affected much by the change in $d(k)$. Moreover, the value of m is affecting the proportions very significantly, as both the proportions exhibit a huge decline for $m = 5$ as compared to the case when $m = 3$.

Furthermore, as ρ increases, i.e. the degree of collinearity increases, the proportions P_{0a} and P_{0b} increases. As ϕ increases, the values of P_{0a} and P_{0b} increases for $\rho = 0.8$ and $m = 3$, and for rest of the values of ρ for $m = 5$, the proportions are found to be minimum at $\phi = 0.6$ in most of the cases.

For $\sigma_e^2 = 10$, the proportion of times H_{0a} and H_{0b} are not rejected is obtained to be 100 (for brevity the results are not reported) for all parametric settings considered here, hence depicting that the TPSRPC estimator performs better than the MR, SRPCR and the SRrk estimators in MSE matrix sense. Further, for $\sigma_e^2 = 0.5$ and 1, the values of P_{0a} and P_{0b} are reported in the Tables 5 to 8.

8. CONCLUDING REMARKS

In this paper, two parameter stochastic restricted principal component (TPSRPC) estimator has been introduced for estimating regression coefficient vector in linear regression model when extraneous information in the form of stochastic linear restrictions is present. Further, it has been shown that this estimator is a general class of estimator which includes the MR, SRRR, SRPCR and SRrk estimators as its special cases. The conditions for the dominance of the proposed estimator over the other estimators have been derived under MSE matrix and MSE criteria. The expressions for finding optimum values of unknown constants k(d) given a value of *d(k)* in the TPSRPC estimator have been derived under MSE. Further, the test statistics have been developed to test the conditions for the dominance of the TPSRPC estimator over the MR, SRPCR and SRrk estimators.

Furthermore, the simulation study suggests that the proposed estimator outperforms the MR, SRRR, SRPCR estimators for all values of ϕ, ρ, m, k and d. The TPSRPC estimator performs better than the SRrk estimator for almost all values of parameters considered here, however, in some cases, the proposed estimator performs equally well to the SRrk estimator in MSE sense. Further, the test results show that for higher values of disturbance variance (σ_e^2), the TPSRPC estimator is more likely to dominate the MR, SRPCR and SRrk estimators than for smaller values of disturbance variance.

The MSE values for the numerical example show that the TPSRPC estimator has a very significant improvement over the MR and SRRR estimators in the MSE sense. Whereas, the SRPCR, SRrk and the TPSRPC estimators exhibit almost equal MSEs for this data set. Further, the test results for the numerical example exhibited that the TPSRPC estimator does not perform better than the MR, SRPCR and the SRrk estimators in MSE matrix sense.

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