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TWO PARAMETRIC GENERALIZED 'USEFUL' R-NORM INFORMATION MEASURE & ITS CODING THEOREM

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ABSTRACT

In the current paper, we propose a new generalized 'useful' R-norm information measure with two parameters α and β. Some basic properties and the particular cases of this measure are discussed. Further, we define the 'useful' R-norm average code-word length and obtain the bounds in terms of the proposed measure. Huffman coding and Shannon-Fano coding schemes are considered to verify the noiseless coding theorem by taking hypothetical data. Also, we analyze the behaviour of the proposed 'useful' R-norm information and its average code-word length at various values of α, β and R.

KEYWORDS

R-norm Information, Utility distribution, Holder's inequality, Mean Code-word Length.

AMS Classification: 94A17, 94A24

1. INTRODUCTION

Let us take into consideration a set of positive real numbers $\Re^+$ such that $\Re^+ = \{R : R > 0, R \neq 1\}$. We also consider set of probability distributions Δ_n $(n \geq 2)$ which is given as $P = \left\{(p_1, p_2, ..., p_n);\ 0 \leq p_i \leq 1 \& \sum_{i=1}^{n} p_i = 1\right\}$. R-norm information measure of the probability distribution P was given by Boekee and Lubbe (1980) for $R \in \Re^+$ as:

$$H_R(P) = \frac{R}{R-1}\left[1 - \left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right];\ R(>0) \neq 1. \tag{1.1}$$

The R-norm information measure (RIM) defined above in (1.1) is a real valued function $\Delta_n \rightarrow \Re^+$ defined on $\Delta_n\left(n \geq 2\right)$. The RIM is not same as the entropy measures given by Shannon (1948), Renyi (1961) and Havrda and Chavrat (1967). Measure (1.1) has the interesting property that it approaches to Shannon's (1948) entropy when $R \to 1$ and when $R \to \infty$, $H_R(P) \to 1 - \max(p_i)$; $\forall\, i = 1, 2, ..., n$.

Belis and Guiasu (1968) considered the qualitative feature of the events in an experiment E and attached a utility distribution $U = \left\{ \left(u_1, u_2, ..., u_n \right); u_i > 0 \right\}$ with the probability distribution P, where u_i represents the importance of the events with probability p_i. Thus, a qualitative-quantitative measure was characterized and Longo (1972) called it as 'useful' information of the experiment E. The 'useful' information measure is defined as:

$$H(P;U) = -\sum_{i=1}^{n} u_i\, p_i \log p_i \;; u_i > 0 \;, 0 \le p_i \le 1 \;\&\; \sum_{i=1}^{n} p_i = 1$$

Later on, Bhaker and Hooda (1993) introduced the following 'useful' information measure:

$$H(P;U) = \frac{-\sum_{i=1}^{n} u_i\, p_i \log p_i}{\sum_{i=1}^{n} u_i\, p_i}$$

If a set N of finite source symbols $X = \left\{ x_1, x_2, ..., x_N \right\}$ are to be encoded using $D\,(D \ge 2)$ code alphabets, then there exists a uniquely decipherable code with lengths $l_1, l_2, ..., l_N$ iff Kraft's (1949) inequality holds i.e.,

$$\sum_{i=1}^{N} D^{-l_i} \le 1. \tag{1.2}$$

Here, D represents the size of code alphabet. Further, if $L = \sum_{i=1}^{N} l_i\, p_i$ be the mean code-word length, then the code satisfying (1.2), the lower bound L is given in terms of Shannon's (1948) entropy i.e., $H(P) \le L$. Guiasu and Picard (1971) gave the following 'useful' mean code-word length:

$$L\left(P;U\right) = \frac{\sum_{i=1}^{N} u_i\, l_i\, p_i}{\sum_{i=1}^{N} u_i\, p_i} \tag{1.3}$$

And the bounds of (1.3) are obtained in terms of $H\left(P;U\right)$. Singh et al. (2003), Kumar (2009), Hooda et al. (2015) studied and gave various generalized 'useful' RIMs and developed the corresponding coding theorems.

2. GENERALIZED 'USEFUL' R-NORM INFORMATION MEASURE

The parametric generalized information measures are more flexible and have more potentiality from the application point of view, thus we consider a two parametric generalization of 'useful' RIM as:

$$H_R^{\alpha,\beta}(P;U)=\frac{R+\alpha-\beta}{R-\beta}\left[1-\left\{\frac{\sum_{i=1}^{n}u_i\,p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n}u_i\,p_i}\right\}^{\frac{\alpha}{R+\alpha-\beta}}\right];$$

$$R>0(\neq 1);\,0<\alpha,\beta\leq 1;\,R\neq\beta;\,u_i>0 \quad (2.1)$$

Particular Cases:

1. For $\alpha=1$, the proposed measure (2.1) tends to the 'useful' RIM corresponding to Kumar and Choudhary's (2012) RIM of degree β.
2. For $\alpha=1$ and $\beta=1$, the measure (2.1) becomes the 'useful' RIM given by Singh et al. (2003).
3. For $\alpha=1$, $\beta=1$ and $R\to 1$, the measure (2.1) reduces to the 'useful' information measure given by Belis and Guiasu (1968).
4. For $u_i=1$, the measure (2.1) reduces to the two parametric RIM given by Safeena, Saima and M.A.K. Baig (communicated).
5. For $u_i=1$ and $\alpha=1$, the measure (2.1) tends to RIM of degree β given by Kumar and Choudhary (2012).
6. For $u_i=1$, $\alpha=1$ and $\beta=1$, the measure (2.1) approaches to RIM given by Boekee and Lubbe (1980).
7. For $u_i=1$, $\alpha=1$, $\beta=1$ and $R\to 1$, the measure (2.1) becomes information measure given by Shannon (1948).

3. PROPERTIES OF GENERALIZED 'USEFUL' RIM

Some of the properties of generalized 'useful' RIM $H_R^{\alpha,\beta}(P;U)$ are given below. These properties are proved numerically by taking a hypothetical data.

1. Symmetric: $H_R^{\alpha,\beta}\left(p_1,p_2....,p_n;u_1,u_2....,u_n\right)$ is a symmetric function of $(P;U)$ where $P=\left(p_1,p_2....,p_n\right)$ & $U=\left(u_1,u_2....,u_n\right)$

Table 3.1
Symmetric Property

P_i	u_i	α	β	R	$H_R^{\alpha,\beta}(P;U)$	Q_i	w_i	$H_R^{\alpha,\beta}(Q;W)$
0.41	4	0.62	0.81	0.31	329.2692	0.03	5	329.2692
0.18	2					0.41	4	
0.15	6					0.18	2	
0.13	1			42	0.5980	0.15	6	0.5980
0.10	3					0.13	1	
0.03	5					0.10	3	

We observe from Table (3.1) that $H_R^{\alpha,\beta}(P;U) = H_R^{\alpha,\beta}(Q;W)$ where $Q = (p_n, p_1, p_2, ..., p_{n-1})$ & $W = (u_n, u_1, u_2, ..., u_{n-1})$.

2. By adding an extra event having probability of occurrence zero or utility zero, there is no change in 'useful' RIM i.e.,

$$H_R^{\alpha,\beta}(p_1, p_2, ..., p_n, 0; u_1, u_2, ..., u_{n+1}) = H_R^{\alpha,\beta}(p_1, p_2, ..., p_n; u_1, u_2, ..., u_n)$$
$$= H_R^{\alpha,\beta}(p_1, p_2, ..., p_{n+1}; u_1, u_2, ..., u_n, 0).$$

Table 3.2

P_i	u_i	α	β	R	$H_R^{\alpha,\beta}(P;U)$	Q_i	w_i	$H_R^{\alpha,\beta}(Q;W)$
0.41	4	0.62	0.81	0.31	329.2692	0.41	4	329.2692
0.18	2					0.18	2	
0.15	6					0.15	6	
0.13	1			42	0.5980	0.13	1	0.5980
0.10	3					0.10	3	
0.03	5					0.03	5	
0	7					0.23	0	

We observe from the Table (3.2) that we get the same value of 'useful' RIM even if an extra event having probability '0' or utility '0' is added i.e., $H_R^{\alpha,\beta}(P;U) = H_R^{\alpha,\beta}(Q;W)$.

3. $H_R^{\alpha,\beta}(P;U) < H_R^{\alpha,\beta}(Q;U)$, here $Q = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$.

Table 3.3

P_i	u_i	α	β	R	$H_R^{\alpha,\beta}(P;U)$	Q_i	$H_R^{\alpha,\beta}(Q;U)$
0.41	4	0.62	0.81	0.31	329.2692	0.167	415.5678
0.18	2					0.167	
0.15	6					0.167	
0.13	1			42	0.5980	0.167	0.8409
0.10	3					0.167	
0.03	5					0.167	

From Table (3.3), we conclude that the 'useful' RIM is maximum when all P_i have same value.

4. $H_R^{\alpha,\beta}(1,0,0,0,0,0;U)=0$.

Table 3.4

P_i	u_i	α	β	R	$H_R^{\alpha,\beta}(P;U)$
1.0	4	0.62	0.81	0.31	0.0
0.0	2				
0.0	6				
0.0	1			42	0.0
0.0	3				
0.0	5				

From Table (3.4), we see that when one of the probability is equal to '1', and others '0', the value of $H_R^{\alpha,\beta}(P;U)$ is '0'.

5. Non-additivity: $H_R^{\alpha,\beta}(P;U)$ satisfies the non-additivity of the form:

$$H_R^{\alpha,\beta}(P*Q;U*W)=H_R^{\alpha,\beta}(P;U)+H_R^{\alpha,\beta}(Q;W)-\frac{R-\beta}{R+\alpha-\beta}H_R^{\alpha,\beta}(P;U)H_R^{\alpha,\beta}(Q;W) \tag{3.1}$$

where $P*Q=\left(p_1q_1,...,p_1q_m;p_2q_1,...,p_2q_m;p_nq_1,...,p_nq_m\right)$

& $U*W=\left(u_1w_1,...,u_1w_m;u_2w_1,...,u_2w_m;u_nw_1,...,u_nw_m\right)$

Proof:

Taking Right side of (3.1), we have

$$H_R^{\alpha,\beta}(P;U)+H_R^{\alpha,\beta}(Q;W)-\frac{R-\beta}{R+\alpha-\beta}H_R^{\alpha,\beta}(P;U)H_R^{\alpha,\beta}(Q;W)$$

$$=\frac{R+\alpha-\beta}{R-\beta}\left[1-\left\{\frac{\sum_{i=1}^{n}u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n}u_i p_i}\right\}^{\frac{\alpha}{R+\alpha-\beta}}\right]$$

$$+\frac{R+\alpha-\beta}{R-\beta}\left[1-\left\{\frac{\sum_{j=1}^{m}w_j q_j^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{j=1}^{m}w_j q_j}\right\}^{\frac{\alpha}{R+\alpha-\beta}}\right]$$

$$-\left(\frac{R-\beta}{R+\alpha-\beta}\right)\left(\frac{R+\alpha-\beta}{R-\beta}\right)^2\left[1-\left\{\frac{\sum_{i=1}^{n}u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n}u_i p_i}\right\}^{\frac{\alpha}{R+\alpha-\beta}}\right]\left[1-\left\{\frac{\sum_{j=1}^{m}w_j q_j^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{j=1}^{m}w_j q_j}\right\}^{\frac{\alpha}{R+\alpha-\beta}}\right]$$

$$=\frac{R+\alpha-\beta}{R-\beta}\left[1-\left\{\frac{\sum_{i=1}^{n}\sum_{j=1}^{m}u_i w_j\left(p_i q_j\right)^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n}\sum_{j=1}^{m}u_i w_j p_i q_j}\right\}^{\frac{\alpha}{R+\alpha-\beta}}\right]$$

$$=H_R^{\alpha,\beta}(P*Q;U*W).$$

Thus, the result is established.

6. Non-Negativity: From the result of Tables (3.1), (3.2), (3.3) and (3.4), we conclude that the proposed generalized ‘useful’ RIM $H_R^{\alpha,\beta}(P;U)$ satisfies all the above properties which further implies that $H_R^{\alpha,\beta}(P;U)$ is non-negative.

4. CODING THEOREMS OF GENERALIZED 'USEFUL' RIM

The generalized 'useful' average code-word length $L_R^{\alpha,\beta}(P;U)$ corresponding to the 'useful' RIM $H_R^{\alpha,\beta}(P;U)$ is given as:

$$L_R^{\alpha,\beta}(P;U)=\frac{R+\alpha-\beta}{R-\beta}\left[1-\left\{\frac{\sum_{i=1}^{n}u_i p_i\, D^{-l_i\left(\frac{R-\beta}{R+\alpha-\beta}\right)}}{\sum_{i=1}^{n}u_i p_i}\right\}\right] \tag{4.1}$$

In case, $\alpha=1$ and $\beta=1$, $L_R^{\alpha,\beta}(P;U)$ reduces to the 'useful' average code-word length given by Singh et al. (2003). Further, if we ignore the utilities, $L_R^{\alpha,\beta}(P;U)$ reduces to the average code-word length given by Boekee and Lubbe (1980).

In the following theorem, we derive the bounds of the generalized 'useful' average code-word length $L_R^{\alpha,\beta}(P;U)$ in terms of the generalized 'useful' RIM $H_R^{\alpha,\beta}(P;U)$.

Theorem 4.1:

Let $l_i\,; i=1,2,...,n$ be the length of code-word x_i , then for $R\in\Re^+$

$$H_R^{\alpha,\beta}(P;U)\ \le\ L_R^{\alpha,\beta}(P;U)\ <\ H_R^{\alpha,\beta}(P;U)D^{\frac{\beta-R}{R+\alpha-\beta}}+\frac{R+\alpha-\beta}{R-\beta}\left[1-D^{\frac{\beta-R}{R+\alpha-\beta}}\right]$$

under the condition $\sum_{i=1}^{n}u_i D^{-l_i}\le\sum_{i=1}^{n}u_i p_i$ (4.2)

Expression (4.2) is the generalization of Kraft's (1949) inequality.

Proof:

We have from Holder's Inequality

$$\left(\sum_{i=1}^{n}x_i^{\,p}\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}y_i^{\,q}\right)^{\frac{1}{q}}\le\sum_{i=1}^{n}x_i\, y_i\,;\ \ x_i\,,y_i\ge 0\ \ \forall\, i=1,2,...,n\ \ \&\ \frac{1}{p}+\frac{1}{q}=1\,. \tag{4.3}$$

Setting $x_i=\left(\frac{u_i p_i}{\sum_{i=1}^{n}u_i p_i}\right)^{\frac{R+\alpha-\beta}{R-\beta}}D^{-l_i}\ ;\ y_i=\left(\frac{u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n}u_i p_i}\right)^{\frac{\alpha}{\beta-R}}$; $p=\frac{R-\beta}{R+\alpha-\beta}\,; q=\frac{\beta-R}{\alpha}$ in (4.3), we get

$$\left[\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R-\beta}{R+\alpha-\beta}\right)}}{\sum_{i=1}^{n} u_i p_i}\right]^{\frac{R+\alpha-\beta}{R-\beta}} \left[\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i p_i}\right]^{\frac{\alpha}{\beta-R}} \leq \frac{\sum_{i=1}^{n} u_i D^{-l_i}}{\sum_{i=1}^{n} u_i p_i}$$

Using condition (4.2), we get

$$\left[\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i p_i}\right]^{\frac{\alpha}{\beta-R}} \leq \left[\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R-\beta}{R+\alpha-\beta}\right)}}{\sum_{i=1}^{n} u_i p_i}\right]^{\frac{R+\alpha-\beta}{\beta-R}} \tag{4.4}$$

Since $R \neq \beta$, thus two cases arise. Either $R > \beta$ or $R < \beta$.

Case 1: When $R > \beta$.

Since $\frac{\beta - R}{R+\alpha-\beta} < 0$, thus by raising power $\frac{\beta - R}{R+\alpha-\beta}$ on both sides of (4.4), we get

$$\left[\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i p_i}\right]^{\frac{\alpha}{R+\alpha-\beta}} \geq \left[\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R-\beta}{R+\alpha-\beta}\right)}}{\sum_{i=1}^{n} u_i p_i}\right] \tag{4.5}$$

The expression (4.5) can be re-written as:

$$1-\left[\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i p_i}\right]^{\frac{\alpha}{R+\alpha-\beta}} \leq 1-\left[\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R-\beta}{R+\alpha-\beta}\right)}}{\sum_{i=1}^{n} u_i p_i}\right] \tag{4.6}$$

Multiplying both sides of (4.6) by $\frac{R+\alpha-\beta}{R-\beta} > 0$, we get

$$\frac{R+\alpha-\beta}{R-\beta}\left[1-\left\{\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i p_i}\right\}^{\frac{\alpha}{R+\alpha-\beta}}\right] \leq \frac{R+\alpha-\beta}{R-\beta}\left[1-\left\{\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R-\beta}{R+\alpha-\beta}\right)}}{\sum_{i=1}^{n} u_i p_i}\right\}\right]$$

Or we can write

$$H_R^{\alpha,\beta}(P;U) \le L_R^{\alpha,\beta}(P;U) \tag{4.7}$$

Case 2: When $R<\beta$

Since $\frac{\beta-R}{R+\alpha-\beta}>0$, thus by raising power $\frac{\beta-R}{R+\alpha-\beta}$ on both sides of (4.4), we get

$$\left[\frac{\sum_{i=1}^{n} u_i\, p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i\, p_i}\right]^{\frac{\alpha}{R+\alpha-\beta}} \le \left[\frac{\sum_{i=1}^{n} u_i\, p_i\, D^{-l_i\left(\frac{R-\beta}{R+\alpha-\beta}\right)}}{\sum_{i=1}^{n} u_i\, p_i}\right]$$

$$\Rightarrow 1-\left[\frac{\sum_{i=1}^{n} u_i\, p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i\, p_i}\right]^{\frac{\alpha}{R+\alpha-\beta}} \ge 1-\left[\frac{\sum_{i=1}^{n} u_i\, p_i\, D^{-l_i\left(\frac{R-\beta}{R+\alpha-\beta}\right)}}{\sum_{i=1}^{n} u_i\, p_i}\right] \tag{4.8}$$

Now, multiplying both sides of (4.8) by $\frac{R+\alpha-\beta}{R-\beta}<0$, and after simplification we get (4.7).

Equality holds in (4.7) if $D^{-l_i} = \dfrac{p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\left(\frac{\sum_{i=1}^{n} u_i\, p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i\, p_i}\right)}$

or

$$l_i = -\log_D\, p_i^{\frac{R+\alpha-\beta}{\alpha}} + \log_D\left(\frac{\sum_{i=1}^{n} u_i\, p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i\, p_i}\right)$$

By choosing l_i such that

$$-\log_D p_i^{\frac{R+\alpha-\beta}{\alpha}} + \log_D \left(\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i p_i} \right)$$

$$\le l_i < -\log_D p_i^{\frac{R+\alpha-\beta}{\alpha}} + \log_D \left(\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i p_i} \right) + 1$$

It implies

$$D^{-l_i} > \frac{p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\left(\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i p_i} \right) D} \tag{4.9}$$

Case 1: $R > \beta$

Since $R-\beta > 0$, thus raising power $\dfrac{R-\beta}{R+\alpha-\beta}$ on both sides of (4.9), we get

$$D^{-l_i\left(\frac{R-\beta}{R+\alpha-\beta}\right)} > \frac{p_i^{\frac{R-\beta}{\alpha}}}{\left[\left(\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i p_i} \right) D \right]^{\frac{R-\beta}{R+\alpha-\beta}}}$$

$$\Rightarrow D^{-l_i\left(\frac{R-\beta}{R+\alpha-\beta}\right)} > p_i^{\frac{R-\beta}{\alpha}} \left[\left(\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i p_i} \right) D \right]^{\frac{\beta-R}{R+\alpha-\beta}} \tag{4.10}$$

Multiplying both sides of (4.10) by $\dfrac{u_i p_i}{\sum_{i=1}^{n} u_i p_i}$ and then taking sum over $i = 1, 2, ..., n$

$$\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R-\beta}{R+\alpha-\beta}\right)}}{\sum_{i=1}^{n} u_i p_i} > \left[\left(\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i p_i}\right) D\right]^{\frac{\beta-R}{R+\alpha-\beta}} \left[\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i p_i}\right]$$

$$\Rightarrow \frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R-\beta}{R+\alpha-\beta}\right)}}{\sum_{i=1}^{n} u_i p_i} > \left[\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i p_i}\right]^{\frac{\alpha}{R+\alpha-\beta}} D^{\frac{\beta-R}{R+\alpha-\beta}}$$

$$\Rightarrow 1 - \frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R-\beta}{R+\alpha-\beta}\right)}}{\sum_{i=1}^{n} u_i p_i} < 1 - \left[\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i p_i}\right]^{\frac{\alpha}{R+\alpha-\beta}} D^{\frac{\beta-R}{R+\alpha-\beta}} \tag{4.11}$$

Multiply both sides of (4.11) by $\frac{R+\alpha-\beta}{R-\beta}$, we get

$$\frac{R+\alpha-\beta}{R-\beta}\left[1-\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R-\beta}{R+\alpha-\beta}\right)}}{\sum_{i=1}^{n} u_i p_i}\right] < \frac{R+\alpha-\beta}{R-\beta}\left[1-\left(\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{\alpha}{R+\alpha-\beta}} D^{\frac{\beta-R}{R+\alpha-\beta}}\right]$$

$$\Rightarrow \frac{R+\alpha-\beta}{R-\beta}\left[1-\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R-\beta}{R+\alpha-\beta}\right)}}{\sum_{i=1}^{n} u_i p_i}\right] < \frac{R+\alpha-\beta}{R-\beta}\left[1-\left(\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R+\alpha-\beta}{\alpha}}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{\alpha}{R+\alpha-\beta}}\right] D^{\frac{\beta-R}{R+\alpha-\beta}}$$
$$+\frac{R+\alpha-\beta}{R-\beta}\left[1-D^{\frac{\beta-R}{R+\alpha-\beta}}\right] \tag{4.12}$$

Thus, we can write the inequality (4.12) as:

$$L_R^{\alpha,\beta}(P;U) < H_R^{\alpha,\beta}(P;U) D^{\frac{\beta-R}{R+\alpha-\beta}} + \frac{R+\alpha-\beta}{R-\beta}\left[1-D^{\frac{\beta-R}{R+\alpha-\beta}}\right] \tag{4.13}$$

Case 2: $\beta > R$

Similarly, we can obtain the result (4.13) for $\beta > R$.

Taking (4.7) & (4.13) together, we obtain the following result:

$$H_R^{\alpha,\beta}(P;U) \le L_R^{\alpha,\beta}(P;U) < H_R^{\alpha,\beta}(P;U) D^{\frac{\beta-R}{R+\alpha-\beta}} + \frac{R+\alpha-\beta}{R-\beta}\left[1-D^{\frac{\beta-R}{R+\alpha-\beta}}\right] \tag{4.14}$$

5. ILLUSTRATION

In this section, we consider the Huffman and Shannon-Fano coding to verify the Shannon's noiseless coding theorem, as proved above, by taking a hypothetical data as given by Bhat and Baig (2016).

Table 5.1
Shannon-Fano Coding Scheme

$P(x_i)$	Shannon code-words	l_i	u_i	α	β	R	$H_R^{\alpha,\beta}(P;U)$	$L_R^{\alpha,\beta}(P;U)$	η	RHS
0.41	00	2	4	0.21	0.31	0.94	0.7602	0.9432	80.60	0.9925
0.18	01	2	2	0.49	0.42		0.9565	1.1123	85.99	1.2524
0.15	100	3	6	0.90	1.00		1.6162	1.7909	90.25	2.4089
0.13	101	3	1							
0.10	110	3	3							
0.03	111	3	5							

Table 5.2
Huffman Coding Scheme

$P(x_i)$	Huffman code-words	l_i	u_i	α	β	R	$H_R^{\alpha,\beta}(P;U)$	$L_R^{\alpha,\beta}(P;U)$	η	RHS
0.41	1	1	4	0.21	0.31	0.94	0.7602	0.8262	92.01	0.9925
0.18	000	3	2	0.49	0.42		0.9565	0.9754	98.06	1.2524
0.15	001	3	6	0.90	1.00		1.6162	1.6268	99.35	2.4089
0.13	010	3	1							
0.10	0110	4	3							
0.03	0111	4	5							

In the Tables (5.1) & (5.2), η represents coefficient of efficiency and is given as

$$\eta = \left(\frac{H_R^{\alpha,\beta}(P;U)}{L_R^{\alpha,\beta}(P;U)} \right) \times 100 \text{ and}$$

$$RHS = H_R^{\alpha,\beta}(P;U) D^{\frac{\beta-R}{R+\alpha-\beta}} + \frac{R+\alpha-\beta}{R-\beta}\left[1 - D^{\frac{\beta-R}{R+\alpha-\beta}}\right].$$

From these Tables (5.1) & (5.2), we obtain the following results:

1. In both coding schemes, the theorem 4.1, i.e., the result (4.14) holds.
2. The mean length of code-words is smaller in the Huffman coding scheme.
3. Huffman coding provides more efficient result than Shannon-Fano coding as η in case of Huffman coding is larger in comparison to Shannon-Fano coding.

6. MONOTONIC BEHAVIOUR OF GENERALIZED TWO PARAMETRIC 'USEFUL' RIM AND ITS CODE-WORD LENGTH

In this section, we study the behaviour of the proposed 'useful' RIM defined in (2.1) at different values of α, β & R. Let $P(x_i) = (0.41, 0.18, 0.15, 0.13, 0.10, 0.03)$, $u_i = (4, 2, 6, 1, 3, 5)$, $(l_i)_S = (2, 2, 3, 3, 3, 3)$ and $(l_i)_H = (1, 3, 3, 3, 4, 4)$. By using these values, we construct the following tables and figures.

Here $(l_i)_S$ and $(l_i)_H$ represent the length of code-words and $L_R^{\alpha,\beta}(P;U)_S$ and $L_R^{\alpha,\beta}(P;U)_H$ represent mean code-word lengths when we are using Shannon and Huffman coding schemes respectively. Green line indicates the value of measure (2.1). Blue and red lines indicate the values of $L_R^{\alpha,\beta}(P;U)_S$ and $L_R^{\alpha,\beta}(P;U)_H$ respectively.

Table 6.1

Behaviour of $H_R^{\alpha,\beta}(P;U)$ and $L_R^{\alpha,\beta}(P;U)$ for alpha=0.5 and beta=0.8

R	**0.89**	**6**	**20**	**34**	**57**	**79**	**108**	**154**	**213**	**300**
$H_R^{\alpha,\beta}(P;U)$	1.3202	0.6412	0.6038	0.5980	0.5947	0.5934	0.5924	0.5917	0.5913	0.5908
$L_R^{\alpha,\beta}(P;U)_S$	1.4759	0.8483	0.8158	0.8105	0.8075	0.8063	0.8054	0.8047	0.8043	0.8039
$L_R^{\alpha,\beta}(P;U)_H$	1.3140	0.7446	0.7170	0.7125	0.7100	0.7090	0.7082	0.7077	0.7073	0.7070

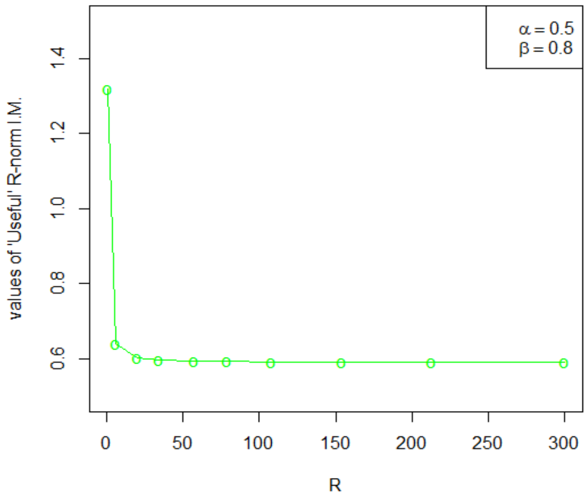

Figure 6.1

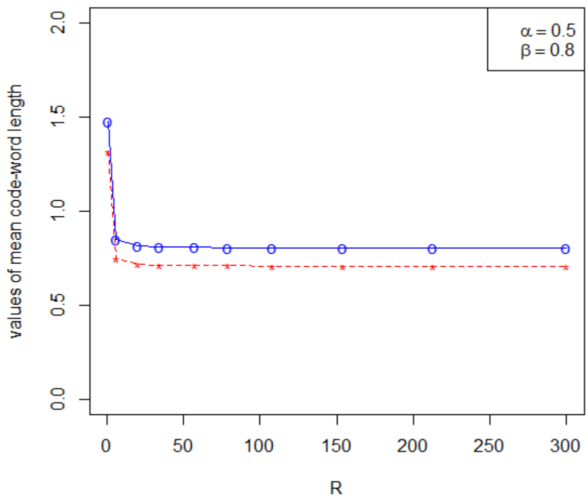

Figure 6.1

From Table (6.1), we observe that the value of measure (2.1) decreases as the value of *R* increases. Same relation prevails between the mean code-word length (4.1) and *R*. This relation is shown graphically in Figure (6.1) & (6.2). Thus, for varying R, there is monotonic decreasing relation between them.

Table 6.2

Behaviour of $H_R^{\alpha,\beta}(P;U)$ and $L_R^{\alpha,\beta}(P;U)$ for R=12 and Alpha=0.50

β	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$H_R^{\alpha,\beta}(P;U)$	0.6123	0.6125	0.6127	0.6129	0.6131	0.6133	0.6135	0.6137	0.6139	0.6142
$L_R^{\alpha,\beta}(P;U)_S$	0.8234	0.8236	0.8237	0.8239	0.8241	0.8243	0.8245	0.8246	0.8248	0.8250
$L_R^{\alpha,\beta}(P;U)_H$	0.7235	0.7236	0.7238	0.7239	0.7241	0.7242	0.7244	0.7245	0.7247	0.7249

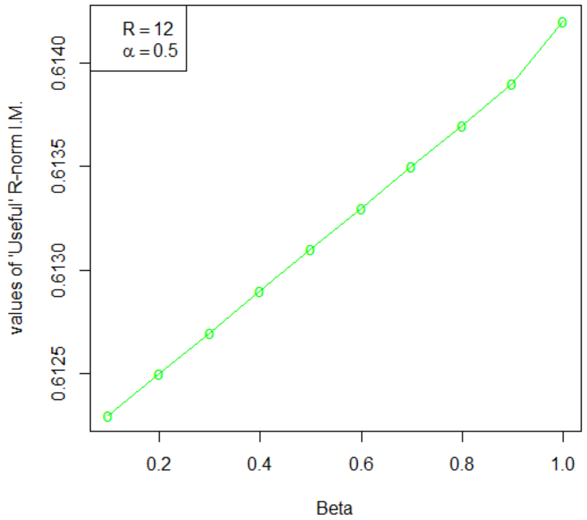

Figure 6.3

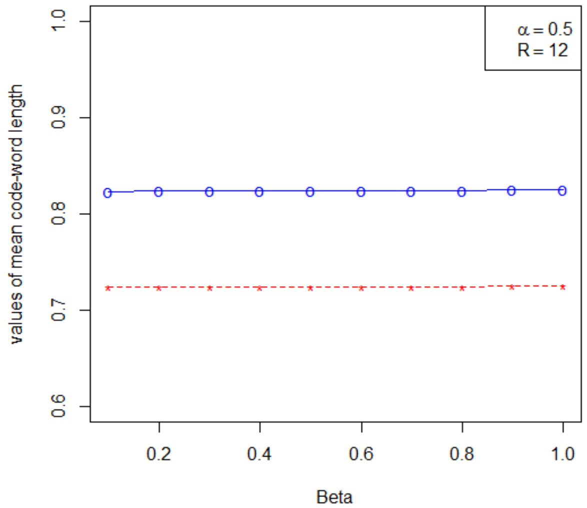

Figure 6.4

We observe from Table (6.2) that the values of measure defined in (2.1) as well as the mean code-word length given in (4.1) increase as the value of β increases. The values are plotted graphically in Figures (6.3) & (6.4), which show that there is an increasing trend. Thus, there is a monotonic increasing relation between them for fixed α and R.

Table 6.3

Behaviour of $H_R^{\alpha,\beta}(P;U)$ and $L_R^{\alpha,\beta}(P;U)$ for R=12 and beta=0.80

α	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**
$H_R^{\alpha,\beta}(P;U)$	0.5947	0.5995	0.6042	0.6090	0.6137	0.6185	0.6232	0.6280	0.6327	0.6375
$L_R^{\alpha,\beta}(P;U)_S$	0.8075	0.8118	0.8161	0.8204	0.8246	0.8288	0.8333	0.8371	0.8412	0.8452
$L_R^{\alpha,\beta}(P;U)_H$	0.7100	0.7137	0.7173	0.7209	0.7245	0.7281	0.7316	0.7351	0.7386	0.7420

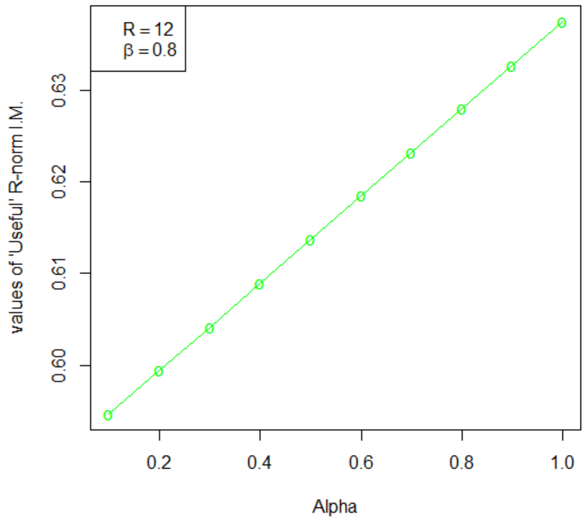

Figure 6.5

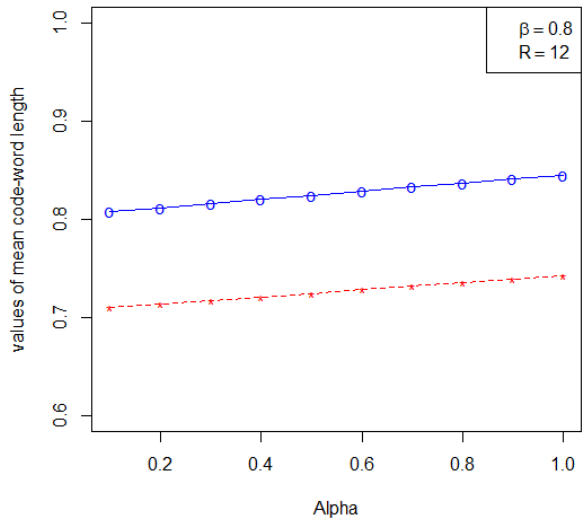

Figure 6.6

It is clear from Table (6.2) and Figures (6.5) and (6.6), that the values of (2.1) and (4.1) increase as we increase the value of α. This implies that (2.1) and (4.1) are monotonically increasing with respect to β and *R*.

We conclude from above tables and figures that the graph for $L_R^{\alpha,\beta}(P;U)_S$ always lie above the graph of $L_R^{\alpha,\beta}(P;U)_H$ which further clarifies that the Huffman coding scheme provides better results than Shannon coding scheme i.e., $L_R^{\alpha,\beta}(P;U)_S > L_R^{\alpha,\beta}(P;U)_H$.

7. CONCLUSION

Here, we have introduced a new generalized two parametric 'useful' RIM and also discussed its important properties. This measure reduces to a number of known RIMs. Further, the mean code-word length is derived and the bounds are obtained. Numerical data is taken to validate the properties and establish the results of the coding theorem. In addition, the monotonic behaviour of the 'useful' RIM and average code-word length is studied by taking different values of the parameters and R.

This measure can be further developed as a fuzzy measure, divergence measure, fuzzy divergence measure, so on.

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