Pak. J. Statist.
2018 Vol. 34(6), 513-530

EXTENDED POISSON-EXPONENTIATED WEIBULL DISTRIBUTION: THEORETICAL AND COMPUTATIONAL ASPECTS

Mohamed K.A. Refaie
Agami High Institute of Administrative Sciences, Alexandria, Egypt.
Email: refaie_top@yahoo.com

ABSTRACT

We introduce a new four-parameter zero truncated Poisson exponentiated Weibull lifetime model. The new model has a robust physical motivation. Comprehensive accounts of some of its statistical properties are derived. The maximum likelihood method is used to estimate the model parameters. We proved empirically the importance of the new model in modeling various types of lifetime data.

KEYWORDS

Zero Truncated Poisson Distribution, Maximum Likelihood Estimation, Order Statistics, Quantile function, Generating Function, Moments.

1. INTRODUCTION AND GENESIS

The zero-truncated Poisson distribution (ZTPD) (or the conditional Poisson distribution) is a discrete distribution whose support is only the set of the positive integers ($\mathbf{I}^{+}$). The ZTPD is the conditional probability distribution of a Poisson-distributed random variable (r.v.), given that the value of the r.v. is $\neq 0$. Thus, it is impossible for a ZTP to be zero, in this paper we will introduce a new flexible version of the ZTPD for modeling several types of lifetime data sets. Suppose that a system has M subsystems functioning independently at a given time where M has ZTPD with parameter λ. It is the conditional probability distribution of a Poisson distributed r.v., given that the value of the r.v. is $\neq 0$. The probability mass function (PMF) of M is given by

$$Pr(M=m)=[exp(-\lambda)\lambda^{m}]/\{[1-exp(-\lambda)]m!\}, m=1,2,3,\ldots. \quad (1)$$

For the ZTP r.v., the expected value $E(M)$ and variance $Var(M|\lambda)$ are respectively given by

$$E(M)=\lambda/[1-exp(-\lambda)],$$

and

$$Var(M|\lambda)=[1+\lambda-E(M)]E(M)$$
$$=(\lambda+\lambda^{2})/[1-exp(-\lambda)]-\{\lambda^{2}/[1-exp(-\lambda)]^{2}\}.$$

The cumulative distribution function (CDF) of the Topp Leone exponentiated Weibull distribution (TLEWD) specified by (due to Rezaei et al. (2017))

$$G(x;\alpha,a,b)=([1-exp(-x^{b})]^{a}\{-[1-exp(-x^{b})]^{a}+2\})^{\alpha}, \quad (2)$$

© 2018 Pakistan Journal of Statistics

respectively. Suppose that the failure time (Y_i) of each subsystem has the TLEWD defined by CDF in (2). Let Y_i denote the failure time of the i^{th} subsystem and let

$$X = min\{Y_1, Y_2, \cdots, Y_{M-1}, Y_M\}, \tag{3}$$

Ramos et al. (2017) has shown that both minimum and maximum can be considered in a general family based on the ZTPD, then the conditional CDF of X given M as described in Ramos et al. (2015) is

$$\begin{aligned} F(x \mid M) &= 1 - P_r(X > x \mid M) = 1 - P_r(Y_1 > x)^M \\ &= 1 - \left(1 - G_{TLEW}(x\,; \alpha\,, a, b)\right)^M, \end{aligned} \tag{4}$$

therefore, the marginal CDF of X is can be expressed as

$$F(x) = \frac{1 - exp[-\lambda([1 - exp(-x^b)]^a\{-[1 - exp(-x^b)]^a + 2\})^\alpha]}{(1 - e^{-\lambda})}, \tag{5}$$

where $F(x) = F_{ZTPTLEWD}(x\,; \lambda\,, \alpha, a, b)$. Equation (5) is called the CDF of the zero truncated Poisson Topp Leone exponentiated Weibull (ZTPTLEW) model. The corresponding PDF of (5) reduces to

$$\begin{aligned} f(x) = {} & 2\lambda\alpha a b\left(1 - e^{-\lambda}\right)^{-1} x^{b-1}\, exp(-x^b)\, [1 - exp(-x^b)]^{a\alpha-1} \\ & \times exp[-\lambda([1 - exp(-x^b)]^a\{-[1 - exp(-x^b)]^a + 2\})^\alpha] \\ & \times \{1 - [1 - exp(-x^b)]^a\}\{-[1 - exp(-x^b)]^a + 2\}^{\alpha-1}, \end{aligned} \tag{6}$$

where $f(x) = f_{ZTPTLEWD}(x\,; \lambda\,, \alpha, a, b)$.

Some other useful extension of model can be found in Afify et al. (2016a, b, c), Aryal and Yousof (2017), Brito et al. (2017), Cordeiro et al. (2017a, b), Aryal et al. (2017), Yousof et al. (2017a-e), Hamedani et al. (2018a, b), Yousof et al. (2018) and Cordeiro et al. (2018), among others.

Now we shall provide a useful linear representation for the ZTPTLEW density function in (6). Expanding the quantity

$$exp[-\lambda([1 - exp(-x^b)]^a\{2 - [1 - exp(-x^b)]^a\})^\alpha],$$

in power series, we have

$$f(x) = \sum_{i=0}^{\infty} \frac{\{1 - [1 - exp(-x^b)]^a\}[1 - exp(-x^b)]^{a[\alpha(i+1)+1]-2}}{\left\{1 - \frac{1}{2}[1 - exp(-x^b)]^a\right\}^{-\alpha(i+1)+1}}. \tag{7}$$

Consider the power series

$$(1 - z)^{\varphi-1} = \sum_{q=0}^{\infty}\{(-1)^q \Gamma(\varphi) z^q \div [q!\,\Gamma(\varphi - q)]\}, \tag{8}$$

which holds for $\varphi > 0$ real non-integer and $|z| < 1$. Using (8) and after some algebra the PDF of the ZTPTLEWD in (7) will be

$$f(x) = \sum_{i,j}^{\infty} \left[\eta_{i,j}^{(1)} \, \pi_{[\alpha(i+1)+j]a}(x\,;b) - \eta_{i,j}^{(2)} \, \pi_{[\alpha(i+1)+j+1]a}(x\,;b) \right], \tag{9}$$

where

$$\pi_\gamma(x\,;b) = \gamma b x^{b-1} \, exp(-x^b) \, [1 - exp(-x^b)]^{\gamma-1}.$$

$$\eta_{i,j}^{(1)} = \frac{2^{\alpha(i+1)-j} \alpha \lambda^{i+1} (-1)^{i+j}}{[-\,exp(-\lambda) + 1] i! \, [\alpha(i+1) + j] a} \binom{-1 + \alpha(1+i)}{j},$$

and

$$\eta_{i,j}^{(2)} = \frac{2^{\alpha(i+1)-j} \alpha \lambda^{i+1} (-1)^{i+j}}{[-\,exp(-\lambda) + 1] i! \, [\alpha(i+1) + j + 1] a} \binom{-1 + \alpha(1+i)}{j}.$$

Equation (9) reveals that the density of X can be expressed as a linear representation of EWD. So, several mathematical properties of the new family can be obtained by knowing those of the exp-W distribution. The CDF of the ZTPTLEWD can also be expressed as a mixture of EWD. Via integrating (9), we obtain the same mixture representation

$$F(x) = \sum_{i,j}^{\infty} \left[\eta_{i,j}^{(1)} \, H_{[\alpha(i+1)+j]a}(x\,;b) - \eta_{i,j}^{(2)} \, H_{[\alpha(i+1)+j+1]a}(x\,;b) \right], \tag{10}$$

where

$$H_\gamma(x\,;b) = [1 - exp(-x^b)]^{\gamma},$$

is the CDF of the EWD with power parameter γ.

Figure 1 (left panel) displays three plots of the ZTPTLEWD for different values of parameters, these plots reveal that the ZTPTLEWD can be right or left-skewed model. The HRF plots of the ZTPTLEWD given in Figure 1 (right panel) can be decreasing or increasing or bathtub (**U**) shapes.

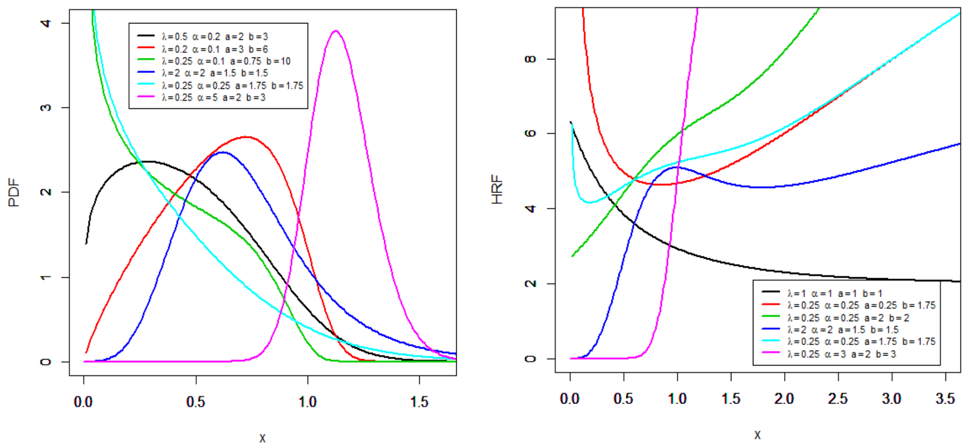

Figure 1: Plots of the ZTPTLEW PDF (left panel) and HRF (right panel) for Some Parameter Values

2. MATHEMATICAL PROPERTIES

2.1 Moments, Incomplete Moments and Generating Function

The n^{th} ordinary moment of X is given by

$$\mu_n^{'} = E(X^n) = \int_{-\infty}^{\infty} x^n \, f(x) dx.$$

Then we obtain

$$\mu'_n = \Gamma\left(\frac{n}{b}+1\right)\sum_{i,j,h}^{\infty}\left(\eta_{i,j,h}^{(1)} - \eta_{i,j,h}^{(2)}\right), \quad (11)$$

where

$$\eta_{i,j,h}^{(1)} = \eta_{i,j}^{(1)}\zeta_h^{([\alpha(i+1)+j]a,n)},$$

$$\eta_{i,j,h}^{(2)} = \eta_{i,j}^{(2)}\zeta_h^{([\alpha(i+1)+j+1]a,n)},$$

and

$$\zeta_h^{(C,m)} = (-1)^h\Gamma(1+C)\left[h!\,\Gamma(C-h)(h+1)^{\left(\frac{m}{b}+1\right)}\right]^{-1}.$$

Setting $n = 1$ in (11), we have the mean of X. The last integration can be computed numerically for most parent distributions. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. The n^{th} central moment of X, say M_n, follows as

$$M_n = E(X-\mu)^n = \sum_{h=0}^{n}(-1)^h\binom{n}{h}(\mu_1^{'})^n\,\mu_{n-h}^{'}.$$

The s^{th} incomplete moment, say $\vartheta_s(t)$, of X can be expressed from (9) as

$$\vartheta_s(t) = \int_{-\infty}^{t} x^s f(x) dx.$$

Then

$$\vartheta_s(t) = \gamma\left(\frac{s}{b}+1, \left(\frac{\alpha}{t}\right)^b\right)\sum_{i,j,h}^{\infty}\left(\eta_{i,j,h}^{(1)} - \eta_{i,j,h}^{(2)}\right), \quad (12)$$

where

$$\eta_{i,j,h}^{(1)} = \eta_{i,j}^{(1)}\zeta_h^{([\alpha(i+1)+j]a,s)},$$

and

$$\eta_{i,j,h}^{(2)} = \eta_{i,j}^{(2)}\zeta_h^{([\alpha(i+1)+j+1]a,s)}.$$

The $\vartheta_1(t)$ is the first incomplete moment given by (12) with $s = 1$. The moment generating function $M_X(t) = E(e^{tX})$ of X can be derived from equation (9) as

$$M_X(t) = \Gamma\left(\frac{r}{b}+1\right)\sum_{i,j,r,h}^{\infty}\left(\eta_{i,j,r,h}^{(1)} - \eta_{i,j,r,h}^{(2)}\right),$$

where

$$\eta_{i,j,r,h}^{(1)} = \eta_{i,j}^{(1)} \zeta_h^{([\alpha(i+1)+j]a,r)} t^r (r!)^{-1},$$

and

$$\eta_{i,j,r,h}^{(2)} = \eta_{i,j}^{(2)} \zeta_h^{([\alpha(i+1)+j+1]a,r)} t^r (r!)^{-1}.$$

2.2 Probability Weighted Moments

The $(s,r)^{th}$ PWM of X following the ZTPTLEWD, say $\rho_{s,r}$, is formally defined by

$$\rho_{r,s} = E\left\{X^s F(X)^r\right\} = \int_{-\infty}^{\infty} x^s F(x)^r f(x)\, dx.$$

Using equations (5) and (6), we can write

$$f(x)F(x)^r = \sum_{i,j}^{\infty} \left[c_{i,j}^{(1)} \pi_{[\alpha(i+1)+j]a}(x) - c_{i,j}^{(2)} \pi_{[\alpha(i+1)+j+1]a}(x)\right],$$

where

$$c_{i,j}^{(1)} = \sum_{k=0}^{\infty} \binom{r}{k} \binom{-1+\alpha(i+1)}{j} \frac{2^{\alpha(i+1)-j} \alpha \lambda^{i+1} (-1)^{i+j+k} (k+1)^i}{i!\,[1-exp(-\lambda)]^{r+1} [\alpha(i+1)+j]a},$$

and

$$c_{i,j}^{(2)} = \sum_{k=0}^{\infty} \binom{r}{k} \binom{-1+\alpha(i+1)}{j} \frac{2^{\alpha(i+1)-j} \alpha \lambda^{i+1} (-1)^{i+j+k} (k+1)^i}{i!\,[1-exp(-\lambda)]^{r+1} [\alpha(i+1)+j+1]a},$$

then, the $(s,r)^{th}$ PWM of X will be

$$\rho_{r,s} = \Gamma\left(\frac{r}{b}+1\right) \sum_{i,j,h=0}^{\infty} \left(c_{i,j,h}^{(1)} - c_{i,j,h}^{(2)}\right),$$

where

$$c_{i,j,h}^{(1)} = c_{i,j}^{(1)} \zeta_h^{([\alpha(i+1)+j]a,r)},$$

and

$$c_{i,j,h}^{(2)} = c_{i,j}^{(2)} \zeta_h^{([\alpha(i+1)+j+1]a,r)}.$$

2.3 Residual Life and Reversed Residual Life Functions

The n^{th} moment of the residual life, say

$$a_n(t) = E[(X-t)^n \mid_{X>t, n=1,2,\ldots}],$$

the n^{th} moment of the residual life of X is given by

$$a_n(t) = \frac{\int_t^{\infty} (x-t)^n dF(x)}{1-F(t)}.$$

Therefore

$$a_n(t) = \gamma\left(\frac{n}{b}+1, \left(\frac{\alpha}{t}\right)^b\right) [1-F(t)]^{-1} \sum_{r}^{n} \sum_{i,j,h}^{\infty} \left(p_{i,j,h}^{(1)} - p_{i,j,h}^{(2)}\right)$$

where

$$p_{i,j,h}^{(1)} = \eta_{i,j}^{(1)} \binom{n}{r} (-t)^{n-r} \zeta_h^{([\alpha(i+1)+j]a,n)},$$

and

$$p_{i,j,h}^{(2)} = \eta_{i,j}^{(2)} \binom{n}{r} (-t)^{n-r} \zeta_h^{([\alpha(i+1)+j+1]a,n)}.$$

The n^{th} moment of the reversed residual life, say

$$A_n(t) = E\left[(t-X)^n \mid_{X\le t, t>0, n=1,2,\dots}\right]$$

uniquely determines $F(x)$. We obtain

$$A_n(t) = \frac{\int_0^t (t-x)^n dF(x)}{F(t)}.$$

Then, the n^{th} moment of the reversed residual life of X becomes

$$A_n(t) = \Gamma\left(\frac{n}{b}+1, \left(\frac{\alpha}{t}\right)^b\right) F(t)^{-1} \sum_{r}^{n} \sum_{i,j,h}^{\infty} \left(q_{i,j,h}^{(1)} - q_{i,j,h}^{(2)}\right).$$

where

$$q_{i,j,h}^{(1)} = \eta_{i,j}^{(1)} (-1)^r t^{n-r} \zeta_h^{([\alpha(i+1)+j]a,n)},$$

and

$$q_{i,j,h}^{(2)} = \eta_{i,j}^{(2)} (-1)^r t^{n-r} \zeta_h^{([\alpha(i+1)+j+1]a,n)}.$$

2.4 Moments of Order Statistics

Suppose that we have a random sample $(X_1, \dots, X_n)$ from the ZTPTLEWD and suppose that $X_{(1)}, \dots, X_{(n)}$ be the corresponding order statistics. The PDF of i^{th} order statistic, say $f_{i\,:\,n}(x)$, can be written as

$$f_{i\,:\,n}(x) = \sum_{j=0}^{n-i} (-1)^j \frac{f(x)}{B(i, -i+1+n)} \binom{n-i}{j} F^{j+i-1}(x), \tag{13}$$

where $B(\cdot,\cdot)$ is the beta function. By substituting Equation (5) and Equation (6) in Equation (13) and using a power series expansion, we have

$$F(x)^{j+i-1} f(x) = \sum_{w,m=0}^{\infty} \left[a_{w,m}^{(1)}\, \pi_{\alpha(w+1)+m}(x) - a_{w,m}^{(2)}\, \pi_{\alpha(w+1)+m+1}(x)\right],$$

where

$$a_{w,m}^{(1)} = \sum_{k=0}^{\infty} \frac{2^{\alpha(w+1)-m} \alpha \lambda^{w+1} (-1)^{w+m+k} (k+1)^w}{w!\,[1-exp(-\lambda)]^{j+i} [\alpha(w+1)+m] a} \binom{j+i-1}{k} \binom{\alpha(w+1)-1}{m},$$

and

$$a_{w,m}^{(2)} = \sum_{k=0}^{\infty} \frac{2^{\alpha(w+1)-m} \alpha \lambda^{w+1} (-1)^{w+m+k} (k+1)^w}{w!\,[1-exp(-\lambda)]^{j+i} [\alpha(w+1)+m+1] a} \binom{j+i-1}{k} \binom{\alpha(w+1)-1}{m}.$$

The PDF of $X_{i\,:\,n}$ can be expressed as

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \frac{(-1)^j \binom{n-i}{j}}{B(i,-i+1+n)} \sum_{w,m=0}^{\infty} \left[a_{w,m}^{(1)} \pi_{[\alpha(w+1)+m]a}(x) - a_{w,m}^{(2)}\, \pi_{[\alpha(w+1)+m+1]a}(x)\right].$$

Then, the density function of the ZTPTLEW order statistics is a mixture of exp-W density. Based on the last equation, the moments of $X_{i\,:\,n}$ can be expressed as

$$E\left(X_{i\,:\,n}^q\right) = \Gamma\left(\frac{q}{b}+1\right) \sum_{j=0}^{n-i} \sum_{w,m,h=0}^{\infty} \frac{(-1)^j \binom{n-i}{j}}{B(i,n-i+1)} \left(a_{w,m,h}^{(1)} - a_{w,m,h}^{(2)}\right),$$

where

$$a_{w,m,h}^{(1)} = a_{w,m}^{(1)} \zeta_h^{([\alpha(w+1)+m]a,q)}$$

and

$$a_{w,m,h}^{(2)} = a_{w,m}^{(2)} \zeta_h^{([\alpha(w+1)+m+1]a,q)}.$$

3. ESTIMATION

For determining the maximum likelihood estimators (MLEs) of $\mathbf{\Theta}$, we have the log-likelihood function

$$\begin{aligned}\ell = \ell(\mathbf{\Theta}) &= n\left[\log \lambda + \log(2) + \log \alpha + \log a + \log b - \log\left(1-e^{-\lambda}\right)\right] \\ &- \sum_{i=1}^{n} x_i^b + (a-1) \sum_{i=1}^{n} \log\left[1-\exp\left(-x_i^b\right)\right] + a(\alpha-1) \sum_{i=1}^{n} \log\left[1-\exp\left(-x_i^b\right)\right] \\ &-(b-1) \sum_{i=1}^{n} \log x_i - \lambda \sum_{i=1}^{n} \left(\left[1-\exp\left(-x_i^b\right)\right]^a \left\{2-\left[1-\exp\left(-x_i^b\right)\right]^a\right\}\right)^\alpha \\ &+ \sum_{i=1}^{n} \log\left\{1-\left[1-\exp\left(-x_i^b\right)\right]^a\right\} + (\alpha-1) \sum_{i=1}^{n} \log\left\{2-\left[1-\exp\left(-x_i^b\right)\right]^a\right\}.\end{aligned}$$

The components of the score vector, $U(\mathbf{\Theta}) = \frac{\partial \ell}{\partial \mathbf{\Theta}}$ are available if needed.

4. SIMULATION STUDIES

We simulate the new model by taking n = 20, 50, 100, 200 and 500. For each sample size n, we evaluate the MLEs of the parameters using the **MATHCAD** (V15) software. Then, we repeat this process 1000 times and calculate the averages of the estimates (AEs), biases (Bias) and mean squared errors (MSEs).

Table 1 gives all the numerical results of the simulation study. The values in Table 1 indicate that the MSEs and the Bias of all parameters decay toward zero when n increases for all settings as expected. The AEs of the parameters tend to be closer to the true parameter values when n increases. This previous fact supports that the asymptotic normal distribution gives an adequate approximation to the finite sample distribution of the MLEs.

Table 1 gives the AEs, Bias and MSEs based on 1000 simulations of the new distribution for some values of parameters.

Table 1
Average Values of Estimates, the corresponding MSEs and Bias (in parentheses)

	n=20	n=50	n=100	n=200	n=500
Parameter	**AE(Bias)(MSE)**	**AE(Bias)(MSE)**	**AE(Bias)(MSE)**	**AE(Bias)(MSE)**	**AE(Bias)(MSE)**
$\alpha = 1.5$	1.6(0.768)(0.123)	1.5(0.62)(0.039)	1.5(0.613)(0.02)	1.5(0.604)(0.009)	1.5(0.607)(0.004)
$\lambda = 0.9$	0.92(-0.576)(1.092)	0.92(0.022)(0.373)	0.91(-0.597)(0.2)	0.91(-0.592)(0.095)	0.9(-0.614)(0.034)
$a = 0.8$	0.82(0.023)(0.017)	0.81(-0.694)(0.006)	0.80(0.099)(0.003)	0.80(-0.099)(0.001)	0.8(-0.097)(0.0005)
$b = 0.9$	0.93(0.029)(0.026)	0.91(0.108)(0.008)	0.91(0.105)(0.004)	0.9(0.102)(0.002)	0.90(0.103)(0.0008)
$\alpha = 2$	2.1(0.099)(0.241)	20.03(1.33)(0.076)	2.2(1.319)(0.039)	2(1.306)(0.018)	2.01(0.009)(0.008)
$\lambda = 0.8$	(0.82)(0.023)(1.27)	0.82(-1.182)(0.428)	0.81(-0.195)(0.233)	0.81(-1.191)(0.11)	0.8(-1.215)(0.04)
$a = 0.9$	0.91(0.129)(0.024)	0.91(0.109)(0.008)	0.91(0.105)(0.004)	0.90(0.102)(0.002)	0.90(0.103)(0.0007)
$b = 0.7$	0.72(0.177)(0.017)	0.71(0.007)(0.005)	0.71(0.195)(0.003)	0.70(0.199)(0.001)	0.070(-0.197)(0.0005)
$\alpha = 2.5$	2.6(0.113)(0.337)	2.53(1.33)(0.109)	2.52(1.321)(0.055)	2.51(1.31)(0.026)	2.5(1.311)(0.009)
$\lambda = 0.9$	(0.92)(-1.577)(1.054)	0.92(-1.578)(0.362)	0.91(-1.594)(0.193)	0.91(0.007)(0.092)	0.9(-1.613)(0.033)
$a = 0.8$	0.82(0.08)(0.015)	0.81(0.0057)(0.005)	0.80(-0.596)(0.003)	0.80(-0.099)(0.001)	0.8(-0.098)(0.0005)
$b = 1.2$	1.2(0.045)(0.055)	1.2(0.413)(0.017)	1.21(0.408)(0.0009)	1.2(0.403)(0.004)	1.2(0.405)(0.0015)

5. APPLICATIONS

We shall provide two real data applications of the ZTPTLEWD to show and demonstrate its potentiality. For comparing the fits of the ZTPTLEWD with other competing models, we consider the Cramér-von Mises (W^*) and the Anderson-Darling (A^*) statistics. These statistics are given as

$$W^* = \frac{1}{12n}\left(1+\frac{1}{2n}\right)+\sum_{j=1}^{n} w_j,$$

and

$$A^* = \left(1+\frac{9}{4n^2}+\frac{3}{4n}\right)\left(n+\frac{1}{n}\sum_{j=1}^{n} a_j\right),$$

respectively, where

$$w_j = \left(z_i - \frac{2j-1}{2n}\right)^2\left(1+\frac{1}{2n}\right),$$
$$a_j = (2j-1)\,log\left[z_i\left(1-z_{n-j+1}\right)\right],$$
$$z_j = F\left(y_j\right),$$

and the y_j's values are the ordered observations.

5.1 Failure Times of 84 Aircraft Windshield

The 1st data consist of 84 observations (the failure times of 84 aircraft windshield). Here, we shall compare the fits of the ZTPTLEWD with those of other competitive models, namely: Marshall Olkin extended Weibull (MOEWD) (by Ghitany et al., 2005), gamma Weibull (GaWD) (by Provost et al., 2011), Weibull Fréchet (WFrD) (by Afify et al., 2016), beta Weibull (BWD) (by Lee et al., 2007), Kumaraswamy Weibull (KwWD) (by Cordeiro et al., 2010), Kumaraswamy transmuted Weibull (KwTWD) (by Afify et al., 2016), modified beta Weibull (MBWD) (by Khan, 2015), Mcdonald Weibull (McWD) (by Cordeiro et al., 2014), transmuted exponentiated generalized Weibull (TEGWD) (by Yousof et al., 2015) and transmuted modified Weibull (TMWD) (by Khan and King, 2013) distributions, whose PDFs (for $x > 0$) are: given in Appendix A. The MLEs and their corresponding standard errors (SEs) (in parentheses) of the model parameters are given in Tables 2. Tables 3 list the values of above statistics for eleven fitted models. The numerical values in Table 3 reveal that the ZTPTLEWD yields the lowest numerical values of these statistics (W^* & A^*) and then provides the best fit to the two data sets.

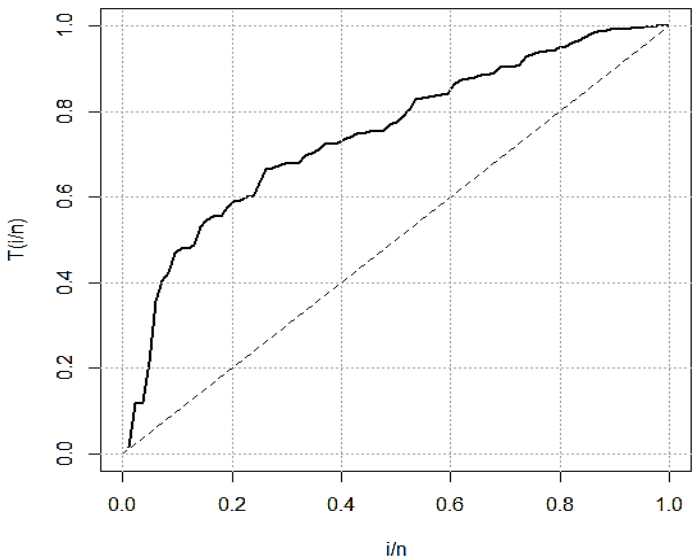

Figure 2: TTT Plot for Data Set I

Table 2
MLEs (Standard Error in Parentheses) for Data Set I

Model	Estimates
ZTPTLEWD(λ,α,a,b)	-4.726, 0.0380, 42.217, 0.879 (1.67), (0.023), (17.942), (0.050)
MOEWD(γ,β,α)	488.899, 0.2832, 1261.97 (189.358), (0.013), (351.073)
GWD(α,β,γ)	2.37697, 0.8481, 3.534 (0.378), (0.0006), (0.665)
KwWD(α,β,a,b)	14.433, 0.204, 34.659, 81.846 (27.095), (0.042), (17.527), (52.014)
WFrD(α,β,a,b)	630.938, 0.302, 416.097, 1.166 (697.942), (0.032), (232.359), (0.357)
BWD(α,β,a,b)	1.360, 0.298, 34.180, 11.495 (1.002), (0.060), (14.838), (6.730)
TMWD$(\alpha,\beta,\gamma,\lambda)$	0.272, 1, 4.6×10^{-6}, 0.469 (0:014), (5.2×10^{-5}), (1.9×10^{-4}), (0.165)
KwTWD$(\alpha,\beta,\lambda,a,b)$	27.791, 0.178, 0.445, 29.525, 168.060 (33.401), (0.017), (0.609), (9.792), (129.165)
MBWD(α,β,a,b,c)	10.150, 0.163, 57.417, 19.386, 2.004 (18.697), (0.019), (14.063), (10.019),(0.662)
McWD(α,β,a,b,c)	1.940, 0.306, 17.686, 33.639, 16.721, (1.011), (0:045), (6.222), (19.994), (9.722)
TEGWD$(\alpha,\beta,\lambda,a,b)$	4.257,0.153, 0.098, 5.231,1173.328 (33.401), (0:017), (0.609),(9.792)

Table 3
The Statistics W & A for Data Set I

Model	$W*$	$A*$
ZTPTLEWD(λ,α,a,b)	**0.051**	**0.538**
MOEWD (γ,β,α)	0.339	4.447
GWD (α,β,γ)	0.255	1.949
KwWD (α,β,a,b)	0.185	1.506
WFrD$(\alpha,\beta,a,b,)$	0.254	1.957
BWD (α,β,a,b)	0.465	3.219
TMWD$(\alpha,\beta,\gamma,\lambda)$	0.806	11.205
KwTWD$(\alpha,\beta,\lambda,a,b)$	01.164	1.363
MBW(α,β,a,b,c)	0.472	3.266
McWD(α,β,a,b,c)	0.199	1.591
TEGWD$(\alpha,\beta,\lambda,a,b)$	1.008	6.233

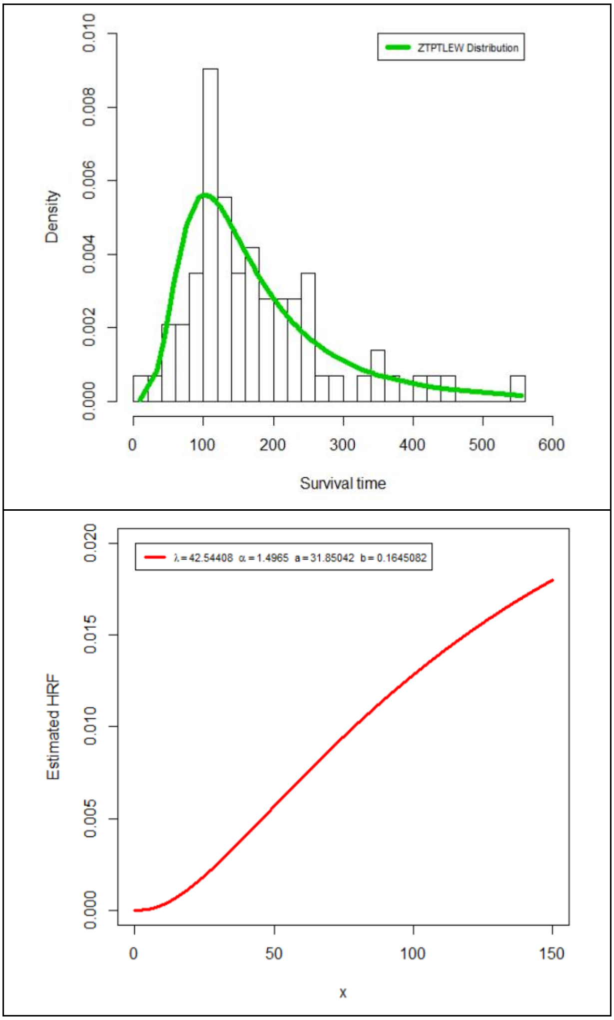

Figure 3: Estimated PDF and Estimated HRF for Data Set I

5.2 Survival Times (In Days) of 72 Guinea Pigs

For the 2nd real data set see Bjerkedal (1960). We shall compare the fits of the ZTPTLEWD with those of other competitive models, namely: Weibull Weibull (WWD) (see Tahir et al., 2016), odd Weibull Weibull (OWWD) (by Bourguignon et al., 2014),Weibull Log Weibull (WLogWD) (see Alzaatreh et al., 2013), the gamma exponentiated-exponential (GaEED) (by Ristic and Balakrishnan 2012) and exponential-exponential geometric (EEGcD) (see Rezaei et al., 2013) distributions, whose PDFs (for $x > 0$) are given in appendix B. The MLEs and their corresponding SEs of the model parameters are given in Tables 4. However, Tables 5 list the numerical values of above statistics for six fitted models. The figures in Table 3 reveal that the new model yields the lowest values of these statistics and then provides the best fit to the two data sets.

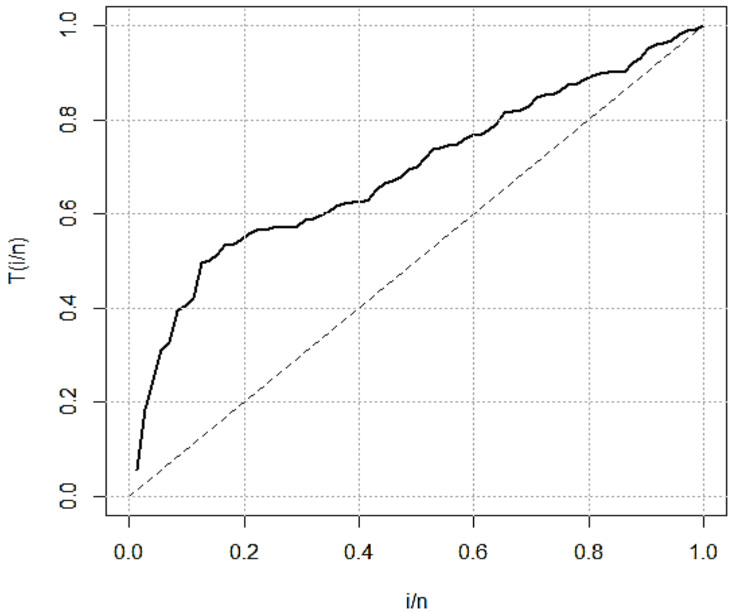

Figure 4: TTT Plot for Data Set II

Figures 2 and 4 give the total time test (TTT) plots the two data sets. Figures 2 and 4 indicate that the empirical HRFs of the two data sets are decreasing.

Table 4
MLEs (Standard Errors in Parentheses) for Data Set II

Model	Estimates
ZTPTLEWD(λ, α, a, b)	42.54, 1.496, 31.85, 0.16
	(0.0), (0.209), (4.238), (0.0)
WWD(β, γ, λ)	2.659, 0.693, 0.027
	(0.713), (0.171), (0.019)
OWWD(β, γ, λ)	11.158, 0.088, 0.457
	(4.545) (0.035) (0.077)
WLogWD(β, γ, λ)	1.787, 0.779, 0.025
	(0.782), (0.333), (0.040)
GaEED$(\lambda, \alpha, \theta)$	2.114, 2.601, 0.009
	(1.329), (0.559), (0.005)
EEGcD(α, θ, p)	2.589, 0.0004, 0.99
	(0.482), 0.004), (0.104)

Table 5
The Statistics $W*$ & $A*$ for Data Set II

Model	$W*$	$A*$
ZTPTLEWD(λ, α, a, b)	**0.094**	**0.583**
WWD(β, γ, λ)	0.142	0.781
OWWD(β, γ, λ)	0.449	2.476
WLogWD(β, γ, λ)	0.435	2.394
GaEED$(\lambda, \alpha, \theta)$	0.315	1.721
EEGcD(α, θ, p)	1.105	0.597

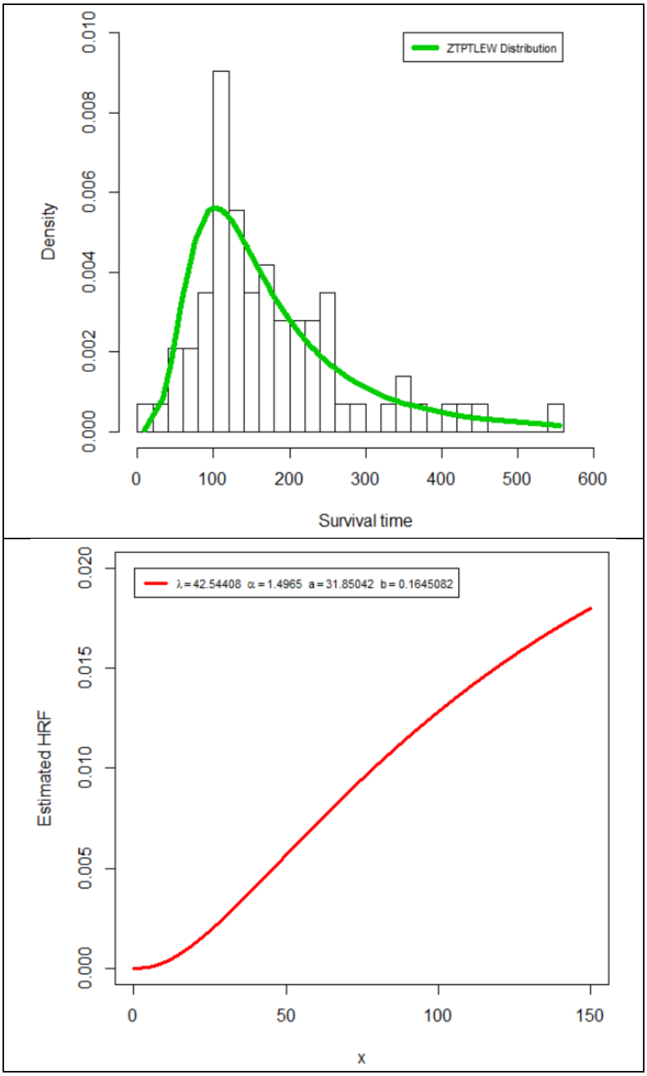

Figure 5: Estimated PDF and Estimated HRF for data set II.

Based on the numerical values in Tables 1 and 2, we conclude that the ZTPTLEWD provide adequate fits as compared to other W models in the both applications with small values for W^* & A^*.

From application 1, the proposed ZTPTLEWD is better than the MOEWD, GaWD, KwWD, WFrD, BWD, TMWD, KwTWD, MBWD, McWD, TEGWD, and adequate alternative to these models.

From application 2, the proposed ZTPTLEWD is better than the WWD, OWWD, WLogWD, GaEED and EEGcD. So, the new model is a sufficient alternative to these models.

6. CONCLUSIONS

We introduced a new four parameter zero truncated Poisson lifetime model. We also provided of some of its statistical properties including ordinary and incomplete moments, generating functions and order statistics. The maximum likelihood (ML) method is used to estimate the unknown model parameters. We assessed the performance of the ML estimators of the new model with respect to sample size n, the assessment was based on a simulation study. We proved empirically (via two real data applications) the importance and flexibility of the new model in modeling two types of lifetime data. The new model is much better than other important competitive extensions.

REFERENCES

1. Afify, A.Z., Cordeiro, G.M., Yousof, H.M. Alzaatreh, A. and Nofal, Z.M. (2016a). The Kumaraswamy transmuted-G family of distributions: properties and applications. *Journal of Data Science*, 14(2), 245-270.
2. Afify, A.Z., Cordeiro, G.M., Yousof, H.M., Abdus Saboor and Ortega, E.M.M. (2016b). The Marshall-Olkin additive Weibull distribution with variable shapes for the hazard rate. *Hacettepe Journal of Mathematics and Statistics*, forthcoming. DOI: 10.15672/HJMS.201612618532.
3. Afify, A.Z., Yousof, H.M., Cordeiro, G.M., Ortega, E.M.M. and Nofal, Z.M. (2016c). The Weibull Fréchet distribution and its applications. *Journal of Applied Statistics*, 43, 2608-2626.
4. Alizadeh, M., Yousof, H.M., Afify, A.Z., Cordeiro, G.M. and Mansoor, M. (2018). The complementary generalized transmuted Poisson-G family of distributions. *Austrian Journal of Statistics*, 47(4), 60-80.
5. Alzaatreh, A., Lee, C. and Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, 71, 63-79.
6. Aryal, G.R., Ortega, E.M., Hamedani, G.G. and Yousof, H.M. (2017). The ToppLeone Generated Weibull distribution: regression model, characterizations and applications. *International Journal of Statistics and Probability*, 6, 126-141.
7. Aryal, G.R. and Yousof, H.M. (2017). The exponentiated generalized-G Poisson family of distributions. *Economic Quality Control*, 32(1), 1-17.
8. Bjerkedal, T. (1960). Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli. *American Journal of Hygiene*, 72, 130-148.

9. Bourguignon, M., Silva, R.B. and Cordeiro, G.M. (2014). The Weibull--G family of probability distributions. *Journal of Data Science*, 12, 53-68.
10. Brito, E., Cordeiro, G.M., Yousof, H.M., Alizadeh, M. and Silva, G.O. (2017). Topp-Leone Odd Log-Logistic Family of Distributions. *Journal of Statistical Computation and Simulation*, 87(15), 3040-3058.
11. Cordeiro, G.M., Afify, A.Z., Yousof, H.M., Pescim, R.R. and Aryal, G.R. (2017a). The exponentiated Weibull-H family of distributions: Theory and Applications. *Mediterranean Journal of Mathematics*, 14, 1-22.
12. Cordeiro, G.M., Afify, A.Z., Yousof, H.M., Cakmakyapan, S. and Ozel, G. (2018). The Lindley Weibull distribution: properties and applications. *Anais da Academia Brasileira de Ciências*, 90(3), 2579-2598.
13. Cordeiro, G.M., Hashimoto, E.M. and Ortega, E.M. (2014). The McDonald Weibull model. *Statistics: A Journal of Theoretical and Applied Statistics*, 48, 256-278.
14. Cordeiro, G.M., Ortega, E.M. and Nadarajah, S. (2010). The Kumaraswamy Weibull distribution with application to failure data. *Journal of the Franklin Institute*, 347, 1399-1429.
15. Cordeiro, G.M., Yousof, H.M., Ramires, T.G. and Ortega, E.M.M. (2017b). The Burr XII system of densities: properties, regression model and applications. *Journal of Statistical Computation and Simulation*, 88(3), 432-456.
16. Hamedani, G.G. Yousof, H.M., Rasekhi, M., Alizadeh, M., Najibi, S.M. (2018a). Type I general exponential class of distributions. *Pakistan Journal of Statistics and Operation Research*, 14(1), 39-55.
17. Hamedani, G.G. Rasekhi, M., Najibi, S.M., Yousof, H.M. and Alizadeh, M. (2018b). Type II general exponential class of distributions. *Pakistan Journal of Statistics and Operation Research*, forthcoming.
18. Khan, M.N. (2015). The modied beta Weibull distribution. *Hacettepe Journal of Mathematics and Statistics*, 44, 1553-1568.
19. Khan, M.S. and King, R. (2013). Transmuted modied Weibull distribution: A generalization of the modified Weibull probability distribution. *European Journal of Pure and Applied Mathematics*, 6, 66-88.
20. Korkmaz, M.Ç., Yousof, H.M. and Hamedani, G.G. (2018). The Exponential Lindley Odd Log-Logistic-G Family: Properties, Characterizations and Applications. *Journal of Statistical Theory and Applications*, 17(3), 554-571.
21. Lee, C., Famoye, F. and Olumolade, O. (2007). Beta-Weibull distribution: some properties and applications to censored data. *Journal of Modern Applied Statistical Methods*, 6, 17.
22. Nofal, Z.M., Afify, A.Z., Yousof, H.M. and Cordeiro, G.M. (2017). The generalized transmuted-G family of distributions. *Communications in Statistics-Theory and Methods*, 46, 4119-4136.
23. Nofal, Z.M., Afify, A.Z., Yousof, H.M., Granzotto, D.C.T. and Louzada, F. (2016). Kumaraswamy transmuted exponentiated additive Weibull distribution. *International Journal of Statistics and Probability*, 5, 78-99.
24. Provost, S.B. Saboor, A. and Ahmad, M. (2011). The gamma--Weibull distribution, *Pak. J. Statist.*, 27, 111-131.

25. Ramos, M.W.A., Marinho, P.R.D., Cordeiro, G.M. da Silva, R.V. and Hamedani, G.G. (2015). The Kumaraswamy-G Poisson family of distributions, J. Stat. Theory Appl. 14, 222–239.
26. Ramos, P.L., Dey, D.K.A., Louzada, F. and Lachos, V.H. (2018). An extended poisson family of life distribution: A unified approach in competitive and complementary risks. arXiv preprint arXiv:1805.07672 (2018).
27. Rezaei, S., Nadarajah, S. and Tahghighnia, N.A. (2013). New three-parameter lifetime distribution. *Statistics*, 47, 835-860.
28. Rezaei, S., Sadr, B.B., Alizadeh, M. and Nadarajah, S. (2017). Topp-Leone generated family of distributions: Properties and applications. *Communications in Statistics-Theory and Methods*, 46(6), 2893-2909.
29. Ristic, M.M. and Balakrishnan, N. (2012). The gamma-exponentiated exponential distribution. *Journal of Statistical Computation and Simulation*, 82, 1191-1206.
30. Yousof, H.M., Afify, A.Z., Alizadeh, M., Butt, N.S., Hamedani, G.G. and Ali, M.M. (2015). The transmuted exponentiated generalized-G family of distributions. *Pak. J. Stat. Oper. Res.*, 11, 441-464.
31. Yousof, H.M., Afify, A.Z., Nadarajah, S., Hamedani, G. and Aryal, G.R. (2018). The Marshall-Olkin generalized-G family of distributions with Applications. *Statistica*, 78(3), 273-295.
32. Yousof, H.M., Afify, A.Z., Cordeiro, G.M. Alzaatreh, A. and Ahsanullah, M. (2017a). A new four-parameter Weibull model. *Journal of Statistical Theory and Applications*, forthcoming.
33. Yousof, H.M., Afify, A.Z., Hamedani, G.G. and Aryal, G. (2017b). The Burr X generator of distributions for lifetime data. *Journal of Statistical Theory and Applications*, 16(3), 288-305.
34. Yousof, H.M., Alizadeh, M., Jahanshahi, S.M.A., Ramires, T.G., Ghosh, I. and Hamedani G.G. (2017c). The transmuted Topp-Leone G family of distributions: theory, characterizations and applications. *Journal of Data Science*, 15, 723-740.
35. Yousof, H.M. Majumder, M., Jahanshahi, S.M.A., Ali, M.M. and Hamedani G.G. (2017d). A new Weibull class of distributions: theory, characterizations and applications. *Communications for Statistical Applications & Methods*, forthcoming.
36. Yousof, H.M., Rasekhi, M., Afify, A.Z., Alizadeh, M., Ghosh, I. and Hamedani G.G. (2017e). The beta Weibull-G family of distributions: theory, characterizations and applications. *Pakistan Journal of Statistics*, 33(2), 95-116.

APPENDIX A

MOEWD:

$$f_{MOEWD}(x) = \alpha\beta\gamma^{\beta}x^{\beta-1}\left[1-(1-\alpha)e^{-(\gamma x)^{\beta}}\right]^{-2} exp\left[-(\gamma x)^{\beta}\right];$$

GaWD:

$$f_{GaWD}(x) = \beta\alpha^{\gamma/\beta+1}\Gamma^{-1}(1+\gamma/\beta)\, exp\left[-\alpha x^{\beta}\right] x^{\beta+\gamma-1};$$

KwWD:

$$f_{KwWD}(x) = ab\beta\alpha^{\beta}x^{\beta-1}\left\{1-exp\left[-(\alpha x)^{\beta}\right]\right\}^{a-1} exp\left[-(\alpha x)^{\beta}\right] \times\left\{1-\left\{1-exp\left[-(\alpha x)^{\beta}\right]\right\}^{a}\right\}^{b-1};$$

WFrD:

$$f_{WFrD}(x) = ab\beta\alpha^{\beta}x^{-\beta-1}\, exp\left(-a\left\{exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]-1\right\}^{-b}\right) exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right] \times\left\{1-exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{-b-1};$$

BWD:

$$f_{BWD}(x) = \beta\alpha^{\beta}B^{-1}(a,b)x^{\beta-1}\, exp\left[-b(\alpha x)^{\beta}\right]\left\{1-exp\left[-(\alpha x)^{\beta}\right]\right\}^{a-1};$$

TMWD:

$$f_{TMWD}(x) = \left(\alpha+\gamma\beta x^{\beta-1}\right) exp\left[-\alpha x-\gamma x^{\beta}\right]\left[1-\lambda+2\lambda\, exp\left(-\alpha x-\gamma x^{\beta}\right)\right];$$

KTWD:

$$f_{KTWD}(x) = ab\beta\alpha^{\beta}x^{\beta-1}\, exp\left[-(\alpha x)^{\beta}\right]\left(1+\lambda-2\lambda\left\{1-exp\left[-(\alpha x)^{\beta}\right]\right\}\right) \times\left[\left\{1-exp\left[-(\alpha x)^{\beta}\right]\right\}\left(1+\lambda-\lambda\left\{1-exp\left[-(\alpha x)^{\beta}\right]\right\}\right)\right]^{a-1} \times\left[1-\left((1+\lambda)\left\{1-exp\left[-(\alpha x)^{\beta}\right]\right\}-\lambda\left\{1-exp\left[-(\alpha x)^{\beta}\right]\right\}^{2}\right)^{a}\right]^{b-1};$$

MBWD:

$$f_{MBWD}(x) = \beta\gamma^{a}\alpha^{-\beta}B^{-1}(a,b)x^{\beta-1}\, exp\left[-b\left(\frac{x}{\alpha}\right)^{\beta}\right]\left\{1-exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]\right\}^{a-1} \times\left(1-(1-\gamma)\left\{1-exp\left[-b\left(\frac{x}{\alpha}\right)^{\beta}\right]\right\}\right)^{-a-b};$$

McWD:

$$f_{McWD}(x) = \beta c\alpha^{\beta}B^{-1}(a/c,b)x^{\beta-1}\left(1-\left\{1-exp\left[-(\alpha x)^{\beta}\right]\right\}^{c}\right)^{b-1} \times exp\left[-(\alpha x)^{\beta}\right]\left\{1-exp\left[-(\alpha x)^{\beta}\right]\right\}^{a-1};$$

TEGWD:

$$f_{TEGWD}(x) = ab\beta\alpha^{\beta}x^{\beta-1}\, exp\left[-a(\alpha x)^{\beta}\right]\left\{1-exp\left[-a(\alpha x)^{\beta}\right]\right\}^{b-1} \times\left(1+\lambda-2\lambda\left\{1-exp\left[-a(\alpha x)^{\beta}\right]\right\}^{b}\right).$$

The parameters of the above densities are all positive real numbers except for the TMW and TExGW distributions for which $|\lambda| \leq 1$.

APPENDIX B

WWD:

$$f_{WWD}(x) = exp\left(-\alpha\{-\,log[1 - exp(-\lambda x^{\gamma})]\}^{\beta}\right)$$

GaEED:

$$f_{GaEED}(x) = \frac{\alpha\theta}{\Gamma(\lambda)} exp(-\theta x)\,[1 - exp(-\theta x)]^{\alpha-1}\{-\alpha\, log[1 - exp(-\theta x)]\}^{\lambda-1}$$

EEGcD:

$$f_{EEGcD}(x) = \frac{\alpha\theta(1-p)\,exp(-\theta x)}{[1 - exp(-\theta x)]^{\alpha-1}\{1 - p + p[1 - exp(-\theta x)]^{\alpha}\}^{2}}$$