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# A NEW WEIBULL-LOMAX (T-X) DISTRIBUTION & ITS APPLICATION

**Sharqa Hashmi**[1,2,§] and **Hina Gull**[1]
[1] Department of Statistics, Lahore College for Women University Lahore, Pakistan
[2] College of Statistical and Actuarial Sciences, University of the Punjab, Lahore, Pakistan
[§] Corresponding author Email: sharqa.hashmi@lcwu.edu.pk

## ABSTRACT

In this article, we have introduced a new distribution named as Weibull-Lomax (T-X) distribution (WLD). Various mathematical properties of Weibull-Lomax distribution are obtained such as moments, survival, hazard rate function, limiting behaviour of its probability density and hazard rate functions. Quantile functions, mode, median and Shannon entropy are also obtained. The relations between WLD and Weibull, Exponential and the type I extreme value distributions are discussed by transformation. The estimation of model parameters have been done using maximum likelihood estimation (MLE) method. The shape of the distribution is discussed. Sufficient conditions for failure rate functions are derived. A simulation study is presented for performance of estimators. Finally an application of real data set is used to illustrate the flexibility of the new model.

## KEYWORDS

T-X family, Moments, Shannon's Entropy**,** Maximum Likelihood Estimation.

## 1. INTRODUCTION

Probability distributions are usually applied to explain real world phenomenon. Probability distributions are widely studied and many of its generalizations are developed to enhance their usefulness and applicability. The Weibull distribution is a well-known continuous distribution used in survival and reliability analysis, introduced by Waloddi Weibull (1951). Weibull distribution with three parameters is widely used in extreme event modelling and maintainability analysis. It is also used in weather forecasting and communication systems engineering.

The flexibility of baseline distribution can be extended by different methods to obtain more sophisticated models. For reference, one can refer to Eugene et al. (2002), Jones (2004), Cordeiro and de Castro (2011), Alzaatreh et al. (2012) and Lee et al. (2013). Alzaatreh et al. (2013) presented a general method that allows for many existing univariate continuous distributions as the generator.

Let $F(x)$ and $r(t)$ are the cumulative distribution function of any random variable $X$ and probability density function of a non-negative continuous random variable $T$

 

defined on $[0,\infty)$ respectively. The cumulative distribution function of the generalized distribution is defined by Alzaatreh et al. (2013) as

$$G(x) = \int_0^{-\log[1-F(x)]} r(t)dt \tag{1}$$

The family of distributions defined by (1) is called "Transformed-Transformer" family or "T-X distribution family" in Alzaatreh et al. (2013). The corresponding probability density function (pdf) to the cdf (1) is

$$g(x) = \frac{f(x)}{1-F(x)} r\{-log\big(1-F(x)\big)\} \tag{2}$$

If a random variable $T$ follows the Weibull distribution with parameter c and γ,

$$r(t) = \left(\frac{c}{\gamma}\right)\left(\frac{t}{\gamma}\right)^{(c-1)} e^{-\left(\frac{t}{\gamma}\right)^{c}}, t \geq 0 \tag{3}$$

where $c > 0$, $\gamma > 0$ and $c$ is the shape parameter and $\gamma$ is the scale parameter. The Weibull-X family has pdf using definition in Eq. (2)

$$g(x) = \left(\frac{c}{\gamma}\right)\frac{f(x)}{1-F(x)}\left\{\frac{-\log(1-F(x))}{\gamma}\right\}^{(c-1)} e^{\left\{-\left(\frac{-\log(1-F(x))}{\gamma}\right)^{c}\right\}} \tag{4}$$

In this paper, we introduce a member of Weibull-X family where $X$ is the Lomax distribution random variable. The Lomax distribution is also called the Pareto Type II distribution with heavy-tailed pdf. It is usually used in business, economics, actuarial science and reliability analysis. Lomax distribution may be used as an alternate distribution of exponential distribution when the data are heavy tailed. Another technique of Weibull-G family is introduced by Bourguignon (2014). Using this method of generation of new distribution, Tahir et al. (2015) proposed Weibull Lomax distribution.

The rest of the paper is organized as: section 2 states the density and distribution function with graphs of Weibull Lomax distribution. In section 3, we study the properties of WL distribution including limiting behaviour and in section 4 we have derived entropy. The maximum likelihood estimates for the model parameters are given in section 5. In section 6, a simulation study is performed. Finally, in section 7 we provide real data application of this new distribution.

## 2. THE WEIBULL-LOMAX DISTRIBUTION

Let $X$ be a non-negative random variable with probability density function $f(.)$ and distribution function $F(.)$ of Lomax distribution and then the pdf of Weibull-Lomax distribution (WLD) $g(x)$ are given below

$$f(x) = \frac{\lambda\kappa}{(1+\lambda x)^{\kappa+1}};\ x > 0, \lambda > 0, \kappa > 0 \tag{5}$$

$$F(x) = 1-(1+\lambda x)^{-\kappa};\ x > 0 \tag{6}$$

Substituting the (5) and (6) in equation (4), we have

$$g(x) = \left(\frac{c}{\gamma}\right)\frac{\lambda\kappa}{(1+\lambda x)}\left\{\frac{\kappa}{\gamma}log(1+\lambda x)\right\}^{c-1} exp\left\{-\left(\frac{\kappa}{\gamma}log(1+\lambda x)\right)^{c}\right\} \tag{7}$$

If replacing $\frac{\kappa}{\gamma} = \beta$, then the above equation can be written as

$$g(x) = \frac{\beta c\lambda}{(1+\lambda x)}[\beta \log(1+\lambda x)]^{c-1} exp\left[-\left(\beta log(1+\lambda x)\right)^{c}\right] \tag{8}$$

where c > 0, β > 0, λ > 0 and c and β are the shape parameters and λ is the scale parameter. The equation (8) is said to be Weibull-Lomax distribution and it will denoted by $WLD(c, \beta, \lambda)$.

The corresponding cdf to the pdf in equation (8) is

$$G(x) = 1 - exp\left[-\left(\beta\ log(1+\lambda x)\right)^{c}\right] \tag{9}$$

## 3. PROPERTIES OF THE WEIBULL-LOMAX DISTRIBUTION (WLD)

**i) Lemma 3.1** (Transformation)**.**

a) If a random variable $Y$ follows the Weibull Distribution with parameter c and $(1/\beta)$, then the random variable $X = \frac{1}{\lambda}(e^{y} - 1)$ follows the $WLD(c, \beta, \lambda)$.

b) If a random variable $Y$ follows the exponential Distribution, then the random variable $X = \frac{1}{\lambda}(e^{\frac{y^{\frac{1}{c}}}{\beta}} - 1)$ follows the $WLD(c, \beta, \lambda)$.

c) If a random variable $Y$ follows the Type-I extreme Value with scale parameter 1/c, then the random variable

$X = \frac{1}{\lambda}\left(e^{\frac{e^{-y}}{\beta}} - 1\right)$ follows the $WLD(c,\beta,\lambda)$.

**ii) Survival and Hazard Functions:**

The survival function are most often used in the reliability and other related fields. The survival function is the probability that the time of death is later than some specified time t. It deals with failure in an electronic and engineering systems and death in biological organisms. The survival function is

$$S(x) = P[X > x] = 1 - F(x)$$

$$S(x) = exp\left[-\left(\beta\ log(1+\lambda x)\right)^{c}\right];\ x > 0, \lambda > 0, \beta > 0, c > 0 \tag{10}$$

The hazard rate is a measure of the tendency to fail, if hazard rate function is large, there will be higher probability of failure. The hazard function is

$$h(x) = \frac{f(x)}{S(x)}$$

$$h(x) = \frac{\beta c\lambda}{(1+\lambda x)}[\beta \log(1+\lambda x)]^{c-1};\ x > 0, \lambda > 0, \beta > 0, c > 0 \tag{11}$$

**iii) Limiting Behavior:**

The limiting behaviours of pdf and hazard function of the Weibull-Lomax distribution are discussed.

**3.1 Theorem:**

Let $X$ be a random variable follows WLD(x; c, β, λ). The behaviour of density curve and hazard curve at origin is given by:

$$\text{i)}\ \lim_{x\to 0} g(x) = \lim_{x\to 0} h_g(x) = \begin{cases} 0 & , c>1 \\ \beta\lambda & , c=1 \\ \infty & , c<1 \end{cases} \tag{12}$$

$$\text{ii)}\ \lim_{x\to\infty} g(x) = 0$$

and

$$\text{iii)}\ \lim_{x\to\infty} h_g(x) = \begin{cases} 0 & , c \le 1\ or\ 1<c<n \\ \infty & , c>n \end{cases}$$

**Proof 3.1:**

i) Since $g(x) = h_g(x)\{1-G(x)\}$ and cumulative distribution function of WLD at origin is $\lim_{x\to 0} G(x) = 0$, we have $\lim_{x\to 0} g(x) = \lim_{x\to 0} h_g(x)$.

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{\beta c\lambda}{(1+\lambda x)}\left[\beta\ \log(1+\lambda x)\right]^{c-1} \exp\left[-\left(\beta\ \log(1+\lambda x)\right)^{c}\right]$$

completes the proof.

ii) The behaviour of density curve of WLD as $x$ approaches to infinity, can be obtained by

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{\beta c\lambda}{(1+\lambda x)}\left[\beta\ \log(1+\lambda x)\right]^{c-1} \exp\left[-\left(\beta\ \log(1+\lambda x)\right)^{c}\right] = 0$$

iii) The behaviour of hazard curve of WLD as $x$ approaches to infinity, can be obtained by

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{\beta c\lambda}{(1+\lambda x)}\left[\beta\ \log(1+\lambda x)\right]^{c-1}$$

If $c \le 1$, we have $\lim_{x\to\infty} h_g(x) = 0$. If c > 1, there is an integer n such that 1 < c < n. By using L' Hospital Rule

$$\lim_{x\to\infty} h_g(x) = \lim_{x\to\infty} \frac{d^{n-1}}{dx^{n-1}} \beta c \frac{\lambda}{(1+\lambda x)} \{\beta \log(1+\lambda x)\}^{c-1}$$

$$= \beta^2 \lambda \lim_{x\to\infty} \frac{c(c-1)(c-2).....(c-n+1)\{\beta \log(1+\lambda x)\}^{c-n}}{(1+\lambda x)}$$

Since c < n, $\lim_{x\to\infty} h_g(x) = 0.$

Plots of WLD pdf and hazard rate functions for some parameter values are displayed in Figures 1 and 2.

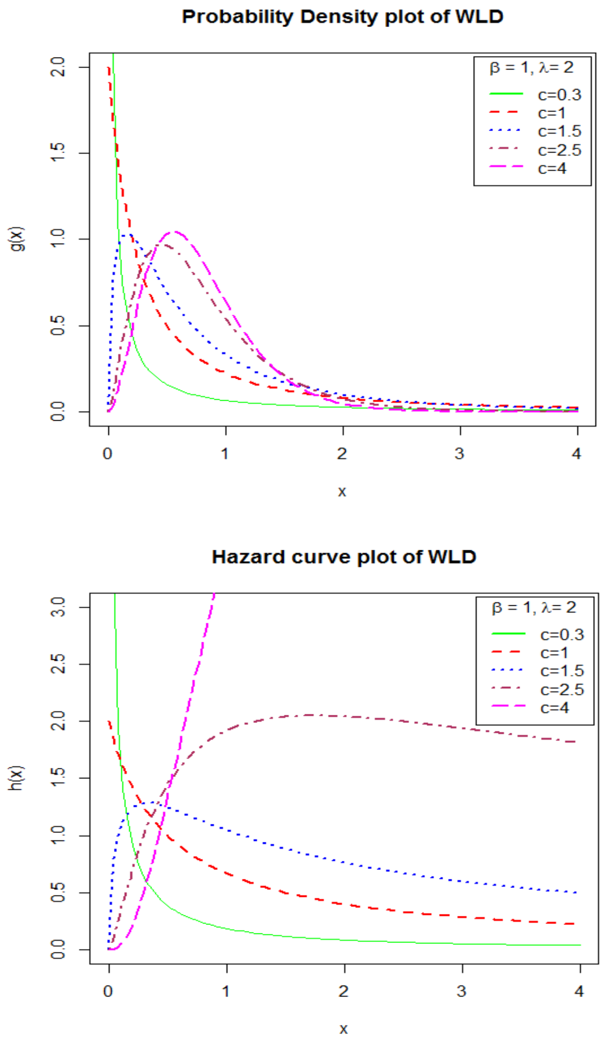


**Figure 1: Plots of the Weibull-Lomax pdf and Hazard Rate Function for Different Values of shape Parameter c when Scale Parameter is β=1 and Location Parameter is λ=2.**

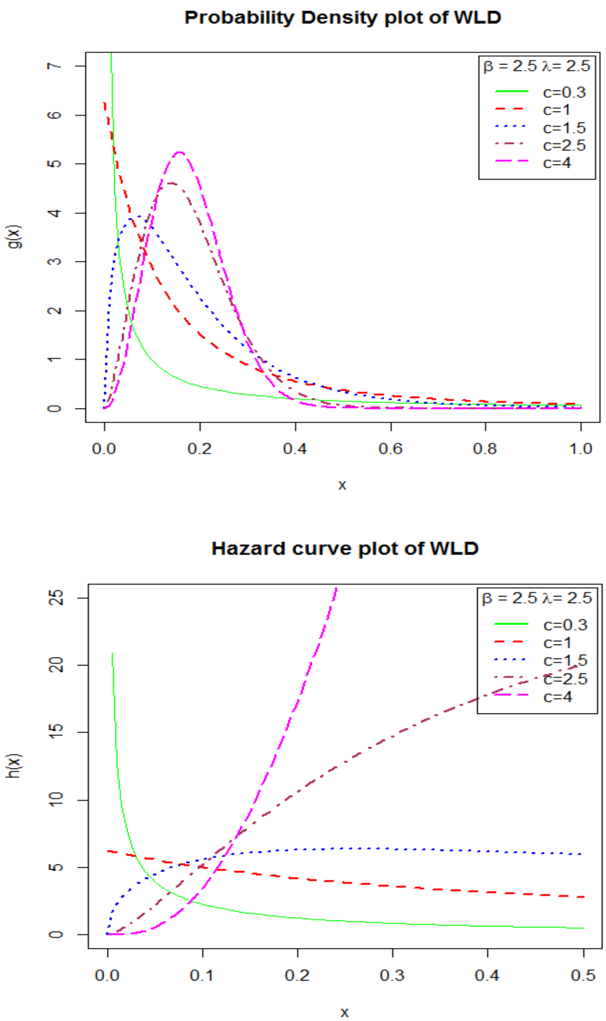


**Figure 2: Plots of the Weibull-Lomax pdf and Hazard Rate Function for Different Values of Shape Parameter c when Scale Parameter is β=2.5 and Location Parameter is λ=2.5**

**iv) Moments**: Moments are important and essential in any statistical analysis, especially in study of the most important characteristics of the distribution including tendency, dispersion, skewness and kurtosis. It can also be helpful in applications. In this subsection, the r[th] moment of WLD about zero will be derived. The r[th] moment of $X$ can be obtained by

$$E(X^r) = \int_0^\infty x^r g(x)\, dx$$

Using the substitution $u = \left(\beta log(1+\lambda x)\right)^c$ in Eq. 8, we have the $r^{th}$ moment expression is

$$E(X^r) = \frac{1}{\lambda^r}\int_0^\infty \left(e^{\frac{u^{1/c}}{\beta}} - 1\right)^r e^{-u} du \tag{13}$$

By using series expansion

$$e^{\frac{u^{1/c}}{\beta}} - 1 = \sum_{i=1}^{\infty} \frac{\left(\frac{u^{1/c}}{\beta}\right)^i}{i\,!}$$

$$\left(e^{\frac{u^{1/c}}{\beta}} - 1\right)^r = \sum_{i=1}^{\infty} \left\{\frac{\left(\frac{u^{1/c}}{\beta}\right)^i}{i\,!}\right\}^r$$

Equation (13) reduces to

$$E(X^r) = \frac{1}{\lambda^r}\int_0^\infty \sum_{i=0}^{\infty} \left\{\frac{u^{i/c}}{\beta^i\, i\,!}\right\}^r e^{-u} du$$

$$= \frac{1}{\lambda^r}\sum_{i=0}^{\infty} \left\{\frac{1}{\beta^i\, i\,!}\right\}^r \int_0^\infty \{u^{i/c}\}^r e^{-u} du$$

$$E(X^r) = \frac{1}{\lambda^r}\sum_{i=1}^{\infty} \frac{\Gamma\left(\frac{ri}{c}+1\right)}{(i\,!\,\beta^i)^r}$$

Put $r = 1,2,3,4$, we get the Ist four moments for WLD.

$$E(X) = \frac{1}{\lambda}\sum_{i=1}^{\infty} \frac{\Gamma\left(\frac{i}{c}+1\right)}{i\,!\,\beta^i} \tag{14}$$

$$E(X^2) = \frac{1}{\lambda^2}\sum_{i=1}^{\infty} \frac{\Gamma\left(\frac{2i}{c}+1\right)}{(i\,!\,\beta^i)^2} \tag{15}$$

$$E(X^3) = \frac{1}{\lambda^3}\sum_{i=1}^{\infty} \frac{\Gamma\left(\frac{3i}{c}+1\right)}{(i\,!\,\beta^i)^3} \tag{16}$$

$$E(X^4) = \frac{1}{\lambda^4}\sum_{i=1}^{\infty} \frac{\Gamma\left(\frac{4i}{c}+1\right)}{(i\,!\,\beta^i)^4} \tag{17}$$

Based on (14) to (17), the measure of skewness and measure of kurtosis can be obtained.

**v) Quantile Function and Median:**

The quantile function for Weibull-Lomax distribution is obtained by inverting Eq. (9). Let $Q(\delta)$, $0<\delta<1$ denote the quantile function for the WLD. Then, $Q(\delta)$ is given by

$$Q(\delta)=\frac{1}{\lambda}\left[exp\left\{\frac{(-\log(1-\delta))^{\frac{1}{c}}}{\beta}\right\}-1\right] \tag{18}$$

Simulating the WL random variable is easy i.e., $X=Q(\delta)$ follows Eq. (8). If we put $\lambda=0.25, 0.50, 0.75$ in equation (18), we get the Quartiles of the WLD. Now put $\delta=0.50$, the median is obtained which is

$$Median=\frac{1}{\lambda}\left[exp\left\{\frac{(-\log(0.5))^{\frac{1}{c}}}{\beta}\right\}-1\right] \tag{19}$$

Table 1 gives the mean, median, variance, skewness and kurtosis of the WLD for some specific values of parameters taking scale parameter $\lambda=1$.

**Table 1**
**Mean, Median, Variance, Skewness and Kurtosis when $\lambda=1$ of the WLD**

| c | β | Theoretical Results | | | | |
|---|---|---|---|---|---|---|
| | | Mean | Median | Variance | Skewness ($\Upsilon_1$) | Kurtosis ($\beta_2$) |
| 3 | 1.2 | 1.184128 | 1.090697 | 0.374684 | 0.927609 | 4.260985 |
| | 1.3 | 1.051277 | 0.975392 | 0.279521 | 0.86258 | 4.050283 |
| | 1.4 | 0.944547 | 0.881634 | 0.215274 | 0.807958 | 3.885395 |
| | 1.5 | 0.857044 | 0.803985 | 0.170165 | 0.761405 | 3.753463 |
| | 1.6 | 0.784074 | 0.738675 | 0.137446 | 0.721238 | 3.645919 |
| 4 | 1.2 | 1.176306 | 1.139068 | 0.212122 | 0.474191 | 3.155391 |
| | 1.3 | 1.046723 | 1.017543 | 0.15971 | 0.429851 | 3.085732 |
| | 1.4 | 0.942212 | 0.918888 | 0.123924 | 0.392071 | 3.031557 |
| | 1.5 | 0.85625 | 0.837298 | 0.098563 | 0.359484 | 2.988634 |
| | 1.6 | 0.784371 | 0.768758 | 0.080024 | 0.33108 | 2.954082 |
| 5 | 1.2 | 1.182053 | 1.169362 | 0.141571 | 0.213704 | 2.861138 |
| | 1.3 | 1.052764 | 1.043903 | 0.106934 | 0.178049 | 2.837696 |
| | 1.4 | 0.948329 | 0.942158 | 0.083188 | 0.147485 | 2.820896 |
| | 1.5 | 0.86232 | 0.858085 | 0.066302 | 0.120988 | 2.808786 |
| | 1.6 | 0.790324 | 0.787512 | 0.053925 | 0.097791 | 2.800054 |
| 6 | 1.2 | 1.190462 | 1.190111 | 0.10283 | 0.036956 | 2.797804 |
| | 1.3 | 1.060662 | 1.061942 | 0.077765 | 0.006262 | 2.796569 |
| | 1.4 | 0.955742 | 0.95807 | 0.060552 | -0.02013 | 2.797964 |
| | 1.5 | 0.869282 | 0.872289 | 0.048296 | -0.04307 | 2.801022 |
| | 1.6 | 0.796874 | 0.80032 | 0.039303 | -0.0632 | 2.805118 |
| 7 | 1.2 | 1.198858 | 1.205212 | 0.078715 | -0.09366 | 2.821662 |
| | 1.3 | 1.06833 | 1.075062 | 0.05955 | -0.12102 | 2.833173 |
| | 1.4 | 0.962784 | 0.969636 | 0.04638 | -0.14459 | 2.845069 |
| | 1.5 | 0.875784 | 0.882609 | 0.036999 | -0.16511 | 2.856921 |
| | 1.6 | 0.802908 | 0.809621 | 0.030113 | -0.18314 | 2.868481 |

It is observed from Table 1 that

Skewness decreases as β or 'c' increases and it is negative for higher values of 'c'.

Kurtosis decreases as β or 'c' increases

For c=4, and β=1.5, WLD approximate normal distribution.

## 4. SHANNON ENTROPY

The Shannon entropy is introduced by Shannon (1948). Entropy is the measure of unpredictability (measure of uncertainty) of information content. The Shannon's Entropy for a random variable $X$ that follows the WLD is

$$E[-\ln\{g(X)\}] = -\log(\beta c\lambda) + \frac{\Gamma(1+c^{-1})}{\beta} + (1-c^{-1})\gamma + 1 \qquad (20)$$

where $\gamma = -\int_0^\infty e^{-u}\log u\,du \approx 0.57722$ is the Euler Gamma constant.

The Shannon entropy of the WLD is obtained by

$$E\left[-\log\left\{g\left(x\right)\right\}\right] = -\int g(x)\log\left\{g(x)\right\}dx.$$

It is defined as

$$E[-\ln(g(X))] = \int -\left(\frac{\beta c\lambda}{(1+\lambda x)}[\beta\log(1+\lambda x)]^{c-1}exp\left[-\left(\beta log(1+\lambda x)\right)^{c}\right]\right) * log\left(\frac{\beta c\lambda}{(1+\lambda x)}[\beta\log(1+\lambda x)]^{c-1}exp\left[-\left(\beta log(1+\lambda x)\right)^{c}\right]\right)dx$$

$$EE[-\ln(g(X))] = -\int\left(\frac{\beta c\lambda}{(1+\lambda x)}[u]^{\frac{c-1}{c}}\exp[-u]\right) * log\left(\frac{\beta c\lambda}{(1+\lambda x)}u^{\frac{c-1}{c}}exp[-u]\right)dx$$

substitution $u = \left\{\beta\log\left(1+\lambda x\right)\right\}^{c}$ in g(x) completes the proof.

## 5. ESTIMATION

We obtain the maximum likelihood estimates (MLEs) of the model parameters of WLD from complete samples. Let a random sample of size n be taken from the WLD with Θ = (β, c, λ). The log likelihood function for the WLD is given by

$$\ln L = n\ln\beta + n\ln c + n\ln\lambda + \left(c-1\right)\sum_{i=1}^{n}\ln\left(\beta\log\left(1+\lambda x_i\right)\right) - \sum_{i=1}^{n}\ln\left(1+\lambda x_i\right) - \sum_{i=1}^{n}\left(\beta\log\left(1+\lambda x_i\right)\right)^{c} \qquad (21)$$

The log-likelihood function can be maximized by using R package (maxBFGS or Adequacy Model) or by solving nonlinear equations (22) to (24) after differentiating Eq. (21).

The equations for c, β and λ are

$$\frac{\partial \ln L}{\partial c}=\frac{n}{c}+\sum_{i=1}^{n}\ln\left(\beta\log\left(1+\lambda x_i\right)\right)-\sum_{i=1}^{n}\left(\beta\log\left(1+\lambda x_i\right)\right)^{c}\ln\left(\beta\log\left(1+\lambda x_i\right)\right) \tag{22}$$

$$\frac{\partial \ln L}{\partial \beta}=\frac{n}{\beta}+\left(c-1\right)\sum_{i=1}^{n}\frac{\log\left(1+\lambda x_i\right)}{\left(\beta\log\left(1+\lambda x_i\right)\right)}-\sum_{i=1}^{n}c\left(\beta\log\left(1+\lambda x_i\right)\right)^{c-1}\log\left(1+\lambda x_i\right) \tag{23}$$

$$\frac{\partial \ln L}{\partial \lambda}=\frac{n}{\lambda}+\left(c-1\right)\sum_{i=1}^{n}\frac{\beta x_i}{\left(1+\lambda x_i\right)\left(\beta\log\left(1+\lambda x_i\right)\right)}-\sum_{i=1}^{n}\frac{x_i}{\left(1+\lambda x_i\right)}-\beta c\sum_{i=1}^{n}\frac{x_i\left(\beta\log\left(1+\lambda x_i\right)\right)^{c-1}}{\left(1+\lambda x_i\right)} \tag{24}$$

The MLE of c, β and λ can be obtained by solving the equations (22), (23) and (24) using $\frac{\partial \ln L}{\partial c}=0, \frac{\partial \ln L}{\partial \beta}=0$ and $\frac{\partial \ln L}{\partial \lambda}=0$ .

## 6. SIMULATION STUDY

A simulation study was carried out N=10,000 times for selected values of n, c, β and λ. Some statistics including mean, median, variance, skewness and kurtosis for different combination of parameters are presented in Table 2. While in Table 3, maximum likelihood estimates of model parameters are computed for different combinations of parametric values taking sample sizes 25, 50, 100, 250 and 500. We generate 10,000 samples by using Monte Carlos simulation. The R software is used to obtain MLEs and other relevant quantities. We compute ML estimates for c, β, and λ based on generated samples.

**Table 2**
**Simulated Descriptive Measures of WLD taking $\lambda = 1$**

| c | β | Simulated Results | | | | |
|---|---|---|---|---|---|---|
| | | Mean | Median | Variance | Skewness | Kurtosis |
| 3 | 1.2 | 1.135932 | 0.958293 | 0.404362 | 1.028999 | 3.828209 |
| | 1.3 | 1.063825 | 0.902278 | 0.257482 | 0.9431478 | 3.46774 |
| | 1.4 | 0.992623 | 0.917507 | 0.160469 | 0.575375 | 2.710963 |
| | 1.5 | 0.934905 | 0.812581 | 0.254237 | 0.566206 | 2.473097 |
| | 1.6 | 0.749679 | 0.601042 | 0.172497 | 0.516189 | 2.602047 |
| 4 | 1.2 | 1.273714 | 1.241534 | 0.250672 | 0.476694 | 3.20647 |
| | 1.3 | 0.987833 | 0.928478 | 0.174413 | 0.506737 | 2.923047 |
| | 1.4 | 0.955646 | 0.965791 | 0.121519 | 0.499924 | 2.788981 |
| | 1.5 | 0.898078 | 0.851078 | 0.126797 | 0.376195 | 2.632746 |
| | 1.6 | 0.730243 | 0.740192 | 0.070753 | 0.310573 | 2.075073 |
| 5 | 1.2 | 1.303075 | 1.212605 | 0.122681 | 0.2016703 | 2.865593 |
| | 1.3 | 1.005451 | 1.093413 | 0.083157 | 0.176355 | 2.547381 |
| | 1.4 | 0.896015 | 0.915055 | 0.068171 | 0.128302 | 2.497167 |
| | 1.5 | 0.830258 | 0.815234 | 0.069047 | 0.117488 | 2.372991 |
| | 1.6 | 0.811848 | 0.819136 | 0.071485 | 0.104332 | 2.348875 |
| 6 | 1.2 | 1.196773 | 1.167525 | 0.121158 | 0.20189 | 2.785721 |
| | 1.3 | 1.150165 | 1.125972 | 0.059905 | 0.08821 | 2.81976 |
| | 1.4 | 0.994339 | 0.936635 | 0.053068 | -0.804599 | 2.553614 |
| | 1.5 | 0.854185 | 0.828309 | 0.049535 | -0.339582 | 2.475932 |
| | 1.6 | 0.799352 | 0.773262 | 0.035921 | -0.705755 | 2.128153 |
| 7 | 1.2 | 1.290445 | 1.292305 | 0.056307 | -0.122648 | 2.764136 |
| | 1.3 | 0.985279 | 1.004339 | 0.057222 | -0.14326 | 2.7432838 |
| | 1.4 | 0.999669 | 1.02807 | 0.054892 | -0.15357 | 2.7366795 |
| | 1.5 | 0.818849 | 0.849047 | 0.046921 | -0.17058 | 2.7164414 |
| | 1.6 | 0.786561 | 0.792407 | 0.025618 | -0.19543 | 2.7075633 |

**Interpretation:**

When λ=1 this table gives the mean, median, variance, skewness and kurtosis of the WLD for different values of c and β. The variance is decreasing for higher values of both parameter c and β. For higher values of c, the distribution is negatively skewed. The distribution is leptokurtic for smaller values of c and β.

To evaluate the performance of the MLEs of the WLD parameters, a simulation study is conducted for different combinations of parameters. The following measures were computed:

- Average Bias $= \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta} - \theta)$
- $Average\ MSE = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta} - \theta)^2$

**Table 3**
**MLEs of Parameters of WLD along with Average Bias and Average MSE**

| c | β | λ | n | $\hat{c}$ | | | $\hat{\beta}$ | | | $\hat{\lambda}$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | MLE | Avg Bias | Avg MSE | MLE | Avg Bias | Avg MSE | MLE | Avg Bias | Avg MSE |
| 0.5 | 1 | 0.5 | 25 | 0.4976 | -0.00243 | 0.00429 | 1.0072 | 0.007215 | 0.000201 | 0.4429 | -0.05711 | 0.010225 |
| | | | 50 | 0.5015 | 0.001523 | 0.000477 | 1.0028 | 0.002797 | 3.56E-05 | 0.4794 | -0.0206 | 0.001813 |
| | | | 100 | 0.5006 | 0.000596 | 0.000184 | 1.0011 | 0.001088 | 1.57E-05 | 0.4921 | -0.00792 | 0.000737 |
| | | | 250 | 0.5002 | 0.000172 | 1.06E-05 | 1.0001 | 0.000135 | 1.20E-06 | 0.499 | -0.00101 | 6.54E-05 |
| | | | 500 | 0.5 | 6.03E-06 | 9.82E-09 | 1 | 3.32E-06 | 2.91E-09 | 0.5 | -2.53E-05 | 1.67E-07 |
| 0.5 | 1 | 1 | 25 | 0.5037 | 0.003668 | 0.006681 | 1.0321 | 0.032092 | 0.001796 | 0.8879 | -0.11206 | 0.020513 |
| | | | 50 | 0.4992 | -0.00081 | 0.00207 | 1.0255 | 0.025497 | 0.001518 | 0.9146 | -0.08544 | 0.014958 |
| | | | 100 | 0.4985 | -0.00146 | 0.001293 | 1.0162 | 0.016228 | 0.000458 | 0.948 | -0.05202 | 0.004593 |
| | | | 250 | 0.4993 | -0.00073 | 0.000254 | 1.0079 | 0.007864 | 0.000164 | 0.9745 | -0.02549 | 0.001713 |
| | | | 500 | 0.5004 | 0.000426 | 0.000189 | 1.0063 | 0.006343 | 0.000153 | 0.9795 | -0.02053 | 0.001577 |
| 0.5 | 3 | 1 | 25 | 0.5026 | 0.002557 | 0.006085 | 3.0046 | 0.004593 | 5.46E-05 | 0.9357 | -0.06432 | 0.009502 |
| | | | 50 | 0.502 | 0.001985 | 0.001053 | 3.002 | 0.002033 | 1.31E-05 | 0.9719 | -0.02811 | 0.002521 |
| | | | 100 | 0.4997 | -0.0003 | 0.000544 | 3.0011 | 0.001075 | 8.05E-06 | 0.9863 | -0.01368 | 0.001209 |
| | | | 250 | 0.5001 | 0.000141 | 2.50E-05 | 3.0002 | 0.000154 | 5.56E-07 | 0.998 | -0.00204 | 9.60E-05 |
| | | | 500 | 0.4999 | -5.01E-05 | 5.05E-06 | 3.0000 | 3.12E-05 | 1.58E-07 | 0.9996 | -0.00036 | 1.58E-05 |
| 0.5 | 3 | 3 | 25 | 0.506 | 0.005968 | 0.007377 | 3.0148 | 0.014798 | 0.000644 | 2.9109 | -0.08905 | 0.039212 |
| | | | 50 | 0.5048 | 0.004772 | 0.002703 | 3.0165 | 0.016464 | 0.000886 | 2.9371 | -0.06287 | 0.011211 |
| | | | 100 | 0.4995 | -0.00053 | 0.001512 | 3.007 | 0.006994 | 7.27E-05 | 2.9718 | -0.02819 | 0.001155 |
| | | | 250 | 0.5007 | 0.000739 | 0.000331 | 3.0046 | 0.004646 | 4.38E-05 | 2.9809 | -0.01906 | 0.000725 |
| | | | 500 | 0.4997 | -0.00028 | 0.000242 | 3.0037 | 0.003672 | 3.18E-05 | 2.9851 | -0.01495 | 0.000522 |

**Remarks:** The above table values indicate that

- Maximum Likelihood Estimates perform well.
- The biases and standard errors of the estimates decrease as the sample size increases.

## 7. APPLICATION

In this section, we provide an application to real data to illustrate the usefulness of WLD. The MLEs of the model parameters are obtained for WLD and some other competent distributions and some accuracy measures are also obtained for checking the adequacy of all fitted distributions.

The data set consists of 84 observations (Tahir et al. 2015) and represents the failure times of Aircraft Windshield. The unit for measurement is 1000h and its descriptive statistics are given in Table 4.

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478,0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699,1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035,1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240,1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305,1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485,1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663

**Table 4**
**Descriptive Statistics for the Failure Times of 84 Aircraft Windshield Data**

| Min. | $Q_1$ | Median | Mean | $Q_3$ | Max. |
|---|---|---|---|---|---|
| 0.04 | 1.866 | 2.385 | 2.563 | 3.376 | 4.663 |

We constructed the Total Time on Test (TTT) plot in Figure 3. It helps to explore the shape of hazard rate function for modelling.

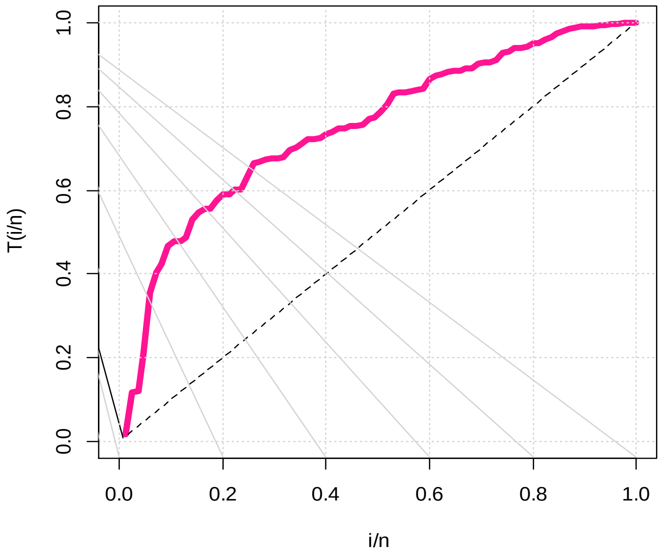


**Figure 3: TTT Plot for the Failure Times of Aircraft Windshield**

The empirical TTT plot is concave and it shows that the shape of hazard rate function is increasing for this data set.

Now, we estimate the unknown parameters by L-BFGS-B method in Table-5. We found the information criterions such as Akaike information criteria (AIC), Consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC) and we also found the goodness of fit tests such as Anderson Darling test (A*) and Cramer-von Mises (W*) to compare the fitted models in Table-6. The least values of these measures, show the better model fit to the data.

The AIC, CAIC, BIC and HQIC statistics are defined as:

a) Log Likelihood $(L)$
b) $AIC = 2k - 2lnL$
c) $BIC = kln(n) - 2lnL$
d) $CAIC = \frac{2kn}{n-k-1} - 2lnL$
e) $HQIC = 2kln\big(ln(n)\big) - 2lnL$

where $L$ is the log-likelihood function, n is the sample size and k is number of parameters.

Tahir et al. (2015) compared WL (W-G) distribution with McL, KwL, GL, BL, EL and Lomax distributions and showed WL (W-G) distribution a better distribution than other fitted distributions. Now we compare Weibull Lomax (T-X) distribution with WL (W-G) distribution. Estimates and accuracy measures of WL (W-G) distribution and other fitted distributions are taken from Tahir et al. (2015).

**Table 5**
**MLE's for Failure Times of Aircraft Windshield Data**

| Distributions | Estimates |
|---|---|
| **WL(T-X)** | **λ=0.02159875, β=38.46932616, c= 2.40649228** |
| WL(W-G) | a=0.0128, b=0.5969, α=6.7753, β=1.5324 |

**Table 6**
**The Measures L, AIC, BIC, CAIC, BIC, HQIC, A* and W***
**for Failure Times of Aircraft Windshield Data**

| Model | *L* | AIC | AICC | BIC | HQIC | A* | W* |
|---|---|---|---|---|---|---|---|
| **WL (T-X)** | **-60.72** | **127.431** | **127.7282** | **134.7599** | **130.3794** | **0.61428** | **0.06076** |
| WL (W-G) | -127.8 | 263.730 | 264.2303 | 273.5009 | 267.6603 | 0.6185 | 0.0932 |

In Tables 5 and 6, we provide model estimates and accuracy measures of both WLD (T-X) and WL (W-G) distributions respectively. It is noted that WLD has less values of AIC, CAIC, BIC, HQIC, A* and W* than the other distribution which shows that the WLD (T-X) is an appropriate model to fit the data set. Now we construct the empirical cdf and estimated cdfs plot.

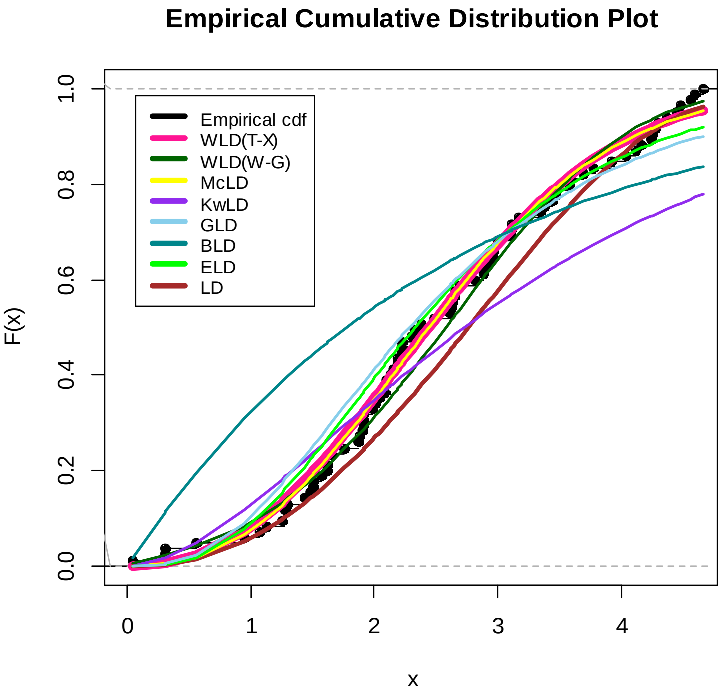


**Figure 4: Empirical cdf and Fitted cdfs plot for Failure Times of Aircraft Windshield Data**

In Figure 4, we can see that WLD (T-X) is the best fit than other distributions. Now, we construct the probability-probability (PP) plot for the fitted distributions. These plots tell "how well a specific distribution fits to the observed data". If the specified distribution is appropriate model, the plot will be approximately linear.

**Probability-Probability Plots on Failure times of Aircraft Windshield**

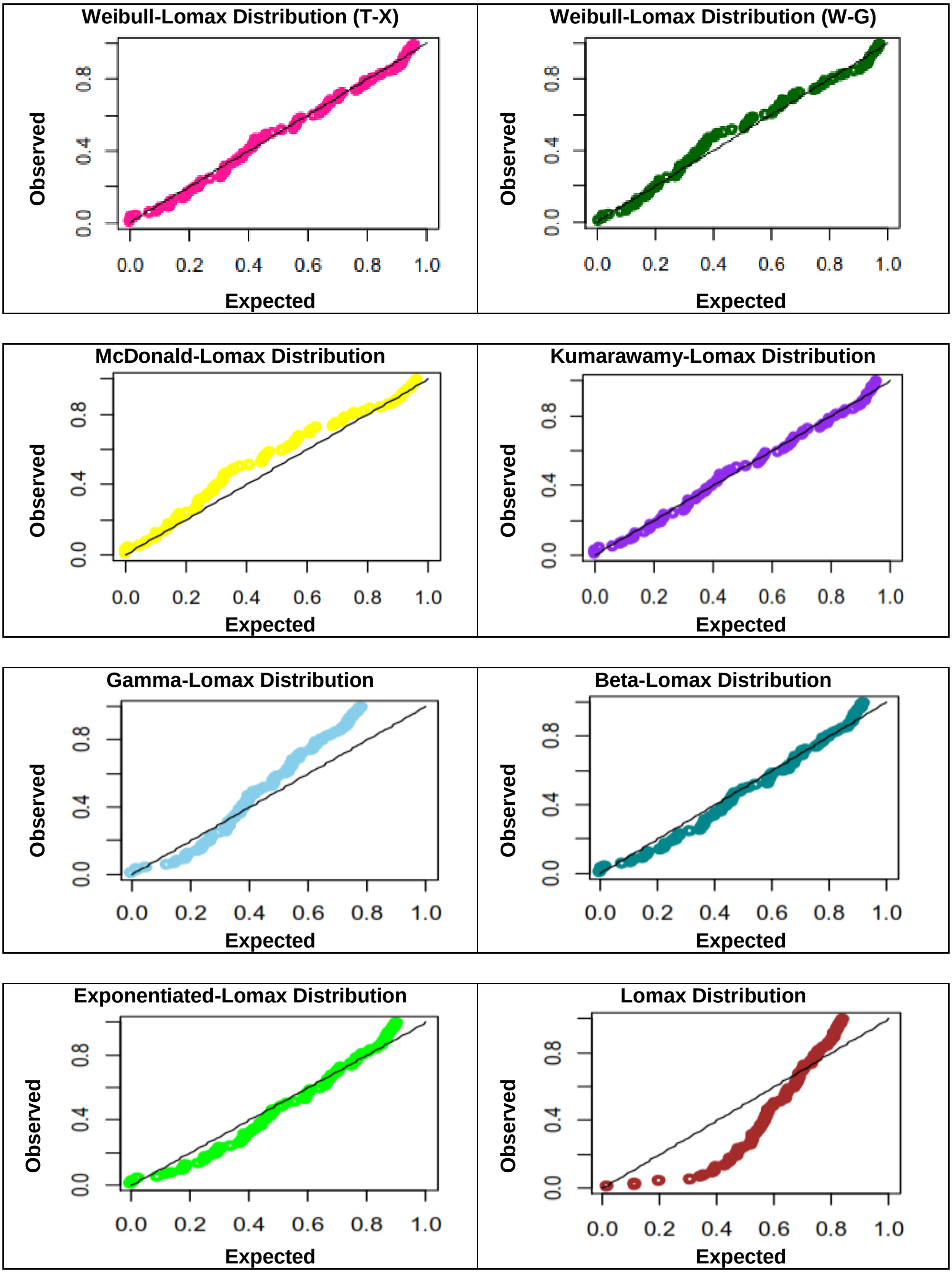


**Figure 5: PP-Plot Failure Times on Aircraft Windshield Data**

It is observed from the Figure 5 of PP plots that WLD (T-X) is the most appropriate fit for this real data set as its PP plot shows a linear trend.

## 8. CONCLUDING REMARKS

In many applied areas, extended generalized forms of distributions are needed which are more flexible. Recent developments focus on new methods by adding parameters to existing distributions. In this paper, we proposed a new distribution which is more flexible to capture skewness and kurtosis behavior, named as Weibull-Lomax distribution by using the T-X generator. We derive the mathematical properties of the distribution such as moments, quantile function, moment generating function and median. The expression of Shannon's entropy is also derived. We estimate the unknown parameters through the maximum likelihood estimator. To illustrate the flexibility of WLD, we fit it to the real data set along with other competent models and it provides better fit than other distributions.

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