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A NEW THREE-PARAMETER FRÉCHET DISTRIBUTION: PROPERTIES AND APPLICATIONS

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ABSTRACT

In this article, we propose a new three-parameter Fréchet distribution named the Odd Lindley Fréchet distribution. The new model can be expressed as a linear mixture of Fréchet densities. We provide some of its mathematical properties. Theestimation of the model parameters is performed by the maximum likelihood method.The importance and usefulness of the proposed distribution for modeling data are illustrated using two real data sets by comparison with some other extentions of the Fréchet distribution.

KEYWORDS

Fréchet distribution; Maximum likelihood estimation; Odd Lindley-G family; Order statistics.

1. INTRODUCTION

The Fréchet distribution, was introduced by Fréchet (1924), (also known as type II extreme value distribution) is one of the important distributions in extreme value theory and it has applications ranging from accelerated life testing through to earthquakes, floods, horse racing, rainfall, queues in supermarkets, wind speeds and sea waves. Further details about the Fréchet distribution and its applications can be explored in Kotz and Nadarajah (2000), Harlow (2002), Zaharim et al. (2009) and Mubarak (2011).

The statistical literature contains many generalizations of the Fréchet distribution. For example, the exponentiated Fréchet (Nadarajah and Kotz, 2003), the beta Fréchet (Nadarajah and Gupta, 2004 and Barreto-Souza et al., 2011), the transmuted Fréchet (Mahmoud and Mandouh, 2013), the Marshall-Olkin Fréchet (Krishna et al., 2013), the gamma extended Fréchet (Silva et al., 2013), the transmuted exponentiated Fréchet (Elbatal et al., 2014), the transmuted Marshall-Olkin Fréchet (Afify et al., 2015), the

Weibull Fréchet (Afify et al., 2016), the beta exponential Fréchet (Mead et al., 2017) and the modified Fréchet (Tablada and Cordeiro, 2017) distributions.

The cumulative distribution function (cdf) of the Fréchet distribution is given by (for $x > 0$)

$$G(x;\alpha,\beta) = \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]. \tag{1}$$

The corresponding probability density function (pdf) of (1) is

$$g(x;\alpha,\beta) = \beta\alpha^{\beta}x^{-\beta-1}\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right], \tag{2}$$

where $\alpha > 0$ is a scale parameter and $\beta > 0$ is a shape parameter.

Let $g(x;\zeta)$ and $G(x;\zeta)$ denote the density and cumulative functions of a baseline model with parameter vector ζ. Gomes-Silva et al. (2017) proposed a new family of distribution called the *Odd Lindley-G* (OLi-G) family.

The cdf of the OLi-G family is given by

$$\begin{aligned} F(x;\theta,\zeta) &= \frac{\theta^2}{1+\theta}\int_0^{\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]}(1+t)e^{-\theta t}dt \\ &= 1 - \frac{1+\theta-G(x;\zeta)}{(1+\theta)[1-G(x;\zeta)]}\exp\left[\frac{-\theta\, G(x;\zeta)}{1-G(x;\zeta)}\right], \end{aligned} \tag{3}$$

where θ is a positive shape parameter.

An easy interpretation of the above family can be given as follows. Let Y be a lifetime random variable having a certain a continuous cdf $G(x;\zeta)$. The odds ratio that an individual (or component) following the lifetime Y will die (failure) at time x is $G(x;\zeta)/\overline{G}(x;\zeta)$. Consider that the variability of this odds of death is represented by the random variable X and assume that it follows the Lindley model with scale θ. We can write Then, we have $P(Y \le x) = P\left[X \le G(x;\zeta)/\overline{G}(x;\zeta)\right] = F(x;\theta,\zeta)$, which is just given by (3).

The OLi-G density function is given by

$$f(x;\theta,\zeta) = \frac{\theta^2}{1+\theta}\frac{g(x;\zeta)}{[1-G(x;\zeta)]^3}\exp\left[\frac{-\theta\, G(x;\zeta)}{1-G(x;\zeta)}\right]. \tag{4}$$

A random variable X with pdf (4) is denoted by $X \sim$OLi-G(θ,ζ).

The rest of this article is organized as follows. Section 2 is devoted to define the OLiFr distribution and provide some plots for its pdf and hazard rate function (hrf). Section 3 deals with the derivation of a useful linear representation for the OLiFr density. In Section 4, we obtain some mathematical properties of the OLiFr distribution including ordinary and incomplete moments, moment generating function and order statistics. The maximum likelihood estimates (MLEs) of the unknown parameters are provided in Section 5. In Section 6, the flexibility of the OLiFr distribution is proved empirically by means of two applications to real data. Finally, we provide some conclusions in Section 7.

2. THE OLiFr DISTRIBUTION

In this section, we define the OLiFr distribution. By inserting the cdf (1) into equation (3), we obtain the cdf of OLiFr distribution (for $x > 0$)

$$F(x) = 1 - \frac{\theta + \left\{1 - \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}}{(1+\theta)\left\{1 - \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}} \exp\left\{\frac{-\theta \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{1 - \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right\}. \quad (5)$$

The pdf corresponding to (5) is

$$f(x) = \frac{\beta \alpha^{\beta} \theta^{2} x^{-\beta-1} \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{(1+\theta)\left\{1 - \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{3}} \exp\left\{\frac{-\theta \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{1 - \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right\}, \quad (6)$$

where α is a scale parameter and β and θ are shape parameters representing the different patterns of the OLiFr distribution. Henceforth, we denote a random variable X having pdf (6) by $X \sim \text{OLiFr}(\theta, \alpha, \beta)$.

The hrf and cumulative hazard rate function (chrf) of X are defined by

$$h(x) = \frac{\theta^{2} \beta \alpha^{\beta} x^{-\beta-1} \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{\left(\theta + \left\{1 - \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}\right)\left\{1 - \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{2}}$$

and

$$H(x) = -\ln\left(\frac{\theta + \left\{1 - \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}}{(1+\theta)\left\{1 - \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}} \exp\left\{\frac{-\theta \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{1 - \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right\}\right),$$

respectively. Figure 1 shows some plots of the OLiFr density for selected values of α, β and θ. The density plots indicate that the OLiFr distribution can be skewed to the left and to the right with small and large values for the skewness and kurtosis measures. The plots of the OLiFr hrf for some parameter values given in Figure 2 reveal that this function can be unimodal, decreasing or increasing, depending on the parameter values. Moreover, Figure 3 displays the hrf regions of OLiFr distribution for $\alpha = 1$.

3. LINEAR REPRESENTATION

In this section, we derive a useful expansion for the OLiFr density function in terms of Fréchet densities.

Applying the exponential series, the OLi-G pdf in equation (4.4), reduces to

$$f(x) = \frac{g(x)}{1+\theta} \sum_{k=0}^{\infty} \frac{(-1)^{k} \theta^{k+2}}{k!} G(x)^{k} [1 - \mathrm{G}(x)]^{-k-3}.$$

The generalized binomial expansion is defined by

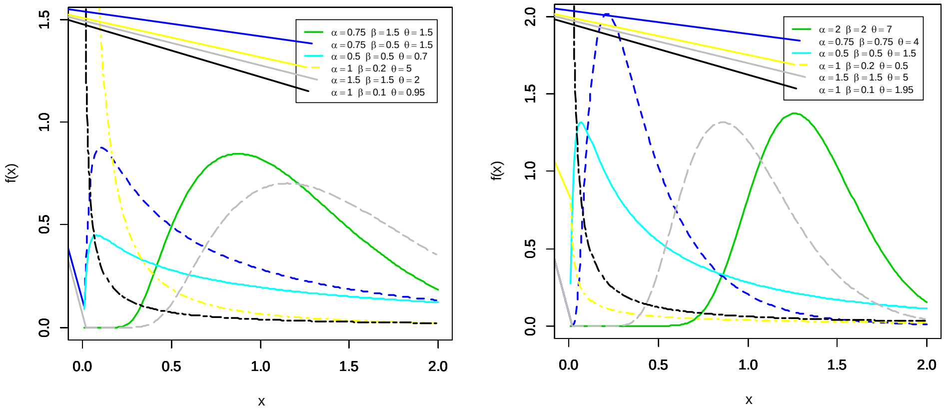

Figure 1: The pdf Plots of the OLiFr Distribution for Some Parameter Values

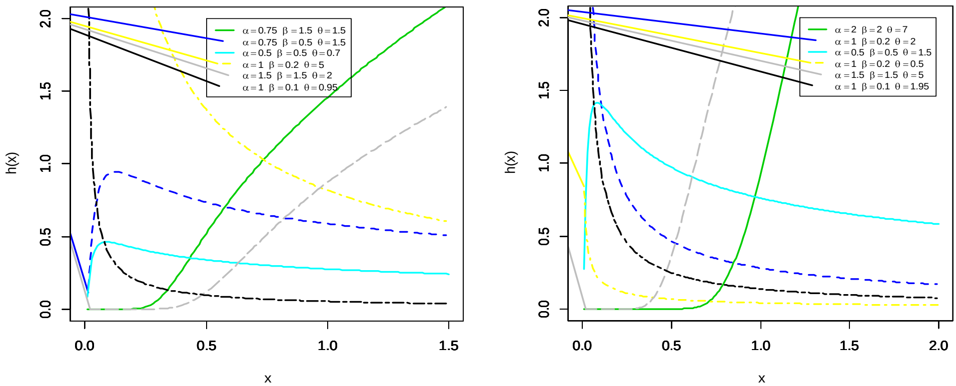

Figure 2: The hrf Plots of the OLiFr Distribution for Some Parameter Values

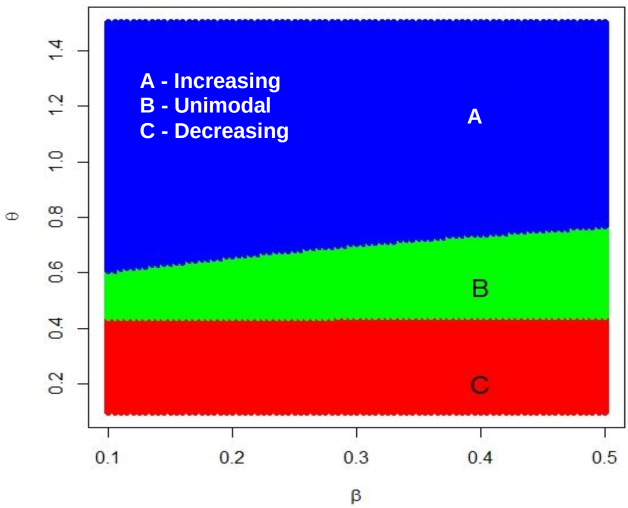

Figure 3: The hrf Regions of the OLiFr Distribution for $\alpha = 1$.

$$(1-z)^{-b} = \sum_{i=0}^{\infty} \frac{\Gamma(b+i)}{i!\,\Gamma(b)}.$$

Applying the above expansion to $[1-\mathrm{G}(x)]^{-k-3}$, we can write

$$f(x) = \frac{g(x)}{1+\theta} \sum_{k,j=0}^{\infty} \frac{(-1)^k \theta^{k+2} \Gamma(k+j+3)}{k!\,j!\,\Gamma(k+3)} G(x)^{k+i}.$$

Taking $G(x)$ and $g(x)$ in the last equation to be the cdf and pdf of the Fréchet distribution in equations (1) and (2), we have

$$f(x) = \frac{\beta\alpha^{\beta}}{1+\theta} \sum_{k,j=0}^{\infty} \frac{(-1)^k \theta^{k+2} \Gamma(k+j+3)}{k!\,j!\,\Gamma(k+3)} x^{-\beta-1} \exp\left[-(k+j+1)\left(\frac{\alpha}{x}\right)^{\beta}\right].$$

Then, the pdf of the OLiFr can be rewritten as

$$f(x) = \textstyle\sum_{k,j=0}^{\infty} b_{k,j} h_{k+j+1}(x), \qquad (7)$$

where

$$b_{k,j} = \frac{(-1)^k \theta^{k+2} \Gamma(k+j+3)}{k!\,j!\,(1+\theta)(k+j+1)\Gamma(k+3)}$$

and $h_{k+j+1}(x)$ is the Fréchet density with shape parameter β and scale parameter $\alpha(k+j+1)^{1/\beta}$. Equation (7) reveals that the OLiFr density can be written as a linear combination of Fréchet densities. So, several of its mathematical properties can be obtained from those of the Fréchet distribution and equation (7).

Let W be a random variable having the Fréchet distribution (2) with parameters α and β. For $r < \beta$, the nth ordinary and incomplete moments of W are given by

$$\mu_n^{'} = \alpha^n\,\Gamma(1-n/\beta) \text{ and } \varphi_{n,W}(t) = \alpha^n \gamma(1-n/\beta, (\alpha/t)^{\beta}),$$

respectively, where $\Gamma(a) = \int_0^{\infty} y^{a-1} e^{-y} dy$ is the complete gamma function and $\gamma(a,z) = \int_0^{z} y^{a-1} e^{-y} dy$ is the lower incomplete gamma function.

4. THE OLiFr PROPERTIES

In this section, we derive some mathematical properties of the OLiFr distribution including ordinary and incomplete moments, moment generating function and order statistics.

4.1 Ordinary and Incomplete Moments

The nth ordinary moment of X is given by

$$\mu_n^{'} = E(X^n) = \sum_{k,j=0}^{\infty} b_{k,j} \int_0^{\infty} x^n h_{k+j+1}(x) dx.$$

For $n < \beta$, we obtain

$$\mu_n^{'} = \textstyle\sum_{k,j=0}^{\infty} b_{k,j} \alpha^n (k+j+1)^{n/\beta} \Gamma(1-n/\beta). \qquad (8)$$

From equation (8), we have the mean of X with $n = 1$.

The skewness and kurtosis measures can be evaluated from the ordinary moments using well-known relationships.

The nth incomplete moment of the OLiFr distribution is defined by

$$\varphi_n(t) = \int_0^t x^n f(x) dx.$$

Using equation (7), we can write

$$\varphi_n(t) = \sum_{k,j=0}^{\infty} b_{k,j} \int_0^t x^n h_{k+j+1}(x).$$

Then, we have (for $n < \beta$),

$$\varphi_n(t) = \sum_{k,j=0}^{\infty} b_{k,j} \alpha^n (k+j+1)^{\frac{n}{\beta}} \gamma\left(1 - n/\beta, (k+j+1)\left(\frac{\alpha}{t}\right)^{\beta}\right).$$

The first incomplete moment, say $\varphi_1(t)$, follows from the above equation with $n = 1$. $\varphi_1(t)$ has an important application related to the Bonferroni and Lorenz curves. These curves are very useful in economics, demography, insurance, engineering and medicine.

Another application of $\varphi_1(t)$ is related to the mean residual life and the mean inactivity time defined by $m_1(t) = [1 - \varphi_1(t)]/R(t) - t$ and $M_1(t) = t - \varphi_1(t)/F(t)$, respectively.

4.2 Moment Generating Function

Afify et al. (2016) derived the moment generating function (mgf) of the Fréchet distribution, $M(t; \alpha, \beta)$, given by (1) and (2).

By setting $y = x^{-1}$, we can write

$$M(t; \alpha, \beta) = \beta \alpha^{\beta} \int_0^{\infty} \exp\left(\frac{t}{y}\right) y^{\beta-1} \exp\left[-(\alpha y)^{\beta}\right] dy.$$

Using the exponential series for the first exponential

$$M(t; \alpha, \beta) = \beta \alpha^{\beta} \sum_{m=0}^{\infty} \frac{t^m}{m!} \int_0^{\infty} y^{\beta-m-1} \exp\left[-(\alpha y)^{\beta}\right] dy.$$

Calculating the integral, we obtain

$$M(t; \alpha, \beta) = \sum_{m=0}^{\infty} \frac{\alpha^m t^m}{m!} \Gamma\left(\frac{\beta - m}{\beta}\right).$$

Consider the Wright generalized hypergeometric function defined by

$$ {}_p\Psi_q\left[\begin{matrix}(\alpha_1,A_1),\dots,(\alpha_p,A_p)\\(\beta_1,B_1),\dots,(\beta_q,B_q)\end{matrix};\ x\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\left(\alpha_j+A_jn\right)}{\prod_{j=1}^{q}\Gamma\left(\beta_j+B_jn\right)}\frac{x^n}{n!}. $$

Hence, we can write $M(t;\alpha,\beta)$ as

$$ M(t;\alpha,\beta)={}_1\Psi_0\left[\begin{matrix}(1,-\beta^{-1})\\-\end{matrix};\alpha t\right]. \tag{9} $$

Using equations (7) and (9), the mgf of X, denoted by $M(t)$, comes out as

$$ M(t)=\sum_{k,j=0}^{\infty} b_{k,j}\quad {}_1\Psi_0\left[\begin{matrix}(1,-\beta^{-1})\\-\end{matrix};\alpha\,(k+j+1)^{1/\beta}t\right]. $$

4.3 Order Statistics

Let $X_1,\dots,X_n$ be a random sample of size n from the OLiFr distribution and $X_{(1)},\dots,X_{(n)}$ be the corresponding order statistics. Then, the pdf of the ith order statistic $X_{i:n}$, say $f_{i:n}(x)$, is given by

$$ f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}f(x)F(x)^{i-1}[1-F(x)]^{n-i}. \tag{10} $$

Gomes-Silva et al. (2017) derived a simple formula for the ith order statistic of the OLi-G family.

According to Gomes-Silva et al. (2017), the pdf of the ith order statistic can be expressed as

$$ f_{i:n}(x)=\sum_{m,p=0}^{\infty}\sum_{j=0}^{k+n-i} v_{j,m,p}g(x)G(x)^{j+m+p}, \tag{11} $$

where

$$ v_{j,m,p}=\frac{n!\,\theta^{j+m+2}}{(i-1)!\,(n-i)!\,m!\,(1+\theta)^{j+1}}\binom{j+m+p}{j+m}\sum_{k=0}^{i-1}(-1)^{k+m}\binom{k+n-i}{j}\binom{i-1}{k} $$

Taking $G(x)$ and $g(x)$ in equation (11) to be the cdf and pdf of the Fréchet distribution, then equation (11) reduces to

$$ f_{i:n}(x)=\sum_{m,p=0}^{\infty}\sum_{j=0}^{k+n-i} v_{j,m,p}\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-(j+m+p+1)\left(\frac{\alpha}{x}\right)^{\beta}\right]. $$

The last equation can be rewritten as

$$ f_{i:n}(x)=\sum_{m,p=0}^{\infty}\sum_{j=0}^{k+n-i} w_{j,m,p}h_{j+m+p+1}(x), \tag{12} $$

where $w_{j,m,p}=v_{j,m,p}/(j+m+p+1)$ and $h_{j+m+p+1}(x)$ denotes the Fréchet density function with shape parameters β and scale parameter $\alpha(j+m+p+1)^{1/\beta}$. Hence, the pdf of the OLiFr order statistics is a linear mixture of Fréchet pdfs. Based on equation (12), we can easily derive some properties of $X_{i:n}$ from those Fréchet properties.

For example, the sth moment of $X_{i:n}$ is given (for $s < \beta$) by

$$E(X_{i:n}^s) = \alpha^s \sum_{m,p=0}^{\infty} \sum_{j=0}^{k+n-i} w_{j,m,p}(j+m+p+1)^{s/\beta}\Gamma(1-s/\beta).$$

5. ESTIMATION AND SIMULATION

In this section, we consider the estimation of the unknown parameters for the OLiFr from complete samples only by maximum likelihood. We investigate the MLEs of the parameters of the OLiFr(α,β,θ) model. Let $\mathbf{x} = (x_1,\dots,x_n)$ be a random sample from this model with unknown parameter vector $\upsilon = (\alpha,\beta,\theta)^T$.

The log-likelihood function for υ, say $\ell = \ell(\upsilon)$, is given by

$$\ell = n\big(\log\beta + \beta\log\alpha + 2\log\theta - \log(1+\theta)\big) - (\beta+1)\sum_{i=1}^{n}\log x_i$$
$$-\sum_{i=1}^{n}\left(\frac{\alpha}{x_i}\right)^{\beta} - 3\sum_{i=1}^{n}\log\left\{1-\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]\right\} - \theta\sum_{i=1}^{n}\frac{\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]}.$$

The above equation can be maximized either directly by using the R (optim function), SAS (PROC NLMIXED sub-routine), Ox program (MaxBFGS) or by solving the nonlinear likelihood equations obtained by differentiating it.

The score vector is given by $\mathbf{U}(\upsilon) = \frac{\partial\ell}{\partial\upsilon} = (\frac{\partial}{\partial\alpha},\frac{\partial}{\partial\beta},\frac{\partial}{\partial\theta})^T$. Then, we have

$$\frac{\partial\ell}{\partial\alpha} = \frac{n\beta}{\alpha} - \frac{\beta}{\alpha}\sum_{i=1}^{n}\left(\frac{\alpha}{x_i}\right)^{\beta} + (\theta-3)\frac{\beta}{\alpha}\sum_{i=1}^{n}\frac{\left(\frac{\alpha}{x_i}\right)^{\beta}\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]},$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + n\log\alpha - \sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\left(\frac{\alpha}{x_i}\right)^{\beta}\log\left(\frac{\alpha}{x_i}\right)$$
$$+(\theta-3)\sum_{i=1}^{n}\frac{\left(\frac{\alpha}{x_i}\right)^{\beta}\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]\log\left(\frac{\alpha}{x_i}\right)}{1-\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]}$$

and

$$\frac{\partial\ell}{\partial\theta} = \frac{2n}{\theta} - \frac{n}{1+\theta} - \sum_{i=1}^{n}\frac{\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]}{1-\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]}.$$

We can obtain the estimates of the unknown parameters by setting the score vector to zero, $\mathbf{U}(\hat{\upsilon}) = \mathbf{0}$. By solving these equations simultaneously gives the MLEs $\hat{\alpha},\hat{\beta}$ and $\hat{\theta}$. These estimates can be obtained numerically using iterative techniques such as the

Newton-Raphson algorithm. For the OLiFr distribution, all the second-order derivatives exist.

For interval estimation of the model parameters, we require the 3×3 observed information matrix $J(\upsilon) = \{J_{rs}\}$ for $r, s = \alpha, \beta, \theta$. Under standard regularity conditions, the multivariate normal $N_3(0, J(\hat{\upsilon})^{-1})$ distribution can be used to construct approximate confidence intervals for the model parameters. Here, $J(\hat{\upsilon})$ is the total observed information matrix evaluated at $\hat{\upsilon}$. Then, approximate $100(1-\phi)\%$ confidence intervals for the model parameters can be determined in the usual way of the first-order asymptotic theory.

Here, the simulation study is conducted to evaluate the performance of MLES of the parameters of OLiFr distribution. The simulation results are evaluated based on the following measures: biases, mean square errors (MSE), coverage probability (CP) and average length (AL). We generate $N = 1{,}000$ samples of size $n = 50, 55, \ldots, 1000$ from OLiFr distribution with $\theta = 0.5, \alpha = 0.5$ and $\beta = 1.5$. The MLEs of the parameters are obtained for each generated sample. The corresponding standard errors are obtained by inverting the observed information matrix. The estimated biases, MSEs are given by

$$\widehat{Bias}_\varepsilon(n) = \frac{1}{N}\sum_{i=1}^{n}(\hat{\varepsilon}_i - \varepsilon), \widehat{MSE}_\varepsilon(n) = \frac{1}{N}\sum_{i=1}^{n}(\hat{\varepsilon}_i - \varepsilon)$$

where $\varepsilon = \theta, \alpha, \beta$. The CPs and ALs are given by

$$CP_\varepsilon(n) = \frac{1}{N}\sum_{i=1}^{n} I\left(\hat{\varepsilon}_i - 1.95996 s_{\hat{\varepsilon}_i}, \hat{\varepsilon}_i + 1.95996 s_{\hat{\varepsilon}_i}\right),$$

$$AL_\varepsilon(n) = \frac{3.919928}{N}\sum_{i=1}^{n} s_{\hat{\varepsilon}_i}$$

where $s_{\hat{\varepsilon}_i}$ is the stanrdard error of the MLEs. Figure 4 displays the simulation results for the above measures. As seen from Figure 3, when the sample size increases, biases and MSEs approach to zero. This fact reveals the consistency property of the MLES. The CP is near to 0.95 for all parameters. Moreover, when the sample size increases, AL decreases for all cases.

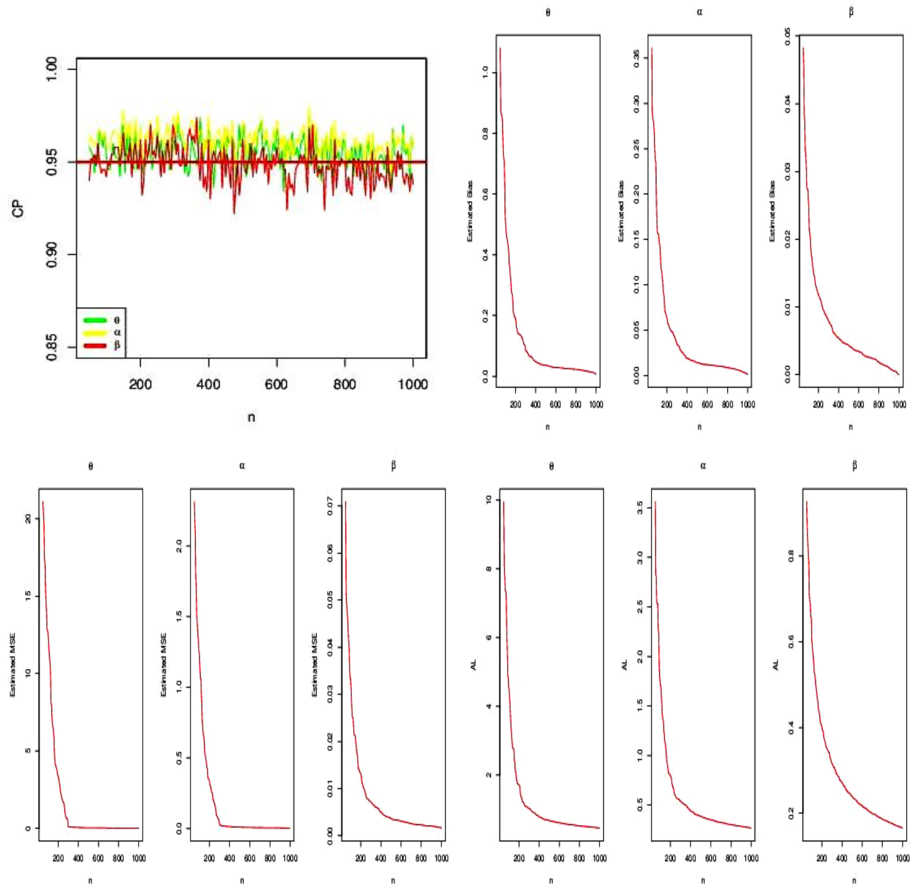

Figure 4: Estimated CPs, Biases, MSEs and ALs for Selected Parameter Values

6. APPLICATIONS

In this section, we present two applications of the OLiFr distribution using real data sets. We shall compare the fit of the OLiFr distribution with the Kumaraswamy Fréchet (KFr), exponentiated Fréchet (EFr), beta Fréchet (BFr), gamma extended Fréchet (GEFr), transmuted Marshall-Olkin Fréchet (TMOFr), transmuted Fréchet (TFr) and Fréchet (Fr) distributions with corresponding pdfs (for $x > 0$):

$$\text{KFr:} f(x;\alpha,\beta,a,b) = ab\beta\alpha^{\beta}x^{-(\beta+1)}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{b-1};$$

$$\text{EFr: } f(x;\alpha,\beta,\theta) = \theta\beta\alpha^{\beta}x^{-(\beta+1)}\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{\theta-1};$$

$$\text{BFr: } f(x;\alpha,\beta,a,b) = \frac{\beta\alpha^{\beta}}{B(a,b)}x^{-(\beta+1)}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{b-1};$$

$$\text{GEFr: } f(x;\alpha,\beta,a,b) = \frac{a\beta\alpha^{\beta}}{\Gamma(b)} x^{-(\beta+1)} \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{a-1} \times \left\{-\log\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{a}\right\}^{b-1};$$

$$\text{TMOFr: } f(x;\alpha,\beta,a,b) = a\beta\alpha^{\beta} x^{-(\beta+1)} \left\{a+(1-a)\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{-2} \times \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1+b-2b\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\left[a+(1-a)\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right]^{-1}\right\};$$

$$\text{TFr: } f(x;\alpha,\beta,b) = \beta\alpha^{\beta} x^{-(\beta+1)} \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{(b+1)-2b\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}.$$

The parameters of the above densities are all positive real numbers except for the TMOFr and TFr distributions for which $|b| \leq 1$.

The first data set represents the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958-1984, rounded to one decimal place. The data are:

1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.0.

The second data set is on breaking stress of carbon fibres (in Gba) given by Nichols and Padgett (2006). The data are:

3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.

In order to compare the fitted distributions, we consider the following criteria: the $-2\hat{\ell}$ (Maximized Log-Likelihood), AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion), BIC (Bayesian Information Criterion), HQIC (Hannan-Quinn Information Criterion), Anderson-Darling (A^*) and Cramér-Von Mises (W^*) statistics. The model with minimum values for these statistics could be chosen as the best model to fit the data.

Table 1
Goodness-of-Fit Statistics for Wheaton River Data

Model	-2ℓ	AIC	$CAIC$	$HQIC$	BIC	W^*	A^*
OLiFr	503.440	509.440	509.793	512.159	516.270	0.13671	0.78090
KFr	506.005	514.005	514.602	517.630	523.112	0.17337	0.97379
EFr	512.243	518.243	518.596	520.962	525.073	0.24225	1.37968
BFr	514.765	522.765	523.362	526.39	531.872	0.28585	1.6124
GEFr	514.651	522.651	523.248	526.277	531.758	0.28449	1.60447
TMOFr	515.440	523.440	524.037	527.066	532.547	0.30523	1.6929
TFr	529.984	535.984	536.337	538.703	542.814	0.45818	2.63602
Fr	534.038	538.038	538.212	539.851	542.591	0.48147	2.80181

Table 2
Goodness-of-Fit Statistics for Breaking Stress of Carbon Fibre Data

Model	$-2\hat{\ell}$	AIC	$CAIC$	$HQIC$	BIC	W^*	A^*
OLiFr	288.300	294.300	294.55	297.463	302.115	0.08634	0.4679
EFr	289.697	295.697	295.947	298.861	303.513	0.55798	0.10372
KFr	289.059	297.059	297.480	301.276	307.479	0.09585	0.51495
BFr	303.133	311.133	311.554	315.350	321.553	0.25137	1.39536
GEFr	303.960	311.960	312.381	316.178	332.381	0.25872	1.43853
TMOFr	301.973	309.973	310.394	314.190	320.393	0.2376	1.26771
Fr	344.308	348.308	348.432	350.417	353.519	0.54849	3.13643
TFr	344.475	350.475	350.725	353.638	358.290	0.55598	3.17823

Tables 1 and 2 provide the values of the MLEs and their corresponding standard errors (in parentheses) of the model parameters, whereas the values of these statistics for the fitted models to both data sets are listed in Tables 3 and 4, respectively.

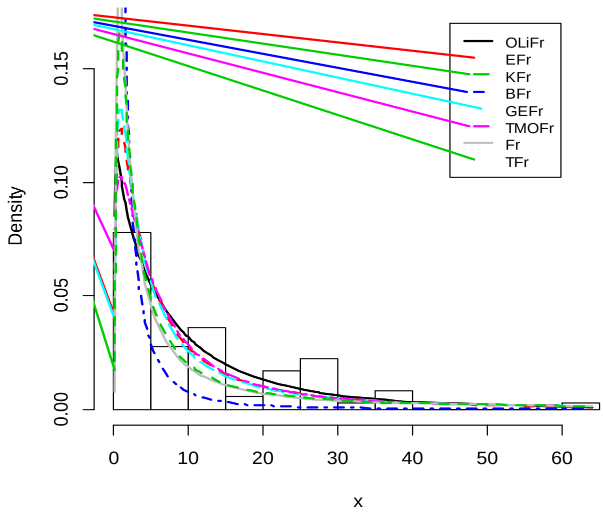

Figure 5: The Fitted pdfs of the OLiFr Model and Other Models for Wheaton River Data

Table 3
MLEs and the Corresponding SEs (Given in Parentheses) for Wheaton River Data

Model	Estimates			
Fr (α, β)	2.8790 (0.553)	0.6521 (0.054)		
OLiFr (α, β, θ)	0.7394 (0.842)	0.6087 (0.058)	0.3929 (0.297)	
EFr (α, β, θ)	391.92 (398.1)	0.2677 (0.033)	14.442 (6.62)	
TFr (α, β, b)	1.5083 (0.437)	0.7107 (0.059)	-0.7289 (0.234)	
KFr (α, β, a, b)	6.3401 (0.011)	0.1332 ($1.6 \cdot 10^{-4}$)	6.6065 (0.011)	478.30 (0.132)
BFr (α, β, a, b)	38.2262 (118.5)	0.1356 (0.082)	11.712 (20.38)	30.316 (34.14)
GEFr (α, β, a, b)	40.4813 (129.17)	0.1345 (0.081)	35.739 (42.97)	11.735 (20.23)
TMOFr (α, β, a, b)	0.1300 (0.169)	1.1923 (0.121)	107.79 (195.8)	-0.0168 (0.559)

The plots of the fitted OLiFr pdf and other fitted pdfs, for both data sets, are displayed in Figures 5 and 6. Figures 7 and 8 display the PP plots and estimated cdfs for the fitted models, respectively. The estimated cdfs, to both data sets, for the competitive models are shown in Figures 9 and 10.

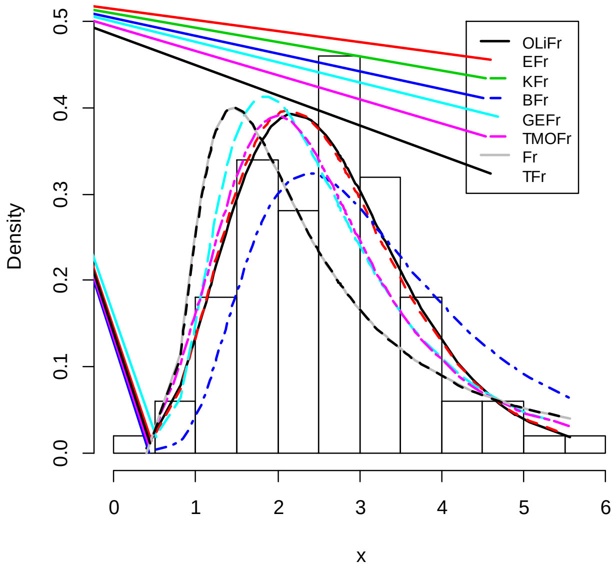

Figure 6: The Fitted pdfs of the OLiFr Model and other Models for Breaking Stress of Carbon Fibre Data

Table 4
MLEs and their Standard Errors (in Parentheses) for Breaking Stress of Carbon Fibre Data

Model	Estimates			
Fr (α, β)	1.8705 (0.112)	1.7766 (0.113)		
OLiFr (α, β, θ)	208.27 (246.8)	0.4200 (0.074)	435.21 (494.9)	
EFr (α, β, θ)	69.148 (57.34)	0.5019 (0.08)	145.32 (122.9)	
TFr (α, β, b)	1.9315 (0.097)	1.7435 (0.076)	0.0819 (0.198)	
KFr (α, β, a, b)	2.0556 (0.071)	0.4654 (0.007)	6.2815 (0.063)	224.18 (0.164)
BFr (α, β, a, b)	1.6097 (2.498)	0.4046 (0.108)	22.014 (21.43)	29.761 (17.47)
GEFr (α, β, a, b)	1.3692 (2.017)	0.4776 (0.133)	27.645 (14.13)	17.458 (14.81)
TMOFr (α, β, a, b)	0.6496 (0.068)	3.3313 (0.206)	101.92 (47.62)	0.2936 (0.270)

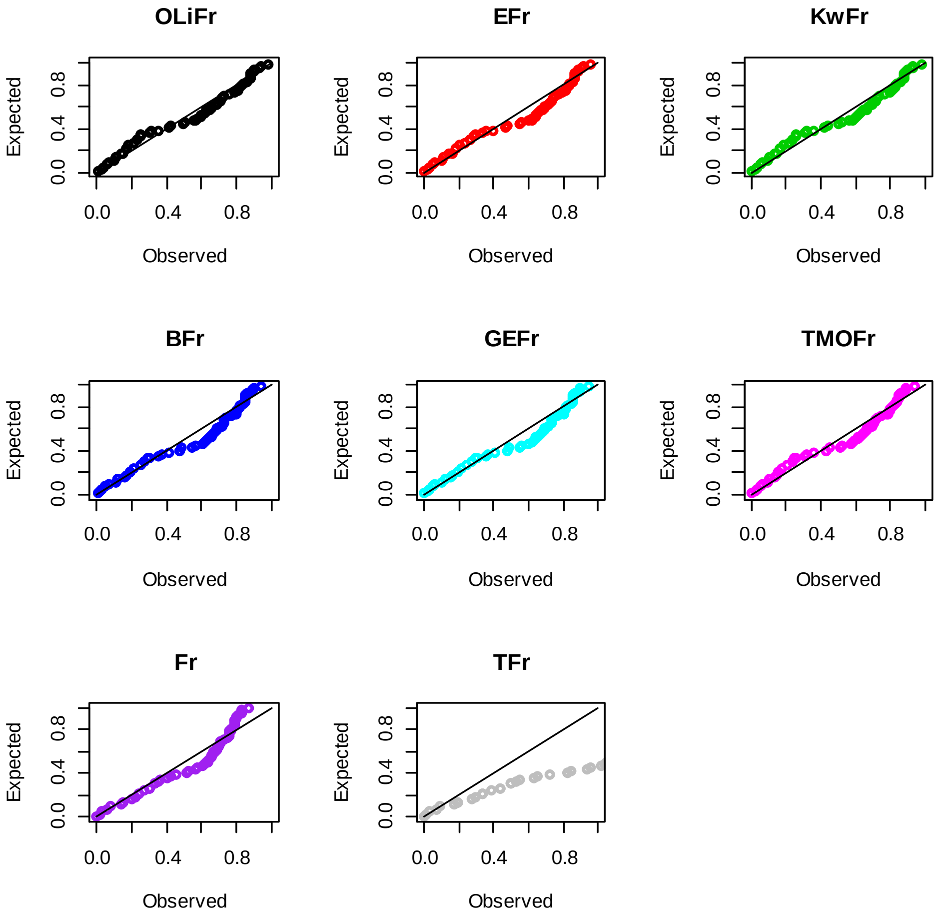

Figure 7: PP Plots for for Wheaton River Data

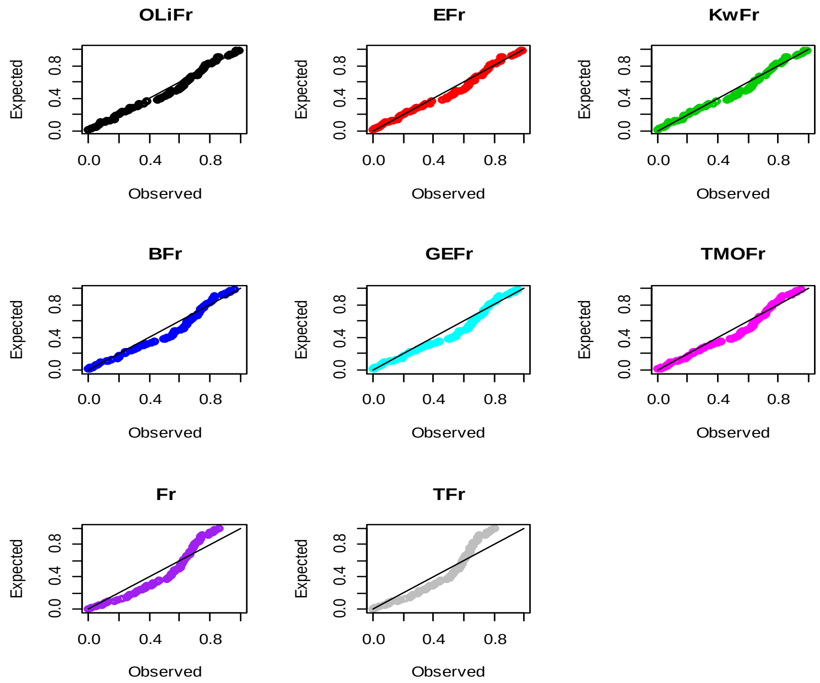

Figure 8: PP plots for Breaking Stress of Carbon Fibre Data

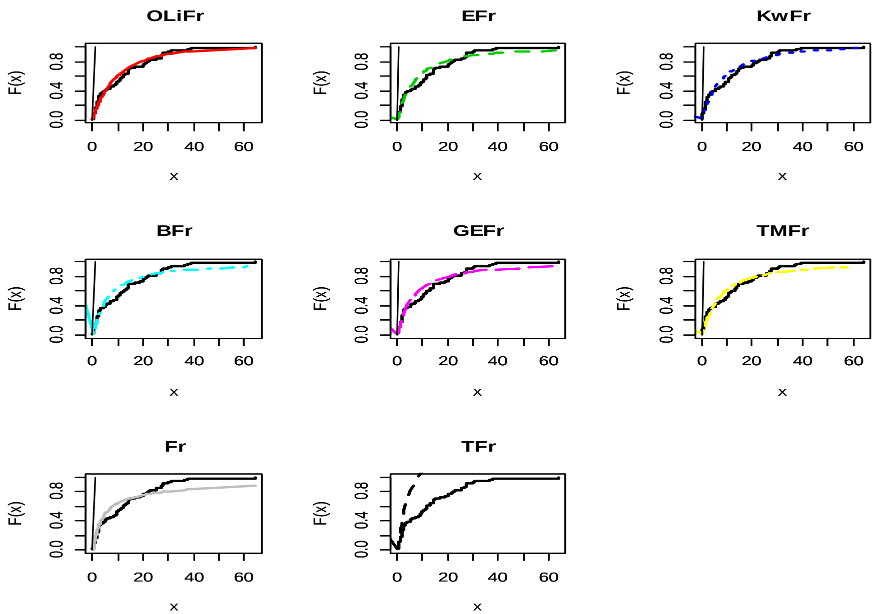

Figure 9: Fitted cdfs on Empirical cdf for Wheaton River Data

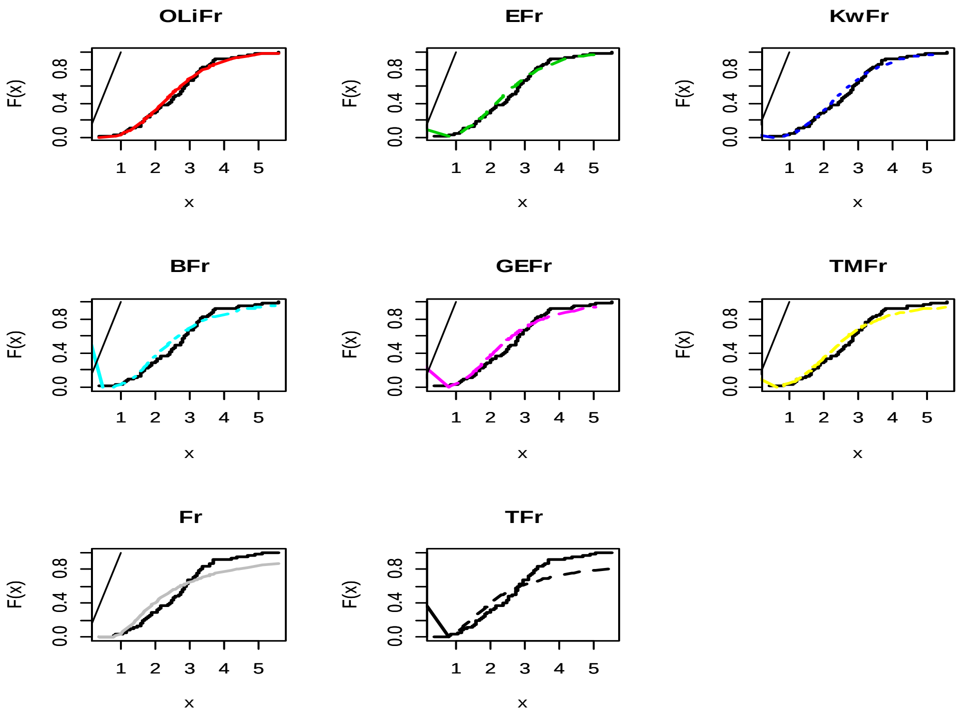

Figure 10: Fitted cdfs on Empirical cdf for Breaking Stress of Carbon Fibre Data

Tables 1 and 2 compare the OLiFr model with the KFr, EFr, BFr, GEFr, TMOFr, TFr and Fr distributions. Its noted that the proposed model has the lowest values for the AIC, $CAIC$, $HQIC$, BIC, W^* and A^* statistics among all fitted models. So, the OLiFr model can be chosen as the best model for both data sets.Moreover, the Kolmogorov-Smirnov (K-S) statistic for OLiFr distribution is obtained for both data sets. The K-S statistic of OLiFr distribution for first data set is 0.0964 and corresponding p-value is 0.5151. The K-S statistic of OLiFr distribution for second data set is 0.0707 and corresponding p-value is 0.6988. The results of K-S reveal that the OLiFr distribution provides superior fits to used data sets. Furthemore, the plots in the Figures from 3 through 8 reveal that the OLiFr distribution gives a better fit than other nested and non-nested models for both data sets.

7. CONCLUSION

In this paper, we propose a new three-parameter model called the odd Lindley Fréchet (OLiFr) distribution, which extends the Fréchet distribution. The OLiFr pdfcan be expressed as a linear mixture of Fréchet densities. We derive explicit expressions for its ordinary and incomplete moments, generating function and order statistics. The model parameters are estimated by maximum likelihood. By means of two real data sets we prove that the new model provides better fits than some other well-known competitive models.

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