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TRANSMUTED SIZE-BIASED EXPONENTIAL DISTRIBUTION AND ITS PROPERTIES

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ABSTRACT

In this paper, transmuted size-biased exponential distribution is developed using quadratic rank transmutation map approach. Mathematical properties like, CDF, r^{th} order moment, moment generating function, characteristic function, measure of skewness and kurtosis, reliability measures, hazard function, Shannon and Renyi entropy functions, mean residual life function are derived. Furthermore, method of moments and maximum likelihood are used to estimate the parameters of proposed distribution. The performance of suggested distribution is compared with its base line distribution while modeling real data sets.

KEY WORDS

Size-Biased Exponential Distribution, Transmutation Map, Hazard Rate Function, Entropy, Order Statistics.

1. INTRODUCTION

In recent developments, researchers focused on generating more flexible, tractable and meaningful distributions and modeled various types of lifetime data with monotone failure rates. In spite of their simplicity in solving many problems of lifetime data and reliability studies, such existing distributions are not useful to model bathtub and multimodal shaped failure rates and also fail to provide sound parametric fit to some practical application. In recent past, new families of probability distributions have been defined that are extension of well-known families of distributions. These newly developed families/classes of distributions provide greater flexibility in modeling complex data.

Weighted in general and size biased in particular distributions arise in practice when observations from a sample are recorded with unequal probability and provide unifying approach for the problems when the observations fall in the non-experimental, non-replicated and non-random categories, first introduced by Fisher (1934) to model ascertainment bias these are later formulized in uniform theory by Rao (1965). Patil and Rao (1977, 1978) discussed that weighted distributions have various statistical applications, especially in analysis of data relating to ecology and human populations. For the first, time Warren (1975) applied weighted distribution to connect with sampling wood cells. Gove (2003) studied some of the latest results on size-biased distributions especially conceder the Weibull family relating to application and parameter estimation with method of moments and maximum likelihood in forestry. A weighted version of exponential distribution is discussed by Mir et al. (2013). They derive some mathematical properties and estimate the parameter with method of moments, maximum likelihood and Bayesian method.

After introducing the concept of extending the probability distributions via weighted distributions, section 2 contains the derivation of *pdf* and CDF with their graphical presentation of transmutation size biased distribution with the help of quadratic rank transmutation map. Section 3 contains the derivation of r^{th} non-moment, moment generating function and mean deviation. The expression for the coefficient of variation, skewness and kurtosis are also reported and their numerical values are calculated. Section 4 is about the quantile function, median and random number generating process. Reliability function, hazard function and their mathematical and graphical presentation are given in Section 5. Section 6 is related to measure of uncertainty of the proposed distribution. Section 7 contains the mean residual life function for variable X has a transmuted sized biased exponential distribution. Section 8 is about order statistics: the lowest, highest and joint order densities of transmuted size biased exponential distribution are specified. Methodology for parameter estimation, Newton Raphson algorithm for maximum likelihood is discussed in Section 9. To compare the suitability of subject distribution with its related distributions, real life data set is selected and its goodness of fit on empirical data is tested by using likelihood function, AIC, AICC, BIC, $K-S$, C_n and LR test in section 10.

2. TRANSMUTED SIZED BIASED EXPONENTIAL DISTRIBUTION

In this section, we derive the probability density function (pdf) and probability distribution function (cdf) with their graphical presentation of transmutation size biased exponential distribution with the help of quadratic rank transmutation map.

In order to generate more distributional flexibility, Shaw and Buckley (2007) suggested quadratic rank transmutation map (QRTM) approach. Transmutation map provides a powerful technique for turning the ranks of one distribution in to the ranks of another. According to this approach, a random variable X is said to have transmuted distribution if its cumulative distribution (CDF) is given by

$$F_T(x) = (1+\lambda)F_B(x) - \lambda F_B(x)^2 , \tag{2.1}$$

where $F_B\left(x\right)$ is the CDF of the base distribution, which on differentiation yields,

$$f_T\left(x\right) = f_B\left(x\right) \left[\left(1+\lambda\right) - 2\lambda F_B\left(x\right)\right], \tag{2.2}$$

where $F_B\left(x\right)$ and $F_T\left(x\right)$ are the CDF's and $f_B\left(x\right)$ and $f_T\left(x\right)$ are the *pdf*'s of the base distribution and transmuted distribution respectively. Note that if $\lambda = 0$, we have the distribution of the base random variable.

Various generalizations have been introduced based on the transmutation map approach. Aryal and Tsokos (2009) used QRTM to drive a flexible family of probability distributions. They take extreme value distribution as the base line distribution by adding a new parameter that produced extra variability. They also fitted the proposed family of distribution to real data set. Merovci (2013) generalized the Lindley distribution using the QRTM. Further, he comprehensively derived the mathematical properties and its reliability performance. The practicality of the transmuted Lindley distribution for modeling data was illustrated using real data. Elbatal et al. (2013) proposed transmuted generalized linear exponential distribution, Merovci (2013) generalized the exponentiated exponential distribution using the quadratic rank transmutation map, Khan and King (2013) proposed transmuted modified Weibull distribution, Merovci (2013) generalized the Raleigh distribution using the QRTM.

Recently, among others, Hussian (2014) proposed transmuted exponentiated gamma distribution, Ahmad et al. (2014) generalized the inverse Raleigh distribution using the QRTM, Merovci and Elbatal (2014) proposed transmuted Lindley-geometric distribution, Merovci and Puka (2014) generalized the pareto distribution using the QRTM, Abdual-Moniem and Seham (2015) proposed transmuted gompertz distribution, exponentiated transmuted modified Weibull distribution is proposed by Paland Tiensuwan (2015), Using QRTM, Afify et al. (2015) proposed transmuted Weibull Lomax distribution.

Definition 2.1

The CDF of a sized-biased exponential distribution is

$$F(x;\beta) = \left(1 - e^{-x/\beta}(1 + x/\beta)\right), \tag{2.3}$$

with the probability density function is

$$f(x;\beta) = \frac{xe^{-x/\beta}}{\beta^2}, \qquad x > 0 \tag{2.4}$$

By substituting (2.3) in (2.1), we obtain the cdf of transmuted sized biased exponential distribution.

$$F(x;\beta,\lambda) = \left(1 - e^{-x/\beta}(1 + x/\beta)\right)\left[1 + \lambda\left(e^{-x/\beta}(1 + x/\beta)\right)\right] \tag{2.5}$$

and its respective pdf is given by

$$f(x;\beta,\lambda) = \frac{xe^{-x/\beta}}{\beta^2}\left(1 - \lambda + 2\lambda e^{-x/\beta}(1 + x/\beta)\right), \quad x > 0 \tag{2.6}$$

where $\beta > 0$ is the scale parameter, $|\lambda| \le 1$ is transmuted parameter and $x \in \mathbb{R}^+$.

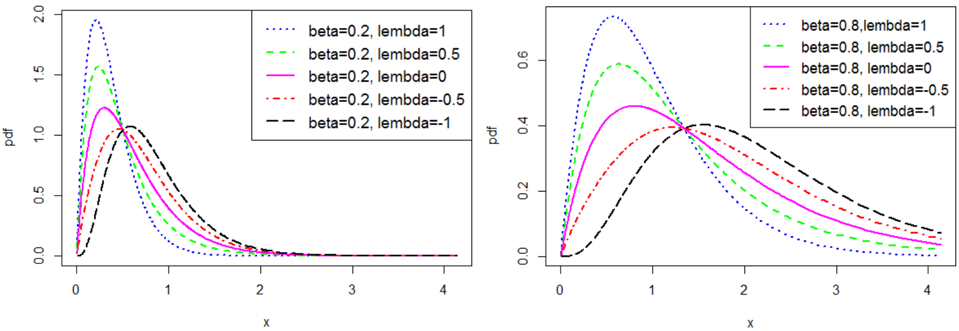

Fig. 1: pdf's Plots of Various Transmuted Size Biased Exponential Distributions

Figure 1 and 2 illustrates some of the possible shapes of the pdf and cdf of transmuted size biased exponential distribution for selected values of the parameters β and λ, respectively.

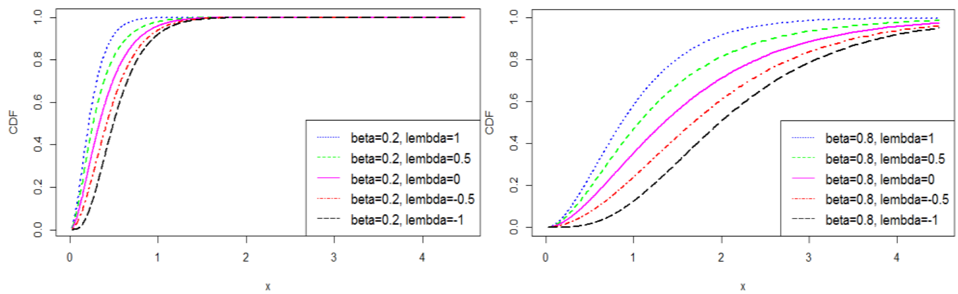

Fig. 2: cdf's Plots of Various Transmuted Size Biased Exponential Distributions

3. STATISTICAL PROPERTIES OF TRANSMUTED SIZED BIASED EXPONENTIAL DISTRIBUTION

In this section, we discuss the statistical properties of the transmuted size biased exponential distribution. Specifically moments, mean, variance, moments ratio, moment generating function, mean deviation, skewness and kurtosis.

Theorem 3.1:

If X has the $T_{SBE}(x;\beta,\lambda)$ with $|\lambda| \leq 1$, then the r^{th} non-central moments are given by

$$\mu_r' = \beta^r \Gamma(r+2)\left[1-\lambda\left(1-\frac{r+4}{2^{r+2}}\right)\right]. \qquad (3.1)$$

Proof:

The r^{th} non-central moment is given by

$$\mu_r' = E\left(X^r\right) = \int_0^\infty x^r f_{TSBE}(x)dx$$

$$= \frac{1}{\beta^2}\left[(1-\lambda)\int_0^\infty x^{r+1}e^{-x/\beta}dx + 2\lambda\left(\int_0^\infty x^{r+1}e^{-2x/\beta}dx + \frac{1}{\beta}\int_0^\infty x^{r+2}e^{-2x/\beta}\,dx\right)\right]$$

$$= \beta^r\Gamma(r+2)\left[1-\lambda\left(1-\frac{r+4}{2^{r+2}}\right)\right].$$

By Setting $r=1$ and $r=2$ in (3.1), we can easily derive the mean (μ) and variance (σ^2) of the transmuted size biased exponential distribution given in statement of the theorem.

The expressions of the Coefficient of variation (CV), Skewness (Skew) and Kurtosis (Kurt) for the transmuted sized biased exponential distribution are respectively given by

$$CV = \frac{\sigma}{\mu_1'} = \frac{\left[(8+3\lambda)(4-3\lambda)\right]^{1/2}}{8-3\lambda},$$

Table 1
CV at Different Values of λ

λ	**1**	**0.5**	**0**	**-0.5**	**-1**
CV	0.663325	0.7497434	0.7071068	0.6293821	0.5378254

$$Skew = \frac{\mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3}{\sigma^3} = \frac{\left[2\left(128-24\lambda-54\lambda^2-27\lambda^3\right)\right]^2}{\left[(8+3\lambda)(4-3\lambda)\right]^3},$$

Table 2
Skewness at Different Values of λ

λ	**-1**	**-0.5**	**0**	**0.5**	**1**
Skew	1.457726	1.476668	2.000000	2.933834	1.589782

From Table 2, we can say that transmuted size biased exponential distribution is positively skewed distribution for almost values of transmuted parameter λ.

$$Kurt = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4}{\sigma^4} = \frac{3\left[2048-3\lambda\left\{256(1+\lambda)+9\lambda^3(8+3\lambda)\right\}\right]}{\left[(8+3\lambda)(4-3\lambda)\right]^2}.$$

Table 3
Kurtosis at Different Values of λ

λ	-1	-0.5	0	0.5	1
Kurt	5.346122	5.309453	6.000000	7.658393	5.330579

Table 3 shows that, transmuted size biased exponential distribution is leptokurtic for almost all values of transmuted parameter λ.

Theorem 3.2:

If X has the $T_{SBE}(x;\beta,\lambda)$ with $|\lambda| \leq 1$, then the moment generating function of X, say $M_X(t)$, is

$$M_X(t)=\left[\frac{(1-\lambda)}{(1-\beta t)^2}+\frac{2\lambda(4-\beta t)}{(2-\beta t)^3}\right]. \qquad (3.2)$$

Proof:

The moment generating function of the random variable X is given by

$$M_X(t)=E\left(e^{tX}\right)=\int_0^\infty e^{tx}\frac{xe^{-x/\beta}}{\beta^2}\left(1-\lambda+2\lambda e^{-x/\beta}\left(1+x/\beta\right)\right)dx$$

$$=\frac{1}{\beta^2}\left[(1-\lambda)\int_0^\infty xe^{-\frac{x}{\beta}(1-\beta t)}dx+2\lambda\left(\int_0^\infty xe^{-\frac{x}{\beta}(2-\beta t)}dx+\frac{1}{\beta}\int_0^\infty x^2e^{-\frac{x}{\beta}(2-\beta t)}dx\right)\right]$$

$$=\left[\frac{(1-\lambda)}{(1-\beta t)^2}+\frac{2\lambda(4-\beta t)}{(2-\beta t)^3}\right].$$

Theorem 3.3:

If X has the $T_{SBE}(x;\beta,\lambda)$ with $|\lambda| \leq 1$, then mean deviation about mean of X, say $E|X-\mu|$, is

$$E|X-\mu|=\frac{\beta e^{-\left(\frac{8-3\lambda}{4}\right)}}{16}\left[\lambda e^{-\left(\frac{8-3\lambda}{4}\right)}\left\{(8-3\lambda)(16-3\lambda)+40\right\}-8(\lambda-1)(16-3\lambda)\right]. \qquad (3.3)$$

Proof:

The mean deviation of the random variable X is given by

$$E\left|X-\mu\right| = \int_{-\infty}^{\infty} \left|X-\mu\right| f\left(x\right) dx .$$

$$= \int_{0}^{\infty} \left|X-\mu\right| \frac{xe^{-x/\beta}}{\beta^2} \left(1-\lambda+2\lambda e^{-x/\beta}\left(1+x/\beta\right)\right) dx . \tag{3.4}$$

Now we have

$$\int_{0}^{\infty} \left|X-\mu\right| xe^{-x/\beta} dx = \mu \int_{0}^{\mu} xe^{-x/\beta} dx - \int_{0}^{\mu} x^2 e^{-x/\beta} dx + \int_{\mu}^{\infty} x^2 e^{-x/\beta} dx - \mu \int_{\mu}^{\infty} xe^{-x/\beta} dx$$

and

$$\mu \int_{0}^{\mu} xe^{-x/\beta} dx = \mu \left[\left. \frac{xe^{-x/\beta}}{-1/\beta} \right|_0^{\mu} - \int_0^{\mu} \frac{e^{-x/\beta}}{-1/\beta} dx \right] = \mu \left[-\mu\beta e^{-\mu/\beta} + \beta \left. \frac{e^{-x/\beta}}{-1/\beta} \right|_0^{\mu} \right] .$$

$$= -\mu^2 \beta e^{-\mu/\beta} - \mu\beta^2 e^{-\mu/\beta} + \mu\beta^2 ,$$

Similarly, after solving (3.4), we have (3.3).

4. QUANTILE AND RANDOM NUMBER GENERATION

The p^{th} quantile x_p of the transmuted size biased exponential distribution can be obtained from (2.5) as

$$F\left(x_q\right) = \left(1+\lambda\right)\left(1-e^{-x_q/\beta}\left(1+x_q/\beta\right)\right) - \lambda\left[1-e^{-x_q/\beta}\left(1+x_q/\beta\right)\right]^2 = p ,$$

and after simple calculation this yields

$$e^{-x_p/\beta}\left(1+x_p/\beta\right) = \frac{\left(\lambda-1\right) \pm \sqrt{\left(\lambda-1\right)^2 - 4\lambda\left(p-1\right)}}{2\lambda} . \tag{4.1}$$

The above equation has no closed form solution in x_p, so we have to use a numerical technique such as a Newton- Raphson method to get the quantile. If we put p = 0.5 in equation (4.1) one gets the median of transmuted size biased exponential distribution. Further from (4.1), the values of x_p for $p \sim Uniform(0,1)$ gives the random values generated from transmuted size biased exponential distribution.

5. RELIABILITY ANALYSIS OF THE TRANSMUTED SIZED BIASED EXPONENTIAL DISTRIBUTION

The reliability function, measure the mortality or failure of a system. In other words it measures that system will survive beyond a specified time based on a certain distribution. By definition survival function is

$$R(t) = 1 - F(t) .$$

Now, reliability function of transmuted size biased exponential distribution is

$$R(t)=e^{-t/\beta}(1+t/\beta)\left[1-\lambda\left(1-e^{-t/\beta}(1+t/\beta)\right)\right]. \quad (5.1)$$

With various choices of parametric values the Figure 3 illustrates the reliability function pattern of transmuted size biased exponential distribution.

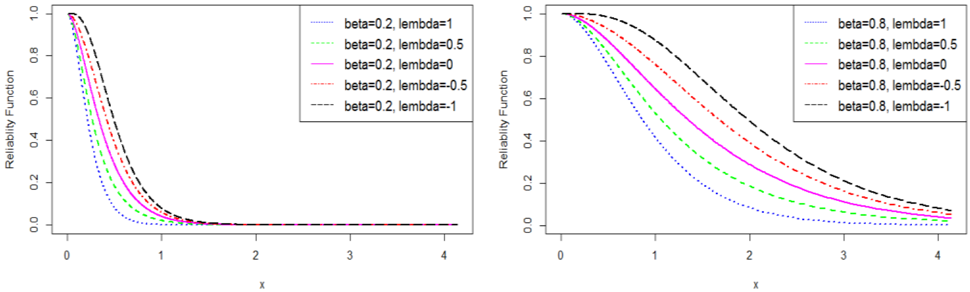

Fig. 3: Plots of Reliability Function for various Values of β and λ

It is important to note that $R(t)+F(t)=1$. One of the characteristic in reliability analysis is the hazard rate function. Hazard rate function is very useful in defining and formulating a model when dealing with lifetime data. It describes the current chance of failure for the population that has not yet failed. The general form of hazard function is defined as

$$h(\mathrm{t})=\frac{f(t)}{R(t)}.$$

The hazard function of transmuted size biased exponential distribution is given as

$$h(t)=\frac{t\left(1-\lambda+2\lambda e^{-t/\beta}(1+t/\beta)\right)}{\left(\beta^2+t\beta\right)\left[1-\lambda\left(1-e^{-t/\beta}(1+t/\beta)\right)\right]}.$$

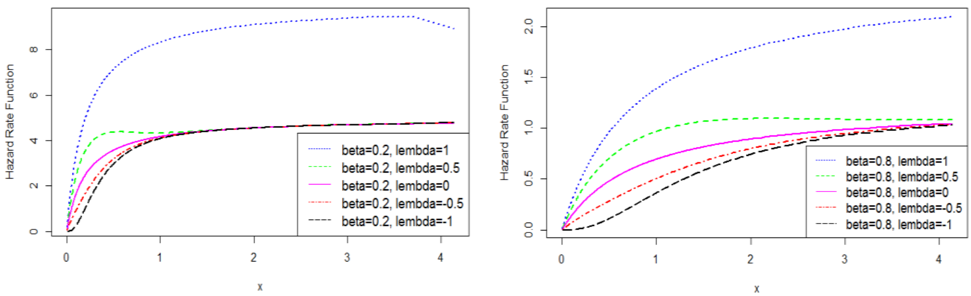

Fig. 4: Hazard Function Plots for Selected Values of β and λ

The cumulative hazard function of the transmuted size biased exponential distribution given as

$$H(t) = -\ln\left|e^{-t/\beta}(1+t/\beta)\left[1-\lambda\left(1-e^{-t/\beta}(1+t/\beta)\right)\right]\right|. \quad (5.2)$$

It is important to note that the units for $H(\mathrm{t})$ is the cumulative probability of failure per unit of time, distance or cycles. The hazard rate plots for TSBED for various values of parameters are given in the figure 4.

6. MEASURES OF UNCERTAINTY OF THE TRANSMUTED SIZED BIASED EXPONENTIAL DISTRIBUTION

Statistical entropy is a good measure of randomness or uncertainty associated with a random variable X and is a measure of a reduction in that uncertainty. The concept of entropy was introduced by Shannon (1948) pioneering work on the mathematical theory of communication in the nineteenth century. Entropy has been used as a major tool in engineering, information theory and other sciences. In this section, we present Shannon entropy and Renyi entropy for the transmuted size biased exponential distribution.

6.1. Shannon Entropy:

If X is continuous random variable has the $T_{SBE}(x;\beta,\lambda)$ distribution. Then the Shannon entropy is defined by

$$E\left[-log\ f(X)\right] = \int_0^\infty \left[-\log f(x)\right] f(x)\,dx.$$

We have

$$E\left[-log\ f(X)\right] = 2\log(\beta) - E\left[\log(X)\right] + E\left[X/\beta\right] - E\left[\log\left(1-\lambda+2\lambda e^{-X/\beta}(1+X/\beta)\right)\right].$$

Now, with the substitution $x/\beta = t$ and $\left(1-\lambda+2\lambda e^{-x/\beta}(1+x/\beta)\right) = y$, we can readily obtain both $E\left[\log(X)\right]$ and $E\left[\log\left(1-\lambda+2\lambda e^{-X/\beta}(1+X/\beta)\right)\right]$ respectively so that Shannon entropy for the transmuted size biased exponential distribution is given by

$$E[-log\ f(X)] = \log(2\beta) - \Gamma'(2) + \frac{\lambda}{4}\left(2\Gamma'(2) - \Gamma'(3)\right) + \left(\frac{8-3\lambda}{4}\right) - \frac{1}{8\lambda}\left[(1+\lambda)^2\left\{\log(1+\lambda)^2 - 1\right\} - (1-\lambda)^2\left\{\log(1-\lambda)^2 - 1\right\}\right], \quad (6.1)$$

where

$$E(X) = \frac{\beta(8-3\lambda)}{4},\ \Gamma'(2) = \int_0^\infty \log(t)\, t e^{-t} dt \text{ and } \Gamma'(3) = \int_0^\infty \log(t)\, t^2 e^{-t} dt.$$

6.2. Renyi Entropy

During the last couple of decades a number of research papers have extended Shannon's original work. Among others Park (1995), Wong and Chen (1990) provided some results on Shannon entropy for order statistics. Renyi (1961) who developed a one-parameter extension of Shannon entropy. If X is continuous random variable has the $T_{SBE}\left(x;\beta,\lambda\right)$ distribution, then the Renyi entropy is defined by

$$I_R\left(\gamma\right)=\frac{1}{(1-\gamma)}log\left[\int_R\left\{f\left(x\right)\right\}^{\gamma}dx\right], \text{ for } \gamma>0 \ , \quad \gamma\neq 1.$$

We have,

$$\int_0^{\infty}\left\{f(x;\beta,\lambda)\right\}^{\gamma}dx=\int_0^{\infty}\frac{x^{\gamma}e^{-x\gamma/\beta}}{\beta^{2\gamma}}\left[1-\lambda\left\{1-2e^{-x/\beta}\left(1+x/\beta\right)\right\}\right]^{\gamma}dx. \tag{6.2}$$

If $k>0$ and $|\,z\,|\leq 1$, we have the series of representations

$$(1-z)^k=\sum_{j=0}^{\infty}\frac{\left(-1\right)^j\Gamma(k+j)}{\Gamma\left(k\right)j!}z^j . \tag{6.3}$$

From (6.2) if $\left|\lambda\left\{1-2e^{-x/\beta}\left(1+x/\beta\right)\right\}\right|\leq 1$, we expand $\left(1-\lambda\left\{1-2e^{-x/\beta}\left(1+x/\beta\right)\right\}\right)^{\gamma}$ as in (6.3) and then (6.2) can be reduced to

$$\int_0^{\infty}f^{\gamma}\left(x;\beta,\lambda\right)dx=\sum_{i=1}^{\infty}\frac{\left(-1\right)^i\Gamma(\gamma+i)}{\Gamma\left(\gamma\right)i!}\int_0^{\infty}\frac{x^{\gamma}e^{-x\gamma/\beta}}{\beta^{2\gamma}}\left[\lambda\left\{1-2e^{-x/\beta}\left(1+x/\beta\right)\right\}\right]^i dx.$$

After similar simplifications, we can easily obtain the Renyi entropy as

$$I_R\left(\gamma\ \right)=\frac{1}{(1-\gamma\ )}log\left[\sum_{i,j,k=1}^{\infty}\frac{\left(-1\right)^{i+j}2^j\lambda^i\Gamma(\gamma+i)\Gamma(i+j)\Gamma(j+k)\Gamma\left(\gamma+k+1\right)}{\beta^{\gamma-1}\left(\gamma+j\right)^{\left(\gamma+k+1\right)}\Gamma\left(\gamma\right)\Gamma\left(i\right)\Gamma\left(j\right)i!\,j!k!}\right]. \tag{6.4}$$

7. MEAN RESIDUAL LIFE FUNCTION

The mean residual function gives an interpretable measure of how much more time to be expected to survive for an individual, given that one already reached the time point *t*. Assuming that X is a continuous random variable has $T_{SBE}\left(x;\beta,\lambda\right)$ with reliability function given in (5.1), the mean residual life function is given by (see, Abdous and Berred, 2005).

$$\mu\left(t\right)=E\left(X-t\,/\,X>t\right)=\frac{\int_t^{\infty}R\left(x\right)dx}{R\left(t\right)}$$

or $$\mu\left(t\right)=\frac{1}{R\left(t\right)}\left[\left(1-\lambda\right)\int_t^{\infty}e^{-x/\beta}\left(1+x/\beta\right)dx+\lambda\int_t^{\infty}e^{-2x/\beta}\left(1+x/\beta\right)^2dx\right].$$

After simplification the mean residual life function for proposed distribution is

$$\mu(t)=\frac{(1-\lambda)(2\beta+t)+\frac{\lambda e^{-t/\beta}}{4\beta}\left[(\beta+t)^2+(2\beta+t)^2\right]}{(1+t/\beta)\left[1-\lambda\left(1-e^{-t/\beta}(1+t/\beta)\right)\right]}. \tag{7.1}$$

8. ORDER STATISTICS

In fact, the order statistics have many applications in reliability and life testing. The order statistics arise in the study of reliability of a system. Let $X_1, X_2, ..., X_n$ be a simple random sample from $T_{SBE}\ (x;\beta,\lambda)$ distribution with cumulative distribution function and probability density function given in (2.5) and (2.6), respectively. Let $X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)}$ denote the order statistics obtained from this sample. In reliability literature, $X_{(j)}$ denote the lifetime of an $(n-j+1)$ out of n system which consists of n independent and identically components. Then the pdf of $X_{(j)}, 1 \le j \le n$ order statistics follows $T_{SBE}(x;\beta,\lambda)$ is given by

$$f\left(x_{(j)}\right)=\frac{n!}{(j-1)!(n-j)!} f\left(x_{(j)}\right)\left[F\left(x_{(j)}\right)\right]^{j-1}\left[1-F\left(x_{(j)}\right)\right]^{n-j}$$

$$=\frac{n!}{(j-1)!(n-j)!}\frac{xe^{-x_{(j)}/\beta}}{\beta^2}\left(1-\lambda+2\lambda e^{-x_{(j)}/\beta}\left(1+x_{(j)}/\beta\right)\right)$$

$$\times\left[\left(1-e^{-x_{(j)}/\beta}\left(1+x_{(j)}/\beta\right)\right)\left[1+\lambda\left(e^{-x_{(j)}/\beta}\left(1+x_{(j)}/\beta\right)\right)\right]\right]^{j-1}$$

$$\times\left[e^{-x_{(j)}/\beta}(1+x/\beta)\left[1-\lambda\left(1-e^{-x_{(j)}/\beta}\left(1+x_{(j)}/\beta\right)\right)\right]\right]^{n-j}.$$

8.1. Distribution of Minimum, Maximum and Median

Let $X_1, X_2, ..., X_n$ be independently identically distributed order random variables from the transmuted size biased exponential distribution having smallest, largest and median order probability density function are given by the following.

The *pdf* of the first or smallest $X_{(1)} = Min\left(X_1, X_2, ..., X_n\right)$ order statistics is given by,

$$f(x_{(1)}) = n[1-F(x_{(1)})]^{n-1} f(x_{(1)})$$

$$=\frac{n\,x_{(1)}e^{-x_{(1)}/\beta}}{\beta^2}\left(1-\lambda+2\lambda e^{-x_{(1)}/\beta}\left(1+x_{(1)}/\beta\right)\right)$$

$$\times\left[e^{-x_{(1)}/\beta}\left(1+x_{(1)}/\beta\right)\left[1-\lambda\left(1-e^{-x_{(1)}/\beta}\left(1+x_{(1)}/\beta\right)\right)\right]\right]^{n-1}.$$

The *pdf* of the last or largest $X_{(n)} = Max\left(X_1, X_2, ..., X_n\right)$ order statistics is given by,

$$
\begin{aligned}
f(x_{(n)}) &= n[F(x_{(n)})]^{n-1} f(x_{(n)}) \\
&= \frac{nx_{(n)}e^{-x_{(n)}/\beta}}{\beta^2}\left(1-\lambda+2\lambda e^{-x_{(n)}/\beta}\left(1+x_{(n)}/\beta\right)\right) \\
&\quad\times\left[\left(1-e^{-x_{(n)}/\beta}\left(1+x_{(n)}/\beta\right)\right)\left[1+\lambda\left(e^{-x_{(n)}/\beta}\left(1+x_{(n)}/\beta\right)\right)\right]\right]^{n-1}.
\end{aligned}
$$

The *pdf* of the median $X_{(m+1)}$ order statistics is given by,

$$
\begin{aligned}
f(x_{(m+1)}) &= \frac{(2m+1)!}{m!m!}\left(F(x_{(m+1)})\right)^m\left(1-F(x_{(m+1)})\right)^m f(x_{(m+1)}) \\
&= \frac{(2m+1)!}{m!m!}\left[\left(1-e^{-x_{(m+1)}/\beta}\left(1+x_{(m+1)}/\beta\right)\right)\left[1+\lambda\left(e^{-x_{(m+1)}/\beta}\left(1+x_{(m+1)}/\beta\right)\right)\right]\right]^m \\
&\quad\times\left[e^{-x_{(m+1)}/\beta}\left(1+x_{(m+1)}/\beta\right)\left[1-\lambda\left(1-e^{-x_{(m+1)}/\beta}\left(1+x_{(m+1)}/\beta\right)\right)\right]\right]^m \\
&\quad\times\left(\frac{x_{(m+1)}e^{-x_{(m+1)}/\beta}}{\beta^2}\left(1-\lambda+2\lambda e^{-x_{(m+1)}/\beta}\left(1+x_{(m+1)}/\beta\right)\right)\right).
\end{aligned}
$$

8.2. Joint Distribution of the j^{th} and k^{th} Order Statistics

The joint density distribution of the j^{th} order statistics $X_{(j)}$ and k^{th} order statistics $X_{(k)}$ from transmuted size biased exponential distribution is given by

$$
\begin{aligned}
f\left(x_{(j)}, x_{(k)}\right) &= C\,[F(x_{(j)})]^{j-1}\,[F(x_{(k)})-F(x_j)]^{k-j-1}[1-F(x_{(k)})]^{n-k}\,f(x_{(j)})f(x_{(k)}) \\
&= C\left[\left(1-e^{-x_{(j)}/\beta}\left(1+x_{(j)}/\beta\right)\right)\left[1+\lambda\left(e^{-x_{(j)}/\beta}\left(1+x_{(j)}/\beta\right)\right)\right]\right]^{j-1} \\
&\quad\times\left[\left(1-e^{-x_{(k)}/\beta}\left(1+x_{(k)}/\beta\right)\right)\left[1+\lambda\left(e^{-x_{(k)}/\beta}\left(1+x_{(k)}/\beta\right)\right)\right]\right. \\
&\quad\left.-\left(1-e^{-x_{(j)}/\beta}\left(1+x_{(j)}/\beta\right)\right)\left[1+\lambda\left(e^{-x_{(j)}/\beta}\left(1+x_{(j)}/\beta\right)\right)\right]\right]^{k-j-1} \\
&\quad\times\left[e^{-x_{(k)}/\beta}\left(1+x_{(k)}/\beta\right)\left[1-\lambda\left(1-e^{-x_{(k)}/\beta}\left(1+x_{(k)}/\beta\right)\right)\right]\right]^{n-k} \\
&\quad\times\frac{x_{(j)}x_{(k)}e^{-\frac{1}{\beta}\left(x_{(j)}+x_{(k)}\right)}}{\beta^4}\left[1-\lambda+2\lambda e^{-x_{(j)}/\beta}\left(1+x_{(j)}/\beta\right)\right]
\end{aligned}
$$

$$\left[1-\lambda+2\lambda e^{-x_{(k)}/\beta}\left(1+x_{(k)}/\beta\right)\right]$$

where,

$$C = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}.$$

9. PARAMETER ESTIMATION AND INFERENCE

In this section, we have used method of moments and method of maximum likelihood to estimate the unknown parameters of the transmuted size biased exponential distribution.

9.1. Method of Moments

To find the estimators of the parameters of a distribution with method of moments (MMs), we equate the population moments to the sample moments. Given a random sample $x_1, x_2, ..., x_n$, of size n from transmutes size biased exponential distribution with pdf (2.6), then from (3.4) we have the following system of two equations

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \beta\left(\frac{8-3\lambda}{4}\right)$$

or

$$8\beta - 3\lambda\beta - 4\bar{x} = 0 \tag{9.1}$$

and

$$\bar{x}' = \frac{1}{n}\sum_{i=1}^{n} x_i^2 = 3\beta^2\left(\frac{8-5\lambda}{4}\right)$$

or

$$24\beta^2 - 15\lambda\beta^2 - 4\bar{x}' = 0 \tag{9.2}$$

(9.1) is being multiplied by 5β, subtracting (9.2) from (9.1) and solving for β, we get

$$\hat{\beta} = \frac{5\bar{x} \pm \sqrt{(5\bar{x})^2 - (16)\bar{x}'}}{8}, \quad \sqrt{(5\bar{x})^2 - (16)\bar{x}'} \text{ should exist.} \tag{9.3}$$

Putting the value of $\hat{\beta}$ in (9.1) and solving for λ, we get

$$\hat{\lambda} = \frac{8\left(\bar{x} \pm \sqrt{(5\bar{x})^2 - (16)\bar{x}'}\right)}{3\left(5\bar{x} \pm \sqrt{(5\bar{x})^2 - (16)\bar{x}'}\right)}, \quad \sqrt{(5\bar{x})^2 - (16)\bar{x}'} \text{ should exist.} \tag{9.4}$$

9.2. Method of Maximum Likelihood Estimation

In this section, we discuss the maximum likelihood estimators (MLE's) and inference for the $T_{SBE}(x;\beta,\lambda)$ distribution. Let $x_1, x_2, ..., x_n$ be a random sample of size n from $T_{SBE}(x;\beta,\lambda)$ distribution then the sample log likelihood function is given by

$$\log(\ell) = -2n\ln\beta + \sum_{i=1}^{n}\ln x_i - \frac{1}{\beta}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\ln\left(1-\lambda+2\lambda e^{-x_i/\beta}\left(1+x_i/\beta\right)\right) \qquad (9.5)$$

To find the parameter estimates, we partially differentiate the log likelihood function with respect to respected parameters (β,λ) and then equating to zero respectively

$$\frac{\partial\log(\ell)}{\partial\beta} = -\frac{2n}{\beta} + \frac{\sum_{i=1}^{n} x_i}{\beta^2} + \sum_{i=1}^{n}\frac{2\lambda x_i^2/\beta^3 e^{-x_i/\beta}}{\left(1-\lambda+2\lambda e^{-x_i/\beta}\left(1+x_i/\beta\right)\right)} = 0,$$

$$\frac{\partial\log(\ell)}{\partial\lambda} = \sum_{i=1}^{n}\frac{2e^{-x_i/\beta}\left(1+x_i/\beta\right)-1}{\left(1-\lambda+2\lambda e^{-x_i/\beta}\left(1+x_i/\beta\right)\right)} = 0.$$

The maximum likelihood estimator $\hat{\theta} = \left(\hat{\beta},\hat{\lambda}\right)'$ of $\theta = (\beta,\lambda)'$ is achieved by solving above nonlinear equations. For numerically maximize the log-likelihood function given in (9.5) we use an appropriate numerical solution algorithm such as the quasi-Newton algorithm. For $n \to \infty$, the MLEs of θ can be treated as being approximately bivariate normal with mean 0 and variance-covariance matrix equal to the inverse of the expected information matrix. That is,

$$\sqrt{n}\left(\hat{\theta}-\theta\right) \to N\left(0, I^{-1}\left(\hat{\theta}\right)\right),$$

where $I^{-1}\left(\hat{\theta}\right)$ is the variance-covariance matrix of the unknown parameters $\theta = (\beta,\lambda)'$. By $I_{ij}\left(\hat{\theta}\right)$, we can approximate the elements of the 2×2 matrix I^{-1}, $I_{ij}\left(\hat{\theta}\right), i, j = 1, 2,$ where $I_{ij}\left(\hat{\theta}\right) = -\log(\ell)_{\theta_i,\theta_j}\big|_{\theta=\hat{\theta}}$. Also hence as $n \to \infty$ the asymptotic distribution of the MLE $(\hat{\beta},\hat{\lambda})$ is given by

$$\begin{pmatrix}\hat{\beta}\\ \hat{\lambda}\end{pmatrix} \sim N\left[\begin{pmatrix}\beta\\ \lambda\end{pmatrix}, \begin{pmatrix}\hat{V}_{11} & \hat{V}_{12}\\ \hat{V}_{21} & \hat{V}_{22}\end{pmatrix}\right]$$

where, $V_{ij} = V_{ij}\big|_{\theta=\hat{\theta}}$ and $\begin{pmatrix}V_{11} & V_{12}\\ V_{21} & V_{22}\end{pmatrix} = \begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}^{-1}$ is the approximate variance covariance matrix with its elements obtained from

$$A_{11} = -\frac{\partial^2 \ln L}{\partial \beta^2},\ A_{12} = A_{21} = -\frac{\partial^2 \ln L}{\partial \beta\, \partial \lambda} \text{ and } A_{22} = -\frac{\partial^2 \ln L}{\partial \lambda^2}.$$

We have,

$$A_{11} = -\frac{1}{\beta^2}2n + \frac{2}{\beta^3}\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{2\lambda x_i^2 e^{-x_i/\beta}\left[\beta(1-\lambda)(x_i - 3\beta) - 2\lambda\beta e^{-x_i/\beta}(2x_i + 3\beta)\right]}{\left[\beta^3\left(1-\lambda+2\lambda e^{-x_i/\beta}(1+x_i/\beta)\right)\right]^2},$$

$$A_{22} = -\frac{\partial^2 \log(\ell)}{\partial \lambda^2} = -\sum_{i=1}^{n}\left(\frac{2e^{-x_i/\beta}(1+x_i/\beta)-1}{\left(1-\lambda+2\lambda e^{-x_i/\beta}(1+x_i/\beta)\right)}\right)^2$$

and

$$A_{12} = A_{21} = -\sum_{i=1}^{n} \frac{2x_i^2 e^{-x_i/\beta}}{\beta^3\left[\left(1-\lambda+2\lambda e^{-x_i/\beta}(1+x_i/\beta)\right)\right]^2}.$$

Approximate $100(1-\alpha)\%$ two sided confidence intervals for β and λ are respectively given by

$$\hat{\beta} \pm z_{\alpha/2}\sqrt{I_{11}(\hat{\theta})} \text{ and } \hat{\lambda} \pm z_{\alpha/2}\sqrt{I_{22}(\hat{\theta})}$$

where $z_{\alpha/2}$ is the upper α^{th} percentiles of the standard normal distribution. Using *R* we can easily calculate the values of the standard error and asymptotic confidence intervals. We can also compute the Hessian matrix and its inverse. In any case, hypothesis is $H_0 : \psi = \psi_0$ verses $H_1 : \psi \neq \psi_0$, where ψ vector is formed with some components of θ and ψ_0 is a specified vector. For example, comparing transmuted size biased exponential distribution with size biased exponential distribution yields the hypothesis

$H_0 : \lambda = 0$ or the two models are equally close to the true data.

$H_1 : T_{SBE(x;\beta,\lambda)}$ is better than $S_{BE}(x;\beta)$ or proposed model is more closer to the true data.

We use the likelihood ratio (LR) test statistic to check whether the transmuted size biased exponential distribution for a given data set is statistically superior to the size biased exponential distribution. We compute the maximized unrestricted and restricted log-likelihood functions to construct the LR test statistic. The (LR) statistic is defined as

$$\omega = -2\log\frac{\ell(\theta_0)}{\ell(\theta_1)} = 2\left[\log\left(\ell(\theta_1)\right) - \log\left(\ell(\theta_0)\right)\right],$$

where θ_1 and θ_0 are the MLEs under H_1 and under H_0 respectively. The LR test statistic ω for testing H_0 versus H_1 is asymptotically distributed as χ_k^2, where k is the length of the parametric vector θ of interest. The LR test rejects H_0 if $\omega > \chi^2_{k;\gamma}$, where $\chi^2_{k;\gamma}$ denotes the upper $100\gamma\%$ quantile of the χ^2_k distribution.

We also compute AIC (Akaike Information Criterion), AICC (Akaike Information Criterion Corrected), BIC (Bayesian information criterion), Kolmogorov–Smirnov (K–S), Statistic and Cramer-von Mises Statistic (C_n). The statistics of the criterion are respectively defined as

$$AIC = 2p - 2\log(\ell), \qquad AICC = AIC + \frac{2p(p+1)}{n-p-1},$$

$$BIC = p.\log(n) - 2\log(\ell), \quad K-S = \max_{1\le i\le n}\left(F(X_i) - \frac{i-1}{n}, \frac{i}{n} - F(X_i)\right),$$

and

$$C_n = \frac{1}{12n} + \sum_{i=1}^{n}\left(F(X_i) - \frac{2i-1}{2n}\right)^2$$

where 'p' is number of parameters in the model and 'n' is sample size.

10. APPLICATION OF TRANSMUTED SIZE-BIASED EXPONENTIAL DISTRIBUTION

In this section, we use real data sets to show that the transmuted size biased exponential distribution (TSBE) is a better model than one based on the size biased exponential distribution (SBED). The data set given in table 4 represents an uncensored data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients reported by Merovci(2013) and lee and Wang (2003). We use several initial values to find the best fit for each model. We obtain the following results

Table 4
Remission Times (in months) of 128 Bladder Cancer Patients

00.08	02.09	03.48	04.87	06.94	08.66	13.11	23.63	00.20	02.23
03.52	04.98	06.97	09.02	13.29	00.40	02.26	03.57	05.06	07.09
09.22	13.80	25.74	00.50	02.46	03.64	05.09	07.26	09.47	14.24
25.82	00.51	02.54	03.70	05.17	07.28	09.74	14.76	26.31	00.81
02.62	03.82	05.32	07.32	10.06	14.77	32.15	02.64	03.88	05.32
07.39	10.34	14.83	34.26	00.90	02.69	04.18	05.34	07.59	10.66
15.96	36.66	01.05	02.69	04.23	05.41	07.62	10.75	16.62	43.01
01.19	02.75	04.26	05.41	07.63	17.12	46.12	01.26	02.83	04.33
07.66	11.25	17.14	79.05	01.35	02.87	05.62	07.87	11.64	17.36
01.40	03.02	04.34	05.71	07.93	11.79	18.10	01.46	04.40	05.85
08.26	11.98	19.13	01.76	03.25	04.50	06.25	08.37	12.02	02.02
03.31	04.51	06.54	08.53	12.03	20.28	02.02	03.36	06.76	12.07
21.73	02.07	03.36	06.93	08.65	12.63	22.69	05.49		

Table 5
Summary Statistics for Remission Times (in months) of 128 Bladder Cancer Patients

Mean	Median	Variance	S.D	Skewness	Kurtosis
9.366	6.395	110.425	10.508	3.287	18.483

Table 6 shows the values of $-2\log(\ell)$, AIC, AICC, BIC, $K-S$ and C_n values where Table 7 shows the MMs and MLEs for the transmuted size biased exponential distribution (TSBED), size biased exponential distribution (SBED), size biased Pareto distribution and size biased Maxwell distribution (SBMD).

Table 6
Criteria of Comparison

Model	$-2\log(\ell)$	AIC	AICC	BIC	$K-S$	C_n
TSBED	843.46	847.5	847.6	853.2	0.11	3.44
SBED	853.59	855.6	855.6	858.5	0.14	4.49
SBMD	1338.73	1340.73	1340.76	1343.58	0.49	12.16
SBPD	1077.05	1079.05	1079.08	1081.90	0.42	7.06

In Table 6, the statistic values of all criteria are small for transmuted size biased exponential distribution. These indicate that the transmuted size biased exponential distribution leads to a better fit than the other distributions.

The LR statistics to test the hypotheses $H_0 : \lambda = 0$ versus $H_1 : \lambda \neq 0$: $\omega = 20.2552 > 3.841 = \chi^2_{1,(0.05)}$, so we reject the null hypothesis.

Table 7
Parameter Estimates

Model	Method of Moments	Maximum Likelihood
TSBED	$\hat{\beta} = 4.6988$ $\hat{\lambda} = 0.6405$	$\hat{\beta} = 5.74$ $\hat{\lambda} = 0.58$
SBED	$\hat{\beta} = 4.6828$	$\hat{\beta} = 4.68$
SBPD	$\hat{\beta} = 2.0086$	$\hat{\beta} = 1.234$
SBMD	$\hat{\beta} = 4.9818$	$\hat{\beta} = 7.023$

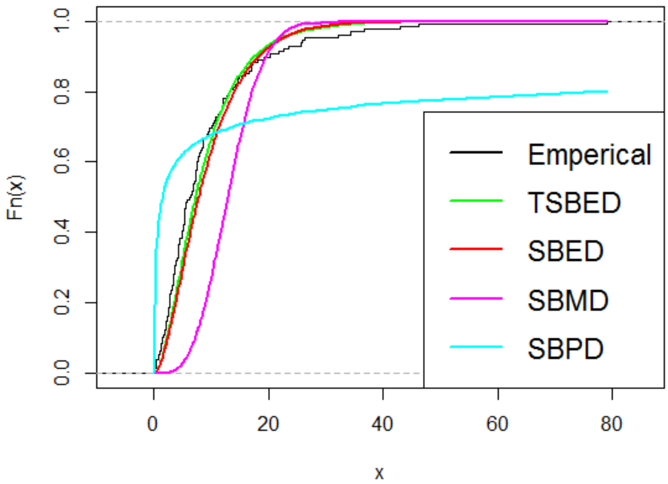

Fig. 5: Empirical, Fitted TSBED, SBED, SBPD and SBMD's CDF of the Remission Times (in months) of 128 Bladder Cancer Patients Data

11. CONCLUSIONS

In this article, we propose a new model i.e. the transmuted size biased exponential distribution which extends the application of size biased exponential distribution in the analysis of data with real support. An obvious reason for generalizing a standard distribution is because the generalized form provides greater flexibility in modeling real data. We derive expansions for expectation, variance, mean deviation, moments and the moment generating function. The parameters estimation is accomplished by the method of maximum likelihood and method of moments. The graph of hazard rate function and reliability behavior of the transmuted size biased exponential distribution demonstrates that the developed distribution can also be used to model reliability or life time data. Also the expression for entropy and mean residual life function for proposed distribution are also derived. The LR and other well-known statistic are used to equate the proposed model with its baseline and some other models. The application of transmuted size biased exponential distribution to real life data show that the new distribution provides quite effective results and better fits than the size biased exponential distribution and also other distributions used in comparison. We expect that this research will serve as a reference and help to advance future research in the subject area.

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