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THE ODD LINDLEY BURR XII DISTRIBUTION WITH APPLICATIONS

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ABSTRACT

This paper proposes a new four-parameter distribution called the odd Lindley Burr XII (OLBXII) distribution. The hazard rate function of the OLBXII distribution can be constant, increasing, decreasing, unimodal or bathtub shape. A comprehensive account of some of its mathematical properties are derived. The density function of the proposed model can be expressed as a linear combination of Burr XII densities. The esimation of the model parameters is carid out using the maximum likelihood method. The importance and flexibility of the proposed model are proved empirically using two real data sets.

KEYWORDS

Burr XII Distribution, Maximum Likelihood, Moments, Odd Lindley G-Family, Order Statistics.

1. INTRODUCTION

There are hundreds of continuous distributions in the statistical literature. These distributions have several applications in many applied fields such as reliability, life testing, biomedical sciences, economics, finance, environmental and engineering, among others. However, These applications have proven that the real data following the well-know models are more often the exception rather than the reality. In order to increase the flexibility of the well-known distribution, many authors have proposed different transformations of these models and have extensively used these extended forms in modelling data in several areas.

The Burr-XII (BXII) distribution (Burr, 1942) has many applications in several areas including failure time modeling, reliability and acceptance sampling plans. Tadikamalla (1980) discussed the BXII model and its related models, namely: Pareto II (Lomax), log-logistic, compound Weibull gamma and Weibull exponential distributions. Shao (2004) showed that the extended BXII distribution can be used to model extreme events and applied it for modeling flood frequency.

Recently, there are many generalized forms of the BXII distribution. For example, the beta BXII (Paranaba et al., 2011), the Kumaraswamy BXII (Paranaba et al., 2013), the beta exponentiated BXII (Mead, 2014), the Marshall-Olkin extended BXII (Al-Saiarie et al., 2014), the McDonald BXII (Gomes et al., 2015), the exponentiated Burr XII Poisson (da Silva et al., 2015) the Kumaraswamy exponentiated BXII (Mead and Afify, 2017) and the Weibull BXII (Afify et al., 2016a) distributions. Further, Nasir et al. (2017) proposed a new family based on the BXII distribution.

The cumulative distribution function (cdf) and probability density function (pdf) of the three parameter BXII distribution are, respectively, defined (for $x > 0$) by

$$G(x;\alpha,\beta,\sigma) = 1 - \left[1 + \left(\frac{x}{\sigma}\right)^{\alpha}\right]^{-\beta} \tag{1}$$

and

$$g(x;\alpha,\beta,\sigma) = \alpha\beta\sigma^{-\alpha}x^{\alpha-1}\left[1 + \left(\frac{x}{\sigma}\right)^{\alpha}\right]^{-\beta-1}, \tag{2}$$

where α and β are positive shape parameters and σ is a scale parameter.

Recently, Cakmakyapan and Ozel (2016) defined the Lindley family of distributions and Gomes-Silva et al. (2017) proposed and studied a new wider family of continuous distribution called the *Odd Lindley-G* (OL-G) family.

Consider the pdf and cdf of a baseline model with parameter vector $\boldsymbol{\zeta}$, denoted by $g(x;\boldsymbol{\zeta})$ and $G(x;\boldsymbol{\zeta})$. Then, the cdf of the OL-G family is given by

$$\begin{aligned} F(x;\theta,\boldsymbol{\zeta}) &= \frac{\theta^2}{1+\theta}\int_0^{\left[\frac{G(x;\boldsymbol{\zeta})}{\overline{G}(x;\boldsymbol{\zeta})}\right]}(1+t)\,e^{-\theta}dt \\ &= 1 - \frac{\theta + \overline{G}(x;\boldsymbol{\zeta})}{(1+\theta)\overline{G}(x;\boldsymbol{\zeta})}\exp\left[-\theta\frac{G(x;\boldsymbol{\zeta})}{\overline{G}(x;\boldsymbol{\zeta})}\right], \end{aligned} \tag{3}$$

where θ is a positive shape parameter.

The pdf of the OL-G family is given by

$$f(x;\theta,\boldsymbol{\zeta}) = \frac{\theta^2}{1+\theta}\frac{g(x;\boldsymbol{\zeta})}{\overline{G}(x;\boldsymbol{\zeta})^3}\exp\left[-\theta\frac{G(x;\boldsymbol{\zeta})}{\overline{G}(x;\boldsymbol{\zeta})}\right]. \tag{4}$$

We use $X \sim$OL-G$(\theta,\boldsymbol{\zeta})$ to denote the random variable X with pdf (4). The hazard rate function (hrf) of the OL-G class is

$$h(x;\theta,\boldsymbol{\zeta}) = \frac{\theta^2\tau(x;\boldsymbol{\zeta})\exp\left[-\theta\frac{G(x;\boldsymbol{\zeta})}{\overline{G}(x;\boldsymbol{\zeta})}\right]}{\overline{G}(x;\boldsymbol{\zeta})\left\{\theta + \overline{G}(x;\boldsymbol{\zeta})\exp\left[-\theta\frac{G(x;\boldsymbol{\zeta})}{\overline{G}(x;\boldsymbol{\zeta})}\right]\right\}},$$

where $\tau(x;\boldsymbol{\zeta})$ is the hrf of the baseline model.

The aim of the present work is to define a new lifetime model called the *odd Lindley Burr XII* (OLBXII) distribution. Its main feature is that one additional shape parameter is inserted in equation (1) to provide great flexibility for the new distribution. We construct the four-parameter OLBXII distribution using the OL-G family of distributions (Gomes-Silva et al., 2017) and derive some of its mathematical properties.

The OLBXII distribution is motivated by the following: The OLBXII distribution exhibits monotone as well as non-monotone hazard rates which makes this distribution to be superior to other lifetime distributions, which exhibit only monotonically increasing/decreasing, or constant hazard rates. It is shown in Section 3 that the OLBXII distribution can be viewed as a mixture of BXII distribution introduced by Burr (1942). It can be viewed as a suitable model for fitting the skewed data which may not be properly fitted by other common distributions. Furthermore, the OLBXII distribution outperforms several of the well-known lifetime distributions with respect to two real data examples.

The remainder of the article is unfolded as follows: in Section 2, we define the new OLBXII distribution, give its special models and provide some possible shapes for its pdf and hrf. We derive a useful linear mixture representation for the pdf and cdf of the OLBXII distribution in Section 3. In Section 4, some mathematical properties of the OLBXII distribution are obtained. The model parameters are estimated via the maximum likelihood estimates (MLEs) in Section 5. In Section 6, we propose some simulations in terms of the sample size n to investigate the behavior of the MLEs. In Section 7, the importance and potentiality of OLBXII distribution are shown using two real data sets. Finally, some conclusions are provided in Section 8.

2. THE OLBXII DISTRIBUTION

By inserting (1) and (2) in equations (3) and (4). Then, we have the cdf and pdf of the OLBXII distribution. The cdf of OLBXII is given by (for $x > 0$)

$$F(x) = 1 - \frac{\theta + \left[1 + \left(\frac{x}{\sigma}\right)^{\alpha}\right]^{-\beta}}{(1+\theta)\left[1 + \left(\frac{x}{\sigma}\right)^{\alpha}\right]^{-\beta}} \exp\left\{\theta - \theta\left[1 + \left(\frac{x}{\sigma}\right)^{\alpha}\right]^{\beta}\right\}. \quad (5)$$

The pdf of the OLBXII distribution is

$$f(x) = \frac{\theta^2 \alpha\beta\sigma^{-\alpha}}{1+\theta} x^{\alpha-1}\left[1 + \left(\frac{x}{\sigma}\right)^{\alpha}\right]^{2\beta-1} \exp\left\{\theta - \theta\left[1 + \left(\frac{x}{\sigma}\right)^{\alpha}\right]^{\beta}\right\}, \quad (6)$$

where σ is a scale parameter and α, β and θ are shape parameters. Figure 1 displays some possible shapes of the pdf of the OLBXII distribution. Figure 2 shows plots of the hrf of OLBXII distribution for given values of the parameters. From Figure 2, one can see that the hrf can be bathtub, upside down bathtub (unimodal), increasing, decreasing or constant. Thus, the proposed OLBXII distribution will be useful to model various shapes of real data. The OLBXII distribution contains the following two special models

- For $\alpha = 1$, we have the odd Lindley Lomax (OLLo) distribution.
- For $\beta = 1$, the OLBXII distribution reduces to the odd Lindley log-logistic (OLLL) distribution.

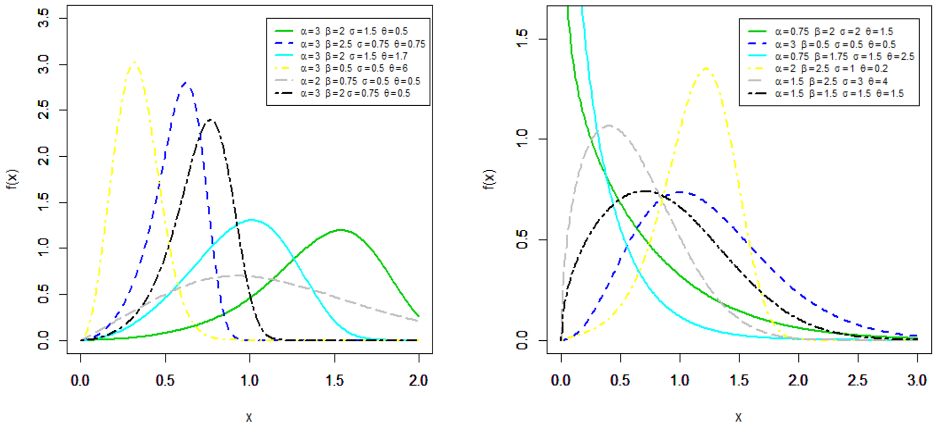

Figure 1: The OLBXII pdf Plots for Selected Values of the Parameters

3. LINEAR REPRESENTATION

A useful expansion for the OLBXII pdf in terms of BXII densities is provided in this section. Using the exponential series, the OL-G density in (4), reduces to

$$f(x) = \frac{g(x)}{1+\theta} \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{k+2}}{k!} G(x)^k \overline{G}(x)^{-k-3}.$$

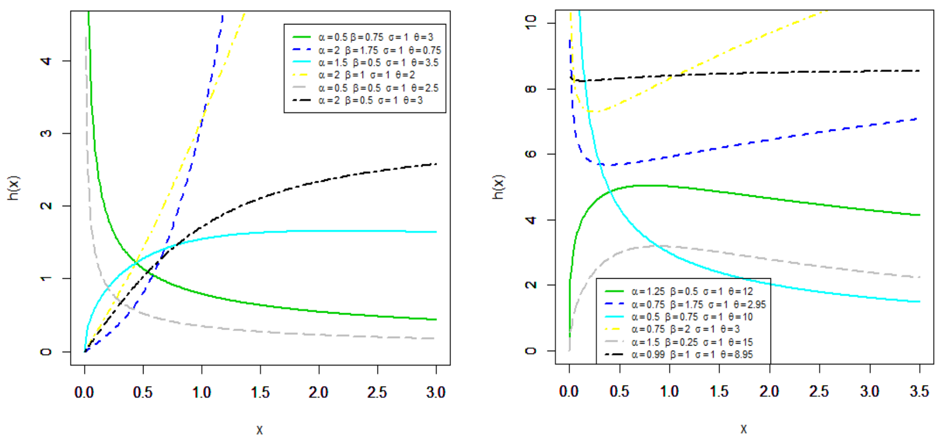

Figure 2: The OLBXII hrf Plots for Given Values of the Parameters

Using the generalized binomial expansion to the term $\overline{G}(x)^{-k-3}$, we have

$$f(x) = \frac{g(x)}{1+\theta} \sum_{k,i=0}^{\infty} \frac{(-1)^k \theta^{k+2} \Gamma(k+i+3)}{k!\, i!\, \Gamma(k+3)} G(x)^{k+i}.$$

Using the pdf and cdf of the BXII distribution in (1) and (2), we obtain

$$f(x) = \frac{\alpha\beta}{\sigma^{\alpha}} x^{\alpha-1} \sum_{k,i=0}^{\infty} \frac{(-1)^k \theta^{k+2} \Gamma(k+i+3)}{k!\, i!\, (\theta+1)\Gamma(k+3)} \left[1 + \left(\frac{x}{\sigma}\right)^{\alpha}\right]^{-\beta-1} \left\{1 - \left[1 + \left(\frac{x}{\sigma}\right)^{\alpha}\right]^{-\beta}\right\}^{k+i}.$$

Applying the generalized binomial expansion to the last term, and after some algebra, it is seen that

$$f(x) = \frac{\alpha\beta}{\sigma^{\alpha}} x^{\alpha-1} \sum_{k,i,j=0}^{\infty} \frac{(-1)^{k+j} \theta^{k+2} \Gamma(k+i+3)}{k!\, i!\, (\theta+1)\Gamma(k+3)} \binom{k+i}{j} \left[1 + \left(\frac{x}{\sigma}\right)^{\alpha}\right]^{-\beta(j+1)-1}.$$

or equivalently, we can write

$$f(x) = \sum_{j=0}^{\infty} \delta_j \, g_{\beta(j+1)}(x), \tag{7}$$

where

$$\delta_j = \sum_{k,i=0}^{\infty} \frac{(-1)^{k+j} \theta^{k+2} \Gamma(k+i+3)}{k!\, i!\, (\theta+1)(j+1)\Gamma(k+3)} \binom{k+i}{j},$$

and $g_{\beta(j+1)}(x)$ is the BXII density with shape parameters α and $\beta(j+1)$, and scale parameter σ. Equation (7) reveals that the OLBXII pdf can be written as a linear combination of BXII densities. So, many of its mathematical properties follows from those of the BXII distribution.

Let Z be a random variable having the BXII distribution in (1) with parameters α, β and σ. For $\alpha\beta > r$, the rth ordinary moment and the rth incomplete moment of X are, respectively, defined by

$$\mu'_{r,Z} = \sigma^r \beta \, B\left(\beta - \frac{r}{\alpha}, 1 + \frac{r}{\alpha}\right) \text{ and } \varphi_{r,Z}(t) = \sigma^r \beta \, B\left(t^{\alpha}; \beta - \frac{r}{\alpha}, 1 + \frac{r}{\alpha}\right),$$

where $B(a,b) = \int_0^{\infty} u^{a-1}(1+u)^{-(a+b)} du$ and $B(w; a, b) = \int_0^{w} u^{a-1}(1+u)^{-(a+b)} du$ are the beta and the incomplete beta functions of the second type, respectively.

By integrating (7), the OLBXII cdf has the following linear representation

$$F(x) = \sum_{j=0}^{\infty} \delta_j \, G_{\beta(j+1)}(x),$$

where $G_{\beta(j+1)}(x)$ is the BXII cdf with shape parameters α and $\beta(j+1)$, and scale parameter σ.

4. THE OLBXII PROPERTIES

In this section, some mathematical properties of the OLBXII are investigated, these properties are: quantile functoin (qf), ordinary and incomplete moments, moment generating function (mgf), mean residual life (MRL), mean inactivity time (MIT) and order statistics.

4.1 Quantile Function

The qf of the OLBXII distribution model, say $Q(\lambda)$ where $0 < \lambda < 1$, is obtained by solving $F\big(Q(\lambda)\big) = \lambda$ in (5) for $Q(\lambda)$ in terms of λ, and it is gaiven by

$$Q(\lambda) = \sigma\left[\left(\frac{1}{\theta}\left(W_{-1}\left[(\lambda-1)(\theta+1)e^{-(\theta+1)}\right]-1\right)\right)^{\frac{1}{\beta}}-1\right]^{1/\alpha}, \quad (8)$$

where $W(x)$ is the negative branch of the Lambert function.

4.2 Ordinary and Incomplete Moments

The nth ordinary moment of X, is given by

$$\mu_n' = E(X^n) = \sum_{j=0}^{\infty} \delta_j \int_0^{\infty} x^n \, g_{\beta(j+1)}(x)dx,$$

for $\alpha\beta > n$, we obtain

$$\mu_n' = E(X^n) = \sum_{j=0}^{\infty} \delta_j \sigma^n \beta(j+1)\, B\left(\beta(j+1) - \frac{n}{\alpha}, 1 + \frac{n}{\alpha}\right). \quad (9)$$

Setting $n = 1$ in (9), gives the mean of X.

The nth incomplete moment, defined by $\varphi_n(t) = \int_0^t x^n f(x)dx$, of the OLBXII distribution follows from equation (7)

$$\varphi_n(t) = \sum_{j=0}^{\infty} \delta_j \int_0^{t} x^s \, g_{\beta(j+1)}(x)\, dx.$$

Then, we have (for $\alpha\beta > n$)

$$\varphi_n(t) = \sum_{j=0}^{\infty} \delta_j \, \sigma^n \beta(j+1)\, B\left(t^{\alpha}; \beta(j+1) - \frac{n}{\alpha}, 1 + \frac{n}{\alpha}\right). \quad (10)$$

The first incomplete moment of X, $\varphi_1(t)$, is obtained from the last equation with $n = 1$. The mean deviations about the mean and the mean deviations about the median of X, depend on $\varphi_1(t)$. Furthermore, $\varphi_1(t)$ has important applications related to the Bonferroni, the Lorenz curves, the MRL and the MIT.

4.3 Moment Generating Function

The mgf of X, $M_X(t) = E[\exp(tX)]$, can be obtained from (7) as

$$M_X(t) = \sum_{j=0}^{\infty} \delta_j \, M_{j+1}(t),$$

where δ_j is defined in Section 3 and $M_{j+1}(t)$ is the mgf of the BXII$(\alpha, \beta(j+1), \sigma)$ distribution. Paranaba et al. (2011) derived a simple representation for the mgf of BXII(α, β, σ) distribution. They defined the mgf of the BXII(α, β, σ) model (for $t < 0$) as

$$M(t) = \alpha\beta \int_0^{\infty} \exp(y\sigma t) y^{\alpha-1} (1 + y^{\alpha})^{-\beta-1} dy.$$

Consider the Meijer G-function given by

$$G_{p,q}^{m,n}\left(x \middle| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j + t) \prod_{j=1}^{n} \Gamma(1 - a_j - t)}{\prod_{j=n+1}^{p} \Gamma(a_j + t) \prod_{j=m+1}^{p} \Gamma(1 - b_j - t)} x^{-t} dt,$$

where $i = \sqrt{-1}$ is the complex unit and L denotes an integration path (Gradshteyn and Ryzhik, 2000, Section 9.3).

Suppose that $\alpha = m/\beta$, where m and β are positive integers. This condition is not restrictive because every positive real number can be approximated by a rational number.

Consider the following result , (Prudnikov et al., 1992, p. 21), which holds for m and β positive integers, $\mu > -1$ and $p > 0$

$$I\left(p, \mu, \frac{m}{\beta}, v\right) = \int_0^{\infty} \exp(-px) x^{\mu} \left(1 + x^{\frac{m}{\beta}}\right)^{v} dx$$
$$= V G_{\beta+m,\beta}^{\beta,\beta+m}\left(\frac{m^m}{p^m} \middle| \begin{matrix} \Delta(m, -\mu), \Delta(\beta, v+1) \\ \Delta(\beta, 0) \end{matrix}\right),$$

where $V = \frac{\beta^{-v} m^{\mu+\frac{1}{2}}}{(2\pi)^{\frac{m-1}{2}} \Gamma(-v) p^{\mu+1}}$ and $\Delta(\beta, a) = \frac{a}{\beta}, \frac{a+1}{\beta}, \ldots, \frac{a+\beta}{\beta}$. We can write (for $t < 0$)

$$M(t) = m \, I\left(-\sigma t, \frac{m}{\beta} - 1, \frac{m}{\beta}, -\beta - 1\right).$$

Then, we can write the mgf of X as

$$M_X(t) = m \sum_{j=0}^{\infty} \delta_j \, I\left(-\sigma t, \frac{m}{\beta(j+1)} - 1, \frac{m}{\beta(j+1)}, -\beta(j+1) - 1\right).$$

4.4 Mean Residual Life and Mean Inactivity Time

The MRL (also known as the life expectancy at age t) is defined by $m_X(t) = E(X - t | X > t)$, for $t > 0$, and it represents the expected additional life length for a unit, which is alive at age t. The MRL has several applications in life insurance, economics and social studies, product quality control, biomedical sciences, demography and product technology.

The MRL of X is defined by

$$m_X(t) = \frac{1-\varphi_1(t)}{1-F(t)} - t. \tag{11}$$

Using equation (10), we can write

$$\varphi_1(t) = \sum_{j=0}^{\infty} \delta_j \, \sigma\beta(j+1)\, B\left(t^{\alpha}; \frac{\alpha\beta(j+1)-1}{\alpha}, \frac{\alpha+1}{\alpha}\right).$$

By substituting the above equation in (11), we obtain

$$m_X(t) = \frac{\sigma\beta}{1-F(t)} \sum_{j=0}^{\infty} \delta_j \, (j+1)\, B\left(t^{\alpha}; \frac{\alpha\beta(j+1)-1}{\alpha}, \frac{\alpha+1}{\alpha}\right) - t.$$

The MIT represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,t)$. The MIT is defined (for $t>0$) by $M_X(t) = E(t-X|X\le t)$.

The MIT of X is given by

$$M_X(t) = t - \frac{\varphi_1(t)}{F(t)}. \tag{12}$$

By substituting $\varphi_1(t)$ in (12), the MIT of X comes out as

$$M_X(t) = t - \frac{\sigma\beta}{F(t)} \sum_{j=0}^{\infty} \delta_j \, (j+1)\, B\left(t^{\alpha}; \frac{\alpha\beta(j+1)-1}{\alpha}, \frac{\alpha+1}{\alpha}\right).$$

4.5 Order Statistics

Suppose that $X_1, \ldots, X_n$ is a random sample of size n from the OLBXII distribution and $X_{(1)}, \ldots, X_{(n)}$ is the corresponding order statistics. Then, the pdf of the ith order statistic $X_{i:n}$, say $f_{i:n}(x)$, is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} f(x) F(x)^{i-1} [1-F(x)]^{n-i}. \tag{13}$$

Gomes-Silva et al. (2017) obtained a simple formula for the ith order statistic of their OL-G family. According to Gomes-Silva et al. (2017), we can write

$$f_{i:n}(x) = \sum_{m,p=0}^{\infty} \sum_{j=0}^{k+n-i} b_{j,m,p}\, g(x) G(x)^{j+m+p}, \tag{14}$$

where

$$b_{j,m,p} = \frac{n!\,\theta^{j+m+2}}{(i-1)!\,(n-i)!\,m!\,(1+\theta)^{j+1}} \binom{j+m+p}{j+m} \sum_{k=0}^{i-1} (-1)^{k+m} \binom{k+n-i}{j} \binom{i-1}{k}.$$

Using equations (1) and (2), we can write equation (14) as

$$f_{i:n}(x) = \sum_{m,p=0}^{\infty} \sum_{j=0}^{k+n-i} b_{j,m,p} \frac{\alpha\beta}{\sigma^{\alpha}} x^{\alpha-1} \left[1 + \left(\frac{x}{\sigma}\right)^{\alpha}\right]^{-\beta-1} \left\{1 - \left[1 + \left(\frac{x}{\sigma}\right)^{\alpha}\right]^{-\beta}\right\}^{j+m+p}.$$

Using the generalized binomial expansion, and after some simplifications, the pdf of $X_{i:n}$ reduces to

$$f_{i:n}(x) = \sum_{s=0}^{\infty} \eta_s g_{\beta(s+1)}(x), \tag{15}$$

where

$$\eta_s = \sum_{m,p=0}^{\infty} \sum_{j=0}^{k+n-i} \frac{(-1)^s}{s+1} b_{j,m,p} \binom{j+m+p}{s},$$

and $g_{\beta(s+1)}(x)$ denotes the BXII pdf with shape parameters α and $\beta(s+1)$, and scale parameter σ. Hence, the pdf of the OLBXII order statistics is a linear mixture of BXII densities. Using (15), we can derive some properties of $X_{i:n}$ based on those BXII properties.

The rth moment of $X_{i:n}$ is given by

$$E(X_{i:n}^r) = \sum_{s=0}^{\infty} \eta_s \sigma^r \beta(s+1)\, B\left(\beta(s+1) - \frac{r}{\alpha}, 1 + \frac{r}{\alpha}\right).$$

5. ESTIMATION

In this section, the unknown parameters of the OLBXII model are estimated, from complete samples only, by maximum likelihood. Suppose that $x_1, \ldots, x_n$ is a random sample of the OLBXII distribution with parameter vector $\upsilon = (\alpha, \beta, \sigma, \theta)^T$.

The log-likelihood function for υ, $\ell = \ell(\upsilon)$, is

$$\ell = 2n\log\theta + n\log\alpha + n\log\beta - n\alpha\log\sigma - n\log(\theta+1) + (\alpha-1)\sum_{i=1}^{n} \log x_i + (2\beta-1)\sum_{i=1}^{n} \log\left(1 + \left(\frac{x_i}{\sigma}\right)^{\alpha}\right) + \theta\sum_{i=1}^{n}\left\{1 - \left[1 + \left(\frac{x_i}{\sigma}\right)^{\alpha}\right]^{\beta}\right\}. \tag{16}$$

The above equation can be maximized either by using the different programs like R (optim function), SAS (PROC NLMIXED) or by solving the nonlinear likelihood equations obtained by differentiating (16).

The score vector, $\mathbf{U}(\upsilon) = \frac{\partial\ell}{\partial\upsilon} = (\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial\sigma}, \frac{\partial\ell}{\partial\theta})^T$, elements are, respectively, given by

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - n\log\sigma + \sum_{i=1}^{n} \log x_i + (2\beta - 1)\sum_{i=1}^{n} \frac{s_i \log\left(\frac{x_i}{\sigma}\right)}{1 + s_i} - \theta\beta \sum_{i=1}^{n} s_i(1 + s_i)^{\beta-1}\log\left(\frac{x_i}{\sigma}\right),$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + 2\sum_{i=1}^{n} \log(1 + s_i) - \theta \sum_{i=1}^{n} (1 + s_i)^{\beta}\log(1 + s_i),$$

$$\frac{\partial \ell}{\partial \sigma} = \frac{-n\alpha}{\sigma} - \frac{2\beta - 1}{\sigma}\sum_{i=1}^{n} \frac{\alpha s_i}{1 + s_i} + \frac{\alpha\beta\theta}{\sigma}\sum_{i=1}^{n} s_i(1 + s_i)^{\beta-1},$$

and

$$\frac{\partial \ell}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{\theta + 1} + \sum_{i=1}^{n} \left[1 - (1 + s_i)^{\beta}\right]. \tag{17}$$

where $s_i = (x_i/\sigma)^{\alpha}$.

The estimate of the unknown parameters can be obtained by setting the score vector to zero, $\mathbf{U}(\hat{\upsilon}) = 0$. We can get the MLEs $\hat{\alpha}, \hat{\beta}, \hat{\sigma}$ and $\hat{\theta}$ by solving the system of equations (17) simultaneously using numerically method with iterative techniques such as the Newton-Raphson algorithm.

For more simplicity, from (17) and for fixed α, β and σ, we can obtain $\hat{\theta}(\alpha, \beta, \sigma)$ as follows

$$\hat{\theta}(\alpha, \beta, \sigma) = \frac{-(K(\alpha, \beta, \sigma) - 1) + \sqrt{(K(\alpha, \beta, \sigma) - 1)^2 + 8K(\alpha, \beta, \sigma)}}{2K(\alpha, \beta, \sigma)},$$

where

$$K(\alpha, \beta, \sigma) = \frac{1}{n}\sum_{i=1}^{n} \left(1 + \left(\frac{x_i}{\sigma}\right)^{\alpha}\right)^{\beta} - 1,$$

and the MLE of α, β and σ denotes by $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\sigma}$, respectivlely, these estimates can be obtained by numerically solving the following non-linear equations

$$\frac{n}{\alpha} - n\log\sigma + \sum_{i=1}^{n} \log x_i + (2\beta - 1)\sum_{i=1}^{n} \frac{s_i \log\left(\frac{x_i}{\sigma}\right)}{1 + s_i} - \beta\hat{\theta}(\alpha, \beta, \sigma)\sum_{i=1}^{n} s_i(1 + s_i)^{\beta-1}\log\left(\frac{x_i}{\sigma}\right) = 0,$$

$$\frac{n}{\beta} + 2\sum_{i=1}^{n} \log(1 + s_i) - \hat{\theta}(\alpha, \beta, \sigma)\sum_{i=1}^{n} (1 + s_i)^{\beta}\log(1 + s_i) = 0,$$

and

$$\frac{-n\alpha}{\sigma}-\frac{2\beta-1}{\sigma}\sum_{i=1}^{n}\frac{\alpha s_i}{1+s_i}+\frac{\alpha\beta\hat{\theta}(\alpha,\beta,\sigma)}{\sigma}\sum_{i=1}^{n}s_i(1+s_i)^{\beta-1}=0. \tag{18}$$

After the numerically iterative techniques are used to compute $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\sigma}$ from the system of non-linear equatoins (18), the MLE of $\hat{\theta}$, $\hat{\theta}(\alpha,\beta,\sigma)$, can be computed from (17) as $\hat{\theta}\left(\hat{\alpha},\hat{\beta},\hat{\sigma}\right)$.

For the OLBXII distribution, all its second order derivatives exist. Therefore, the 4×4 observed information matrix at $\upsilon=(\alpha,\beta,\sigma,\theta)^T$, $J(\upsilon)=\{J_{ij}\}$ for $i,j=\alpha,\beta,\sigma,\theta$, the 4×4 total observed information matrix evaluated at $\hat{\upsilon}$, $J(\hat{\upsilon})=\{\hat{J}_{ij}\}$ for $i,j=\hat{\alpha},\hat{\beta},\hat{\sigma},\hat{\theta}$, and the multivariate normal $N_4(0,J(\hat{\upsilon})^{-1})$ distribution can be used to approximate the confidence intervals for the model parameters.

6. SIMULATION STUDY

To investigate the behavior of the MLEs in the previous section, we propose some simulations in terms of the sample size n. One can simply simuate the OLBXII random variable as follows: Let the randome variable U is uniformelly distributed on the unite inteval $(0,1)$, then by using the relation (8) the randome variable $X=Q(U)$ has the pdf in (6).

By using R 4.0.1 programming language R (R Core Team (2017)), $2{,}000$ random samples from the distribution OLBXII has been generated with different sample sizes $n=100,200$ and $n=400$. We set the true values of the parameters as follows: $\alpha=(0.5,1.2)$, $\beta=(0.5,1)$, $\sigma=(0.75,1.5)$ and $\theta=(1.5,2)$. Tables 1 and 2 show the average $MLEs$ and $MSEs$ were computed for each sample size and each parameter, it can be seen that the estimates are stable and close the true parameter values for these sample sizes. Furthermore, as the sample size increases the MSEs decreases in all cases.

7. DATA ANALYSIS

We prove the flexibility of the OLBXII distribution using two real data sets. The fitted models are compared using the following criteria namely: the Akaike information criterion (AIC), corrected Akaike information criterion ($CAIC$), Hannan-Quinn information criterion ($HQIC$), Bayesian information criterion (BIC), Cramér-von Mises (W^*) and Anderson-Darling (A^*) statistics.

We compare the fits of the OLBXII distribution with some competitive models namely: the Weibull Burr XII (WBXII), Weibull Fréchet (WFr) (Afify et al., 2016b), beta Burr XII (BBXII), Kumaraswamy exponentiated Burr XII (KEBXII) and BXII distributions.
The pdfs of the fitted models are given (for $x>0$) by

WBXII: $f(x)=\alpha\,\beta\,a\,b\,x^{\alpha-1}(1+x^{\alpha})^{\beta b-1}e^{-a\left[(1+x^{\alpha})^{\beta}-1\right]^{b}}\left[1-(1+x^{\alpha})^{-\beta}\right]^{b-1}$;

WFr: $f(x) = a\, b\, \beta\, \alpha^{\beta} x^{-\beta-1} e^{-b\left(\frac{\alpha}{x}\right)^{\beta}} \left\{1 - e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\right\}^{-b-1} e^{-a\left[e^{-\left(\frac{\alpha}{x}\right)^{\beta}} - 1\right]^{-b}}$;

BBXII: $f(x) = \frac{c\, \theta\, \beta^{-c}}{B(a,b)} x^{c-1} \left[1 + \left(\frac{x}{\beta}\right)^{c}\right]^{-\theta b-1} \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)^{c}\right]^{-\theta}\right\}^{a-1}$;

KEBXII: $(x) = \frac{a\, b\, c\, \theta\, \beta x^{c-1}}{(1+x^{c})^{\theta+1}} \left[1 - (1 + x^{c})^{-\theta}\right]^{a\beta-1} \left\{1 - \left[1 - (1 + x^{c})^{-\theta}\right]^{a\beta}\right\}^{b-1}$.

All the parameters are positive real numbers.

Table 1
Average Values of $MLEs$ and the Corresponding $MSEs$ ($\alpha = 0.5$)

n	$\hat{\alpha}$	MSE	β	$\hat{\beta}$	MSE	σ	$\hat{\sigma}$	MSE	θ	$\hat{\theta}$	MSE
100	0.5079	0.0040	0.5	0.4903	0.0143	0.75	0.7553	0.2077	1.5	1.5053	0.0060
200	0.5051	0.0024		0.4963	0.0087		0.7703	0.1329		1.5038	0.0030
400	0.4991	0.0010		0.5027	0.0039		0.7551	0.0752		1.5009	0.0014
100	0.5050	0.0039		0.5021	0.0198		0.7851	0.2509	2	2.0058	0.0114
200	0.5023	0.0018		0.5037	0.0086		0.7501	0.1357		2.0050	0.0063
400	0.5012	0.0009		0.4960	0.0048		0.7331	0.0893		2.0037	0.0026
100	0.5041	0.0045		0.5068	0.0178	1.5	1.6432	0.9556	1.5	1.5067	0.0057
200	0.5058	0.0022		0.4985	0.0087		1.5472	0.5781		1.4978	0.0030
400	0.5047	0.0012		0.4951	0.0043		1.5001	0.2878		1.4992	0.0015
100	0.5004	0.0037		0.5091	0.0203		1.5700	1.0820	2	2.0042	0.0123
200	0.4986	0.0019		0.5126	0.0098		1.6612	0.7037		2.0028	0.0056
400	0.5000	0.0009		0.5061	0.0043		1.5807	0.3217		2.0028	0.0027
100	0.5057	0.0038	1	1.0089	0.1201	0.75	0.7896	0.3157	1.5	1.5023	0.0061
200	0.4977	0.0020		1.0316	0.0778		0.8267	0.2593		1.5033	0.0028
400	0.4987	0.0010		1.0095	0.0368		0.7684	0.1495		1.4981	0.0014
100	0.4984	0.0034		1.1013	0.1587		0.9886	0.3962	2	2.0113	0.0122
200	0.5019	0.0019		1.0172	0.0932		0.7883	0.2834		2.0053	0.0058
400	0.4993	0.0008		1.0014	0.0468		0.7502	0.1682		2.0061	0.0031
100	0.4987	0.0039		1.0722	0.1515	1.5	1.8400	1.4788	1.5	1.5071	0.0057
200	0.5014	0.0019		1.0196	0.0748		1.6151	0.9856		1.5033	0.0034
400	0.5002	0.0010		1.0048	0.0386		1.5039	0.5902		1.4997	0.0016
100	0.4981	0.0031		1.0842	0.1803		1.8591	1.6581	2	2.0022	0.0114
200	0.5030	0.0016		1.0029	0.0859		1.5009	1.0930		2.0030	0.0067
400	0.5012	0.0008		1.0003	0.0456		1.5003	0.6752		1.9989	0.0029

Table 2
Average Values of $MLEs$ and the Corresponding $MSEs$ $(\alpha = 1.2)$

n	$\hat{\alpha}$	MSE	β	$\hat{\beta}$	MSE	σ	$\hat{\sigma}$	SD	θ	$\hat{\theta}$	MSE
100	0.4970	0.0053	0.5	0.5078	0.0195	0.75	0.8127	0.2490	1.5	1.4995	0.0072
200	0.4973	0.0023		0.5102	0.0091		0.8254	0.1532		1.5044	0.0035
400	0.5018	0.0011		0.4945	0.0041		0.7453	0.0713		1.5015	0.0016
100	0.5021	0.0036		0.5142	0.0192		0.8521	0.2660	2	1.9921	0.0126
200	0.5020	0.0020		0.5058	0.0100		0.7797	0.1610		2.0071	0.0057
400	0.5016	0.0010		0.5007	0.0043		0.7551	0.0831		1.9987	0.0028
100	0.5074	0.0048		0.5066	0.0171	1.5	1.7186	0.9930	1.5	1.5051	0.0062
200	0.4995	0.0021		0.5072	0.0084		1.6392	0.5849		1.5030	0.0029
400	0.4999	0.0009		0.5019	0.0037		1.5052	0.2615		1.5005	0.0015
100	0.5023	0.0036		0.5128	0.0185		1.7305	1.0827	2	2.0098	0.0117
200	0.4986	0.0018		0.5139	0.0101		1.6343	0.6546		2.0033	0.0062
400	0.4992	0.0009		0.5033	0.0048		1.5493	0.3423		1.9988	0.0029
100	0.5101	0.0041	1	1.0130	0.1342	0.75	0.7585	0.3395	1.5	1.5012	0.0060
200	0.5008	0.0018		1.0143	0.0671		0.7775	0.2243		1.4995	0.0033
400	0.5031	0.0011		0.9936	0.0365		0.7416	0.1387		1.4996	0.0017
100	0.5029	0.0034		1.0910	0.2023		0.9474	0.4112	2	2.0016	0.0128
200	0.4972	0.0015		1.0399	0.0827		0.8405	0.2775		2.0036	0.0055
400	0.5011	0.0007		1.0138	0.0438		0.7829	0.1620		2.0039	0.0029
100	0.5014	0.0034		1.0197	0.1255	1.5	1.5086	1.3427	1.5	1.5019	0.0072
200	0.4993	0.0021		1.0032	0.0718		1.5022	0.9270		1.5019	0.0031
400	0.5020	0.0010		1.0105	0.0368		1.5051	0.6234		1.5004	0.0014
100	0.4930	0.0031		1.1063	0.1843		2.0136	1.7014	2	2.0065	0.0128
200	0.4986	0.0018		1.0201	0.0958		1.5958	1.1309		2.0053	0.0063
400	0.5022	0.0008		1.0004	0.0469		1.5001	0.6770		2.0018	0.0029

The first data set is reported in Lee and Wang (2003) and it refers to the remission times (in months) of a random sample of 128 bladder cancer patients. The second data set is reported in Smith and Naylor (1987) and it consists of 63 observations of the strengths of 1.5 cm glass fibres (the units of measurement are not given) originally obtained by workers at the UK National Physical Laboratory.

Tables 3 and 5 list the MLEs and their corresponding standard errors (in parentheses) of the fitted models for the two data sets, respectively. The values of goodness-of-fit statistics, for both data sets, are provided in Tables 4 and 6.

The values in Tables 2 and 4, reveal that the OLBXII distribution has a close fit for both data sets and it can be considered a very competitive model to other distributions. The plots in Figures 3, 4 and 5 show that the OLBXII distribution is the most convenient model to for both data sets.

Table 3
MLEs and the Corresponding SEs (Given in Parentheses) for Cancer Data

Model	Estimates				
BXII (α,β)	2.3354 (0.354)	0.2337 (0.04)			
OLBXII $(\alpha,\beta,\sigma,\theta)$	1.4442 (0.195)	0.0936 (0.259)	10.0613 (7.361)	18.6135 (61.571)	
WBXII (α,β,a,b)	0.789 (0.418)	0.2008 (0.312)	6.7391 (43.919)	2.4552 (1.402)	
WFr (α,β,a,b)	51.2054 (155.863)	0.2206 (0.086)	19.5182 (49.01)	2.4642 (1.081)	
KEBXII (a,b,c,θ,β)	3.017 (8.796)	67.6736 (102.6)	0.3383 (0.376)	0.8386 (1.674)	2.8394 (8.279)
BBXII (a,b,c,θ,β)	1.0891 (0.451)	1.3905 (2.405)	1.5728 (0.441)	0.8665 (1.017)	6.3741 (1.582)

Table 4
The Statistics AIC, $CAIC$, $HQIC$, BIC, W^* and A^* for Cancer Data

Model	-2ℓ	AIC	$CAIC$	$HQIC$	BIC	W^*	A^*
OLBXII	819.448	827.448	827.774	832.083	838.856	0.0228	0.1508
WBXII	821.812	829.812	830.138	834.448	841.221	0.0498	0.3265
KEBXII	821.637	831.637	832.129	837.431	845.897	0.0480	0.3189
BBXII	822.297	832.297	832.789	838.091	846.557	0.0412	0.2972
WFr	823.148	831.148	831.473	835.783	842.556	0.0621	0.4051
BXII	906.950	910.950	911.046	913.267	916.654	0.6940	5.3709

Table 5
MLEs and the Corresponding SEs (Given in Parentheses) for Glass Fibres Data

Model	Estimates				
BXII (α,β)	7.4821 (1.2850)	0.3207 (0.0650)			
OLBXII $(\alpha,\beta,\sigma,\theta)$	3.4985 (1.6744)	1.8422 (1.3971)	1.4062 (0.3990)	0.3509 (0.3766)	
WBXII (α,β,a,b)	1.6077 (0.3760)	2.7409 (1.0100)	0.0026 (0.0032)	1.8888 (0.7680)	
WFr (α,β,a,b)	0.3865 (0.7990)	0.2436 (0.2850)	1.4762 (4.7820)	16.8561 (20.4850)	
KEBXII (a,b,c,θ,β)	4.022 (24.1410)	137.8974 115.5110)	1.0241 (0.6650)	1.3285 (1.2970)	4.0102 (26.0651)
BBXII (a,b,c,θ,β)	26.1629 (14.5880)	14.7050 (12.8850)	0.9271 (0.2130)	5.5864 (5.2150)	8.2620 (8.1320)

Table 6
The Statistics AIC, $CAIC$, $HQIC$, BIC, W^* and A^* for glass Fibres Data

Model	$-2\hat{\ell}$	AIC	$CAIC$	$HQIC$	BIC	W^*	A^*
OLBXII	27.295	35.295	35.985	38.667	43.868	0.15175	0.85244
WBXII	28.607	36.607	37.297	39.979	45.180	0.19257	1.05507
WFr	31.001	39.001	39.691	42.373	47.574	0.27786	1.48538
KEBXII	39.041	49.041	50.093	53.255	59.757	0.43694	2.3495
BBXII	51.710	61.710	62.763	65.925	72.426	0.64538	3.50125
BXII	97.442	101.442	101.642	103.128	105.729	1.17788	7.36685

The plots of the fitted OLBXII pdf and other fitted pdfs are displayed in Figure 5, for cancer and glass fibres data, respectively. Figure 6 show the QQ plots of the fitted models.

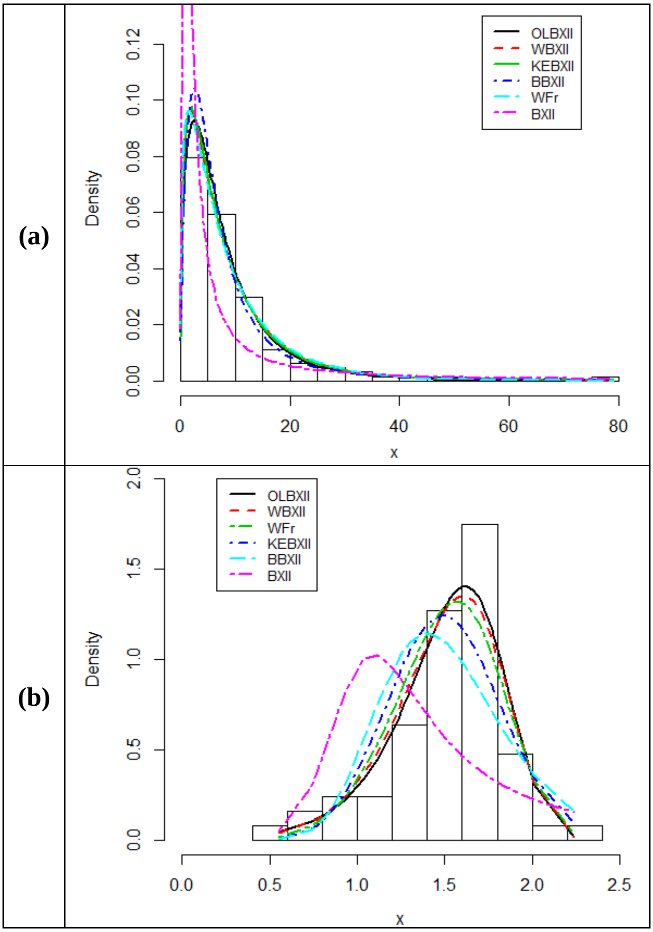

Figure 3: The Fitted Pdf of OLBXII Distribution and other Fitted Distributions for Cancer Data (a) and Glass Fibres data (b)

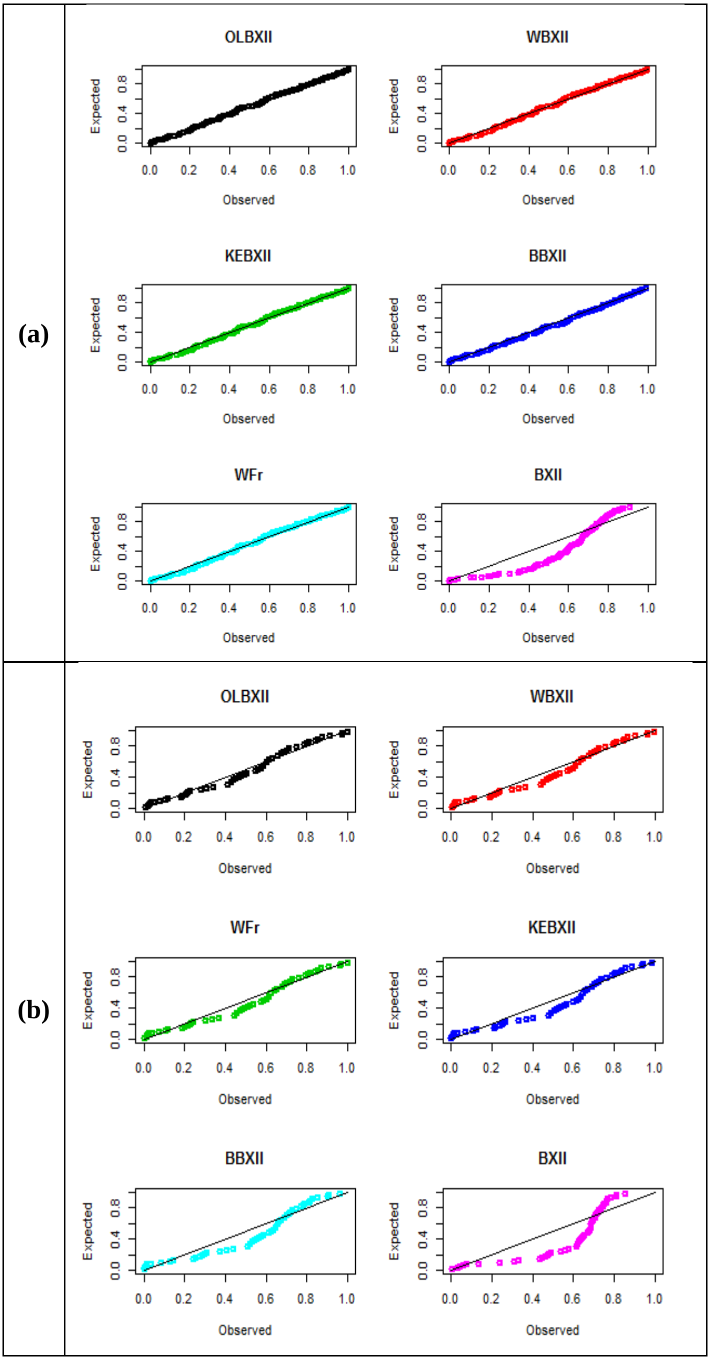

Figure 4: Q-Q plots for the Fitted Distributions for Cancer Data (a) and Glass Fibres Data (b)

8. CONCLUSIONS

We propose and study a new four-parameter lifetime distribution, named the *odd Lindley Burr XII* (OLBXII) distribution, which extends the Burr XII (BXII) distribution. We prove that the OLBXII pdf can be given as a linear mixture of BXII densities. We derive explicit expressions for some of its properties including the ordinary and incomplete moments, generating function, mean residual life, mean inactivity time and order statistics. We estimate the model parameters via the maximum likelihood estimation method. We prove empirically that the OLBXII model can give a better fits than other competitive models by means of two real data sets.

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