**Pak. J. Statist.**
**2019 Vol. 35(1), 65-74**

# ASYMPTOTIC PROPERTIES OF MAXIMUM PSEUDO-LIKELIHOOD ESTIMATORS OF GPD MODEL PARAMETERS WITH INTERVAL CENSORING

**Meghlaoui Dakhmouche**[1] and **Bahi Oussama**[2§]
Laboratory of Mathematical Modelling and Simulation, University of Frères Mentouri Constantine 1, 25000 Constantine, Algeria
Email: [1] m_dakhmouche@yahoo.fr
[2] bahi.oussama2@gmail.com
[§] Corresponding author

## ABSTRACT

In this article, we are interested in asymptotic properties of the maximum pseudo-likelihood estimators of the parameters of a GPD model with interval censoring. A first step consists in an exact calculation of the probability distribution of the random vector $(Y, Z, \Delta)$ associated with the censored model by interval, and its probability density is defined. Then, the efficiency and the asymptotic normality of the maximum pseudo-likelihood estimators are stated in two propositions. A second step, based on simulations and the Barzilai-Borwein algorithm, provides the efficiency and asymptotic normality of the maximum pseudo-likelihood estimators of the parameters of a GPD model.

## KEYWORDS

Interval censoring, pseudo-likelihood, GPD model, asymptotic normality, BB-Algorithm.

## 1. INTRODUCTION

The censored data come from the fact that we do not have access to the complete information. The observation of the random variable of interest $X$ is frequently subject to various outside influences, independent or not, from the studied phenomenon. It is therefore very important to have good analytical methods for this kind of data, which is frequently found, mainly in reliability, epidemiology, insurance and finance (Beirlant, 2008). The most common types of censoring are the right censoring, the left censoring (Lawless, 2003) and the double censoring (Turnbull, 1974). Moreover, Klein and Moeschberger (Klein, 1997) introduced the concept of interval censoring. In this case, we cannot observe the variable of interest $X$, but we know that its value is in a random interval $(\mathrm{Y}, \mathrm{Z})$. In the literature, many contributions have mainly been concerned with the properties of nonparametric estimators in models subject to interval censoring (Gentleman, 1994).

In this contribution, we are interested in asymptotic properties of the maximum pseudo-likelihood estimators of the parameters of a GPD model with interval censoring. In a model censored by interval, the variable of interest $X$ may not be observed. Thus, the statistical model will be poorly specified and the maximum likelihood method becomes weak. In fact, in the presence of interval censoring, a triplet $(Y, Z, \Delta)$ of independent random variables is observed. The exact calculation of the probability distribution of the random vector $(Y, Z, \Delta)$ associated with the interval censored model shows that the distribution is the same as that proposed by Klein and Moeschberger (Klein, 1997). Then we define a pseudo-likelihood function from which we derive the pseudo-maximum likelihood method of estimation which is more adapted to discrete variables (Gourieroux et al. 1984). The idea of this method is to replace the likelihood function of the model with that of a poorly specified pseudo-model but with interesting properties. In two propositions, we show that the pseudo-maximum likelihood estimator still possesses the properties of convergence and asymptotic normality. Finally, the results obtained in the general framework are applied to a GPD model with interval censoring.

## 2. PARAMETRIC MODEL WITH INTERVAL CENSORING

Let $\left(\Re_+, B_{R_+}, P_\theta\right)_{\theta\in\Theta}$ be a model associated with a random variable $X$ of a distribution function $F_X$. We suppose that the set $\Theta$ is compact in $\Re^{d\geq 1}$ and contains the true value $\theta_0$ of $\theta$. Moreover, we assume that the model is identifiable, that is the map $\theta \to P_\theta$ is injective for all $\theta$ in $\Theta$. In the presence of interval censoring, instead of observing the variable of interest $X$, we observe the triplet $(Y, Z, \Delta)$ with values in the set $D = \Re_+^2 \times \{1, 2, 3\}$. The components $Y$ and $Z$ are two random variables absolutely continuous with respect to the Lebesgue measure. We assume that the joint density of $Y$ and $Z$ does not depend on $\theta$, and that the variable of interest $X$ is independent of the pair $(Y, Z)$. The component $\Delta$ is a discrete random variable defined such as

$$\Delta = \begin{cases} 1 \text{ if } Y < X \leq Z \\ 2 \text{ if } X > Z \\ 3 \text{ if } X \leq Y. \end{cases}$$

The joint probability distribution of the triplet $(Y, Z, \Delta)$ is such that

$$\begin{aligned} &\mathrm{P}\left(y \leq Y < y + dy,\ z \leq Z < z + dz, \Delta = \delta\right) \\ &\quad = \mathrm{P}\left(y \leq Y < y + dy,\ z \leq Z < z + dz\right) \\ &\qquad \mathrm{P}\left\{\Delta = \delta / \left(y \leq Y < y + dy,\ z \leq Z < z + dz\right)\right\}. \end{aligned} \tag{1}$$

It obvious that,

$$\lim_{dy\to 0, dz\to 0} \frac{\mathrm{P}\left(y \leq Y < y + dy,\ z \leq Z < z + dz\right)}{dydz} = f_{(Y,Z)}\left(y, z\right),$$

where $f_{(Y,Z)}$ is the joint density function of the couple $(Y,Z)$. Moreover, for $\delta = 1,2,3$ we have

$$\lim_{dy\to 0, dz\to 0} \mathrm{P}\left\{(\Delta=1)/(y \le Y < y+dy, z \le Z < z+dz)\right\} = \mathrm{P}(y < X \le z),$$
$$\lim_{dy\to 0, dz\to 0} \mathrm{P}\left\{(\Delta=2)/(y \le Y < y+dy, z \le Z < z+dz)\right\} = \mathrm{P}(X > z),$$
$$\lim_{dy\to 0, dz\to 0} \mathrm{P}\left\{(\Delta=3)/(y \le Y < y+dy, z \le Z < z+dz)\right\} = \mathrm{P}(X \le y).$$

The distribution of the conditional random variable $(\Delta = \delta / Y = y, Z = z)$ can be summarized as:

$$\mathrm{P}(\Delta=\delta / Y = y, Z = z) = \lim_{dy\to 0, dz\to 0} \mathrm{P}(\Delta = \delta / y \le Y < y+dy, z \le Z < z+dz)$$
$$= \mathrm{P}(y \le X < z)^{1_{\{\delta=1\}}(\delta)} (X < y)^{1_{\{\delta=2\}}(\delta)} (X \ge z)^{1_{\{\delta=3\}}(\delta)}.$$

Let $f_{(Y,Z,\Delta)}$ be the density function of the triplet $(Y, \mathrm{Z}, \Delta)$ defined such that

$$\lim_{dy\to 0, dz\to 0} \mathrm{P}(y \le Y < y+dy,\ z \le Z < z+dz, \Delta=\delta) = f_{(Y,Z,\Delta)}(y,z,\delta),$$

where

$$f_{Y,Z,\Delta}(y,z,\delta) = f_{Y,Z}(y,z)\mathrm{P}(y \le X < z)^{1_{\{\delta=1\}}(\delta)} \mathrm{P}(X \le y)^{1_{\{\delta=2\}}(\delta)} \mathrm{P}(X > z)^{1_{\{\delta=3\}}(\delta)}$$
$$= f_{Y,Z}(y,z)\left(F_X(z) - F_X(y)\right)^{1_{\{\delta=1\}}(\delta)} \left(F_X(y)\right)^{1_{\{\delta=2\}}(\delta)} \left(1 - F_X(z)\right)^{1_{\{\delta=3\}}(\delta)}.$$

Let $(Y_k, Z_k, \Delta_k)_{k=1,\ldots,n}$ be a random sample of the triplet $(Y, Z, \Delta)$, whose distribution function depends on the parameter $\theta$. The likelihood function of this sample is

$$\mathrm{L}(\theta) = \prod_{k=1}^{n} \mathrm{F}_{(Y,Z,\Delta)}(y_k, z_k, \delta_k)$$
$$= \prod_{k=1}^{n} \mathrm{F}_{(Y,Z)}(y_k, z_k) \prod_{k=1}^{n} U(y_k, z_k, \delta_k; \boldsymbol{\theta}),$$

where

$$U(y_k, z_k, \delta_k; \boldsymbol{\theta}) = \left(F_X(z) - F_X(y)\right)^{1_{\{\delta=1\}}(\delta)} \left(F_X(y)\right)^{1_{\{\delta=2\}}(\delta)} \left(1 - F_X(z)\right)^{1_{\{\delta=3\}}(\delta)}.$$

The maximum likelihood estimator of $\theta$ is obtained by maximizing the log-likelihood $l(\boldsymbol{\theta}) = \log L(\boldsymbol{\theta})$. But maximizing $l(\boldsymbol{\theta})$ is equivalent to maximizing the log-pseudo-likelihood

$$\tilde{l}(\boldsymbol{\theta}) = \sum_{k=1}^{n} \log U(y_k, z_k, \delta_k; \boldsymbol{\theta}).$$

So, the maximum pseudo-likelihood estimator $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is

$$\hat{\theta} = \arg\max_{\theta\in\Theta} \tilde{l}\left(\theta\right). \tag{2}$$

In the following, we will show that the estimator of the maximum pseudo-likelihood preserves its asymptotic properties. We assume in the sequel that the function $U$ satisfies the following conditions:

- $\left(C_1\right)$ for all $\left(y,z,\delta\right)\in D$ the function $U(y,z,\delta;\boldsymbol{\theta})$ is continuous for all $\boldsymbol{\theta}\in\Theta$,
- $\left(C_2\right)$ there is a positive function $d: D\to\Re$, with $\mathrm{E}\left[\mathrm{d}\left(y,z,\delta;\boldsymbol{\theta}\right)\right]<\infty$ and such as $\left|\log U(\mathrm{Y},\mathrm{Z},\Delta;\boldsymbol{\theta})\right|\le d(y,z,\delta;\boldsymbol{\theta})$ a.s; $\forall\boldsymbol{\theta}\in\Theta$,
- $\left(C_3\right)$ for all $\left(y,z,\delta\right)\in D$, the function $U(y,z,\delta;\boldsymbol{\theta})$ is smooth and twice continuously differentiable with respect to $\boldsymbol{\theta}$ in a neighborhood of $\boldsymbol{\theta}_0$,
- $\left(C_4\right)$ for all $\left(i,j\right)\in\left\{1,...,d\right\}^{\otimes 2}$, Fisher's information matrix $\boldsymbol{I}\left(\boldsymbol{\theta}\right)$ exist,
- $\left(C_5\right)$ for all $\left(i,j\right)\in\left\{1,...,d\right\}^{\otimes 2}$, the Hessian matrix $J\left(\boldsymbol{\theta}\right)$ exist and is invertible.

## 2.1 Efficiency of the Maximum Pseudo-Likelihood Estimator

Let $\left(\hat{\boldsymbol{\theta}}_n\right)_{n\ge 1}$ be a sequence of maximum pseudo-likelihood estimators.

**Proposition 1:**

*If the function* $U(y,z,\delta;\boldsymbol{\theta})$ *satisfies the conditions* $\left(C_1\right)$ *and* $\left(C_2\right)$, *then*

$$\hat{\boldsymbol{\theta}}_n \xrightarrow{\mathbf{P}} \theta_0$$

**Proof:**

Let $M_n\left(\theta\right)$ and $M\left(\theta\right)$ be such that:

$$M_n\left(\theta\right) = \frac{1}{n}\sum_{k=1}^{n}\log U\left(Y_k, Z_k, \Delta_k;\theta\right)$$

and

$$M\left(\theta\right) = E\left(\log U\left(Y_k, Z_k, \Delta_k;\theta\right)\right).$$

From the equation 2 and the definition of $M_n\left(\theta\right)$, it is clear that $\hat{\theta}_n$ maximizes $M_n\left(\theta\right)$ to conclude with the Van der Vaart's Theorem (Van Der Vaart, 1998) p.45, it remains to show that

$$\sup\left|M_n\left(\theta\right) - M\left(\theta\right)\right| \xrightarrow{\mathbf{P}} 0, \tag{3}$$

and

$$\sup_{\theta\in C} M\left(\theta\right) < M\left(\theta_0\right), \tag{4}$$

where

$$C = \left\{\theta : \left\|\theta - \theta_0\right\| \ge \boldsymbol{\varepsilon}\right\}.$$

The function $U$ is continuous by definition $(C_1)$ and is bounded by construction $(C_2)$. So the conditions of the Lemma of Newey and McFadden (Newey, 1994) are satisfied, from where the condition 10. It remains to check that the last condition of the Theorem of Van DerVaart is satisfied. Let us begin by showing that

$$M(\boldsymbol{\theta}) \leq M(\boldsymbol{\theta}_0),\ \forall \boldsymbol{\theta} \in \Theta. \quad (5)$$

From the definition of $M(\boldsymbol{\theta})$ it is clear that

$$M(\boldsymbol{\theta}) - M(\boldsymbol{\theta}_0) = E\left[\log \frac{U(Y,Z,\Delta;\boldsymbol{\theta})}{U(Y,Z,\Delta;\boldsymbol{\theta}_0)}\right].$$

Using Jensen's inequality, we can write

$$M(\boldsymbol{\theta}) - M(\boldsymbol{\theta}_0) = E\left[\log \frac{U(Y,Z,\Delta;\boldsymbol{\theta})}{U(Y,Z,\Delta;\boldsymbol{\theta}_0)}\right] \leq \log E\left[\frac{U(Y,Z,\Delta;\boldsymbol{\theta})}{U(Y,Z,\Delta;\boldsymbol{\theta}_0)}\right].$$

Furthermore

$$\log E\left[\frac{U(Y,Z,\Delta;\boldsymbol{\theta})}{U(Y,Z,\Delta;\boldsymbol{\theta}_0)}\right] = \log\left(\iint \sum_{\delta \in \{1,2,3\}} \frac{U(\mathrm{y},\mathrm{z},\delta;\boldsymbol{\theta})}{U(\mathrm{y},\mathrm{z},\delta;\boldsymbol{\theta}_0)} f_{Y,Z}(y,z) U(y,z,\delta;\boldsymbol{\theta}_0)\, dydz\right)$$
$$= \log\left(\iint \sum_{\delta \in \{1,2,3\}} f_{Y,Z,\Delta}(y,z,\delta;\boldsymbol{\theta}_0)\, dydz\right) = 0.$$

whence the inequality 5.

Since $M(\boldsymbol{\theta})$ is continuous on the compact $C$, then there exists $\bar{\boldsymbol{\theta}} \in C$ such that

$$\sup_{\boldsymbol{\theta} \in C} M(\boldsymbol{\theta}) = M(\bar{\boldsymbol{\theta}}).$$

The identifiability of the model, the strict concavity of the logarithm and the inequality 5 together lead to:

$$M(\bar{\boldsymbol{\theta}}) < M(\boldsymbol{\theta}_0).$$

Finally, the three conditions of Van der Vaart's Theorem are verified from where the proposition 1.

## 2.2 Asymptotic normality of the maximum pseudo likelihood estimator

To facilitate the rest of the paper, let us define the score function such that:

$$D(Y_k, Z_k, \Delta_k; \boldsymbol{\theta}) = \left(\frac{\partial}{\partial \theta_1} \log \mathrm{U}(Y,Z,\Delta;\boldsymbol{\theta}), \ldots, \frac{\partial}{\partial \theta_d} \log \mathrm{U}(Y,Z,\Delta;\boldsymbol{\theta})\right).$$

**Lemma 2:**

*If the function* $\mathrm{U}(Y,Z,\Delta;\boldsymbol{\theta})$ *satisfies the condition* $(C_3)$, *then*

$$E\left[D(Y,Z,\Delta;\boldsymbol{\theta}_0)\right]=0.$$

**Proof:**

From the relation 5 it is obvious that $\boldsymbol{\theta}_0$ maximizes $\mathbf{M}(\boldsymbol{\theta})$. Since the function $\mathrm{U}(Y,Z,\Delta;\boldsymbol{\theta})$ satisfies the condition $(C_3)$, therefore for all $j\in\{1,...,d\}$ we have:

$$E\left(\frac{\partial}{\partial\theta_j}\log U\left(Y,Z,\Delta;\boldsymbol{\theta}_0\right)\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\right)$$

$$=\frac{\partial}{\partial\theta_j}E\left(\log U\left(Y,Z,\Delta;\boldsymbol{\theta}_0\right)\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\right)=\left(\frac{\partial}{\partial\theta_j}M\left(\theta\right)\right)\Bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}=0,$$

whence the Lemma 2.

**Lemma 3:**

*If the function* $\mathrm{U}(Y,Z,\Delta;\boldsymbol{\theta})$ *satisfies the condition* $(C_4)$, *then*

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}D(Y,Z,\Delta;\boldsymbol{\theta}_0)\xrightarrow{D}N\left(0,I\left(\boldsymbol{\theta}_0\right)\right).$$

**Proof:**

Considering the Lemma 2, it is obvious that the covariance matrix of the vector $D(Y,Z,\Delta;\boldsymbol{\theta})$ coincides with the Fisher information matrix $I(\boldsymbol{\theta}_0)$, so according to the central limit Theorem, we have

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}D(Y_k,Z_k,\Delta_k;\boldsymbol{\theta}_0)\xrightarrow{D}N\left(0,I\left(\boldsymbol{\theta}_0\right)\right).$$

Reconsider the sequence $\left(\hat{\boldsymbol{\theta}}_n\right)_{n\geq 1}$ of the maximum pseudo-likelihood estimators.

**Proposition 4:**

*If the function* $\mathrm{U}(Y,Z,\Delta;\boldsymbol{\theta})$ *satisfies the condition* $(C_1),(C_2),(C_3),(C_4)$ *and* $(C_5)$, then

$$\sqrt{n}\left(\hat{\boldsymbol{\theta}}_n-\boldsymbol{\theta}_0\right)\xrightarrow{D}N\left(0,\Sigma_{\boldsymbol{\theta}_0}\right),$$

where

$$\Sigma_{\boldsymbol{\theta}_0}=J\left(\boldsymbol{\theta}_0\right)^{-1}I\left(\boldsymbol{\theta}_0\right)J\left(\boldsymbol{\theta}_0\right)^{-1}.$$

**Proof:**

The score function $D\left(Y_k, Z_k, \Delta_k; \theta_n\right)$ vanishes at $\hat{\theta}$ since $\hat{\theta}$ maximizes $\log \mathrm{U}\left(Y, Z, \Delta; \theta\right)$. Then, using the Taylor-Young formula in the neighborhood of $\theta_0$, we can write

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} D\left(Y_k, Z_k, \Delta_k; \hat{\theta}_n\right) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} D\left(Y_k, Z_k, \Delta_k; \theta_0\right) + J_n\left(\theta_0\right)\sqrt{n}\left(\hat{\theta} - \theta_0\right) + R_n = 0,$$

where $J_n\left(\theta_0\right)$ is such as

$$J_n\left(\theta_0\right) = \left(J_n^{i,j}\left(\theta_0\right)\right)_{1 \le i, j \le d} = \left(\frac{1}{n} \sum_{k=1}^{n} \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log U\left(Y_k, Z_k, \Delta_k; \theta_0\right)\right)_{1 \le i, j \le d}.$$

$J_n^{i,j}\left(\theta_0\right)$ is defined as empirical mean of a function of some random variables. Then, using the weak law of large numbers we can say that

$$J_n^{i,j}\left(\theta_0\right) \xrightarrow{P} J\left(\theta_0\right).$$

Therefore, we can say that $J_n\left(\theta_0\right)$ is almost surely invertible.

Finally, by Lemma 3 and using the equation 7, we can write that

$$\sqrt{n}\left(\hat{\theta}_n - \theta_0\right) \xrightarrow{D} N\left(0, J\left(\theta_0\right)^{-1} I\left(\theta_0\right) J\left(\theta_0\right)^{-1}\right).$$

## 3. GPD MODEL WITH INTERVAL CENSORING

In a considerable number of applications, fitting the tail data is the main concern. The Generalized Pareto distribution (GPD) was developed as a distribution that can model tails of a wide variety of distributions (Reiss, 2007). The scarcity of observations in the tail of a distribution requires elaborating robust estimation methods. Parametric models with interval censoring may be appropriate for this kind of situation. Let us retake the model considered in Section 2. Let $X_1, X_2, ..., X_n$ be a sample from a GPD distribution in the Frechet domain of attraction, with distribution function $F\left(x; \sigma, \Upsilon\right)$, where $\sigma$ and $\Upsilon$ are scale and shape parameters, $F\left(x; \sigma, \Upsilon\right)$ defined such as

$$F\left(x; \sigma, \Upsilon\right) = 1 - \left(1 + \Upsilon \frac{x}{\sigma}\right)^{-\frac{1}{\Upsilon}},$$

where $x > -\frac{\sigma}{\Upsilon}$.

We assume that the model under consideration is subject to interval censoring. Therefore, the parameters estimation can be carried out using the maximum pseudo-likelihood method.

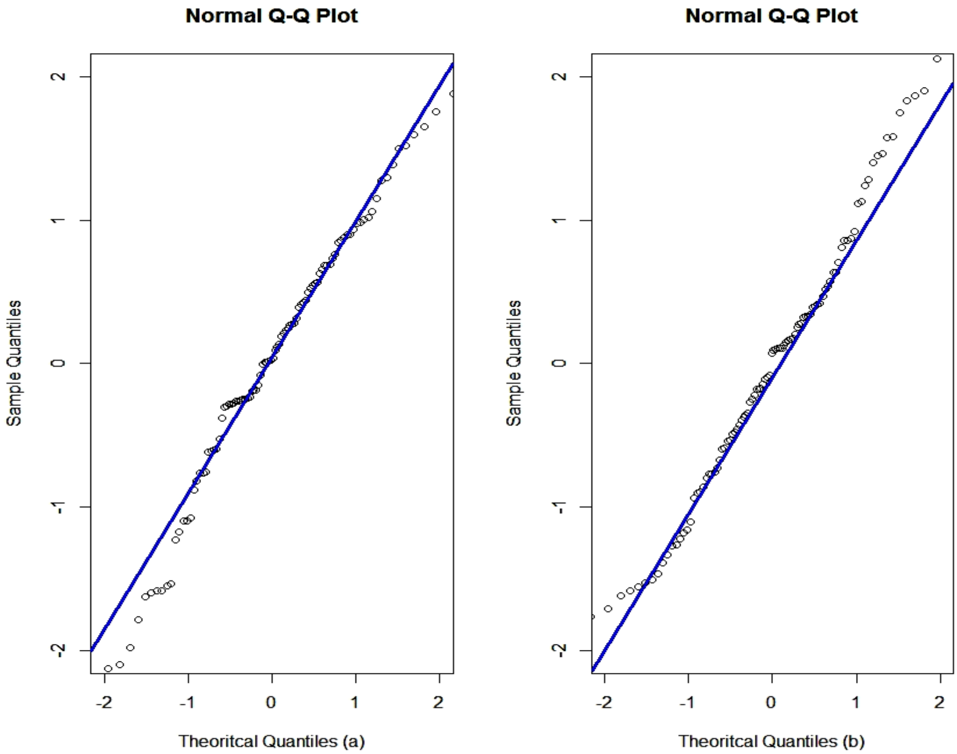
Let $\tilde{l}(\sigma,\Upsilon)$ be the pseudo-likelihood function defined such as

$$\tilde{l}(\sigma,\Upsilon)=\sum_{k=1}^{n}\left[1_{\{\delta_k=1\}}(\delta_k)\log\left(\left(1+\frac{\sigma}{\Upsilon}z_k\right)^{-\frac{1}{\Upsilon}}-\left(1+\frac{\sigma}{\Upsilon}y_k\right)^{-\frac{1}{\Upsilon}}\right)\right.$$
$$\left.-1_{\{\delta_k=2\}}(\delta_k)\frac{1}{\Upsilon}\log\left(1+\frac{\sigma}{\Upsilon}y_k\right)^{-\frac{1}{\Upsilon}}+1_{\{\delta_k=3\}}(\delta_k)\log\left\{1-\left(1+\frac{\sigma}{\Upsilon}z_k\right)^{-\frac{1}{\Upsilon}}\right\}\right].$$

The maximum pseudo-likelihood estimators $\hat{\sigma}$ and $\hat{\Upsilon}$ maximize equation 9. This optimization is carried out using the Barzilai-Borwein algorithm (BB) Varadhan and Gilbert (2009) which is found in the form of a package in the software **R** (Team, 2013). To check the asymptotic properties of the maximum pseudo-likelihood estimators of the parameters of a GPD model, we will proceed as follows. Let $X$ be the variable of interest of a $\text{GPD}(1,0.2)$ distribution censored by interval. But in the presence of an interval censoring instead of observing the variable of interest $X$, we observe the triplet $(Y,Z,\Delta)$. Thereby the sampling will be applied to the couple $(Y,Z)$. Let $n_1,n_2$ and $n_3$ be three increasing sample sizes. For each size, $N$ samples of the couple $(Y,Z)$ are replicated. The estimates of the parameters $\sigma$ and $\Upsilon$ of the GPD are obtained as empirical averages of the $N$ estimates. The numerical results of this procedure are summarized in Table 1.

| $N=10000$ | $n_1=100$ | $n_2=500$ | $n_1=1000$ |
|---|---|---|---|
| $\hat{\Upsilon}$ | 0.2092073 | 0.1913203 | 0.1996281 |
| $MAE$ | 0.1976629 | 0.10715128 | 0.07878693 |
| $MSE$ | 0.06230731 | 0.01686642 | 0.009497064 |
| $\hat{\sigma}$ | 1.011984 | 1.010003 | 1.009373 |
| $MAE$ | 0.20418 | 0.1068269 | 0.0796333 |
| $MSE$ | 0.06533427 | 0.01753554 | 0.00977675 |

In Table 1, we clearly see that the values of $\hat{\sigma}$ and $\hat{\Upsilon}$ converge to the true values $\Upsilon=0.2$ and $\sigma=1$ with very small mean square errors. To check the asymptotic normality of $\hat{\sigma}$ and $\hat{\Upsilon}$, we simulate $N=100$ samples of size $n=100$, and we draw the QQ-plot for each parameter in Figure 1. According to the two graphs in Figure 1, it is clear that the distributions of $\hat{\sigma}$ (Theoretical Quantiles b) and $\hat{\Upsilon}$ (Theoretical Quantiles a) are almost normal.




## 4. CONCLUSIONS

In this work we have shown that estimation by the method of maximum pseudo-likelihood in GPD models yields estimators still possessing properties of efficiency and of asymptotic normality. Moreover, the asymptotic normality of the model parameters makes it possible to construct confidence intervals for each parameter. This estimation method is similar to the M-estimator method, which is well adapted to the discrete variables. In extreme value models, tail index estimation is fundamental. This method of parametric estimation can provide us with sufficient statistics to construct hypothesis tests on the tail index.

## REFERENCES


1. Beirlant, J.A. (2008). *Statistics of extremes: Theory and applications.* John Wiley & Sons.
2. Gentleman, R.A. (1994). Maximum likelihood for interval censored data: Consistency and computation. *Biometrika*, 81(3), 618-623.
3. Gourieroux, C., Monfort, A., & Trognon, A. (1984). Pseudo maximum likelihood methods: Theory. *Econometrica: Journal of the Econometric Society*, 52(3), 681-700.
4. Klein, J.P. (1997). *Survival analysis: techniques for censored and truncated data.* Berlin; New York: Springer-Verlag Inc.

5. Lawless, J.F. (2003). *Statistical models and methods for lifetime data.* Chichester: John Wiley and Sons, New York.
6. Newey, W.K. (1994). Large sample estimation and hypothesis testing. Dans *Handbook of Econometrics*, Vol.\ IV, 2111-2245. North-Holland, Amsterdam.
7. Reiss, R.D.A. (2007). *Statistical analysis of extreme values with applications to insurance, finance, hydrology and other fields* (éd. Third). Birkhäuser Verlag, Basel.
8. Team, R.C. (2013). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria.
9. Turnbull, B.W. (1974). Nonparametric estimation of a survivorship function with doubly censored data. *Journal of the American Statistical Association*, 69(345), 169-173.
10. Van Der Vaart, A.W. (1998). *Asymptotic statistics.* Cambridge University Press, Cambridge.
11. Varadhan, R. and Gilbert, P. (2009). BB: An R package for solving a large system of nonlinear equations and for optimizing a high-dimensional nonlinear objective function. *Journal of Statistical Software*, 32(4), 1-26.