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ON THE GENERALIZED LOG BURR III DISTRIBUTION: DEVELOPMENT, PROPERTIES, CHARACTERIZATIONS AND APPLICATIONS

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ABSTRACT

In this paper, we present a generalized log Burr III (GLBIII) distribution developed on the basis of a generalized log Pearson differential equation (GLPE). The density function of the GLBIII is exponential, arc, J, reverse-J, bimodal, left-skewed, right-skewed and symmetrical shaped. The hazard rate function of GLBIII distribution has various shapes such as constant, increasing, decreasing, increasing-decreasing, upside-down bathtub and modified bathtub. Descriptive measures such as quantile function, sub-models, ordinary moments, moments of order statistics, incomplete moments, reliability and uncertainty measures are theoretically established. The GLBIII distribution is characterized via different techniques. Parameters of the GLBIII distribution are estimated using maximum likelihood method. A simulation study is performed to illustrate the performance of the maximum likelihood estimates (MLEs). Goodness of fit of this distribution through different methods is studied. The potentiality and usefulness of the GLBIII distribution is demonstrated via its applications to two real data sets.

KEY WORDS

Moments; Uncertainty; Reliability; Characterizations and Maximum Likelihood.

1. INTRODUCTION

In recent decades, various continuous univariate distributions have been established but many data sets from reliability, insurance, finance, hydrology, climatology, biomedical sciences and other areas do not follow these distributions. Therefore, modified, extended and generalized distributions and their applications to problems in these areas is a clear necessity of day.

The modified, extended and generalized distributions are attained by the introduction of some transformation or addition of one or more parameters to the baseline distribution. These new developed distributions provide better fit to data than the sub and competing models.

Burr (1942) proposed 12 distributions as Burr family to fit cumulative frequency functions on frequency data. Burr distributions XII, III and X are frequently used. Burr-

III (BIII) distribution has wide applications in modeling insurance data in finance and business and failure time data in reliability, survival and acceptance sampling plans.

Many modified, extended and generalized forms of BIII distribution are presented in literature such as two parameter family of distributions (Mielke; 1973), inverse Burr (Kleiber and Kotz; 2003), BIII (Gove et al.; 2008), three-parameter Burr III (Shao et al.; 2008), Dagum (Benjamin et al.; 2013) Modified BIII (Ali et al.; 2015), McDonald BIII (Gomes et al.; 2015), family of size (Sinner et al.; 2016), mixture of two BIII (Moisheer; 2016), gamma Burr III (Kehinde et al.; 2017), gamma BIII (Cordeiro et al.; 2017), Odd-Burr generalized family of distributions (Alizadeh et al.; 2017) odd Burr-III family (Jamil et al.; 2017), Kumaraswamy odd Burr G family (Nasir et al.; 2018) and generalized BIII (Kehinde et al.; 2018).

The main purpose of this article is to develop a more flexible model of BIII type called the GLBIII distribution. The GLBIII density is exponential, arc, J, reverse-J, bimodal, left-skewed, right-skewed and symmetrical shaped. The GLBIII distribution has constant, increasing, decreasing, increasing-decreasing, upside-down bathtub and modified bathtub hazard rate function. The flexible nature of the GLBIII distribution helps to serve as the best alternative model to other current models to model real data in reliability, life testing, survival analysis and other related areas of research. The GLBIII distribution offers better fits than sub and competing models.

The article is composed of the following sections. In section 2, the GLBIII distribution is developed on the basis of the generalized log Pearson differential equation. Some basic structural properties, some plots and sub-models are also studied. In section 3, moments, incomplete moments and some other properties are derived. In section 4, reliability and uncertainty measures are derived. In section 5, characterizations of GLBIII distribution is studied based on (i) conditional expectation; (ii) ratio of truncated moments; (iii) reverse hazard function and (iv) elasticity function. In Section 6, the parameters of the GLBIII are estimated using maximum likelihood method. In Section 7, a simulation study is performed to illustrate the performance of the maximum likelihood estimates (MLEs) of the GLBIII distribution. In Section 8, the potentiality and usefulness of the GLBIII distribution is demonstrated via its applications to the real data sets. Goodness of fit of the GLBIII distribution through different methods is studied. The ultimate comments are given in Section 9.

2. DEVELOPMENT OF THE GLBIII DISTRIBUTION

The generalized log Pearson differential equation (Bhatti et al. 2018a & Bhatti et al. 2018b) is

$$\frac{1}{f(x)}\frac{df}{dx}=\frac{1}{x}\frac{a_0+a_1(\ln x)+a_2(\ln x)^2+...+a_m(\ln x)^m}{b_0+b_1(\ln x)+b_2(\ln x)^2+...+b_n(\ln x)^n},\ x>1, \tag{1}$$

taking $a_2=a_3=....a_{m-2}=0,\ b_0=0, b_2=b_3=...b_{n-1}=0,\ m=n=-2a+1$, we have

$$\frac{d}{dx}\left[\ln f(x)\right]=\frac{1}{x}\left[\frac{a_0+a_1(\ln x)+a_{m-1}(\ln)^{-2a}+a_m(\ln)^{-2a+1}}{b_1(\ln x)+b_n(\ln)^{-2a+1}}\right]. \tag{2}$$

For $a_m = -b_n$, $a_1 = -b_1$, and integrating both sides of (2) we obtain

$$f(x) = k\frac{(\ln x)^{\frac{a_0}{b_1}}}{x}\left(1+\frac{b_n}{b_1}(\ln x)^{-2a}\right)^{\left(\frac{a_0 - a_{m-1}b_1}{2aa_0b_n}\right)}, \ x > 1, \tag{3}$$

where $k = \dfrac{2a}{\left(\dfrac{b_1}{b_n}\right)^{-\frac{a_0+b_1}{(2a)b_1}} B\left(\dfrac{a_0+b_1}{(-2a)b_1}, \dfrac{a_0 - a_{m-1}b_1 - a_0b_n - b_1b_n}{(-2a)b_1b_n}\right)}$ is the normalizing constant and $B(.,.)$ is the beta function.

Again taking $a_0 = -(2a+1)b^{-2a}, a_1 = -b^{-2a}, a_{m-1} = (2ap-1), a_m = -1, b_1 = b^{-2a}$, $b_n = 1$ in (3), so the pdf of the GLBIII distribution is

$$f(x) = \frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{(-2a-1)}\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]^{-(p+1)}, x > 1, a > 0, b > 0, p > 0. \tag{4}$$

The cdf of the GLBIII distribution with parameters a, b and p is

$$F(x) = \left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]^{-p}, x \geq 1, a > 0, b > 0, \mathrm{p} > 0. \tag{5}$$

2.1 Transformations and Compounding

The GLBIII distribution is derived through (i) ratio of exponential and gamma random variables and (ii) compounding generalized inverse log Weibull (GILW) and gamma distributions.

2.1 Lemma (i)

If the random variable Z_1 follows exponential distribution with parameter value 1 and the random variable Z_2 follows gamma i.e., $Z_2 \sim gamma(\theta; p)$, then for $Z_1 = \left(\frac{\ln X}{b}\right)^{-2a} Z_2$, we have $X = \exp\left[b\left(\frac{Z_2}{Z_1}\right)^{\frac{1}{2a}}\right] \sim GLBIII(a, b, p)$.

Lemma (ii):

Let $w(x; a, b, \theta) = \dfrac{2a\theta}{bx}\left(\dfrac{\ln x}{b}\right)^{-2a-1} e^{-\theta\left(\frac{\ln x}{b}\right)^{-2a}}, x > 1,$ be pdf of GILW distribution and let θ have gamma distribution with pdf $g(\theta, p) = \dfrac{1}{\Gamma(p)}\theta^{p-1}e^{-\theta}$, $\theta > 0$, then integrating the effect of θ with the help of

$$f\left(x;a,b,p\right)=\int_0^{\infty} GILW\left(x;a,b,\theta\right)g\left(\theta,p\right)d\theta$$, we have $X \sim GLBIII\left(a,b,p\right)$.

Proof for Lemma (ii) is given in Appendix A.

2.2 Basic Structural Properties

The survival function, hazard function, cumulative hazard function, reverse hazard function, the Mills ratio and elasticity of a random variable X with the GLBIII distribution with parameters a,b and p are given, respectively, by

$$S(x)=1-\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]^{-p}, \tag{6}$$

$$h\left(x\right)=\frac{\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{-2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]^{-\left(p+1\right)}}{1-\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]^{-p}}, \tag{7}$$

$$H(x)=-\ln\left(1-\left(1+\left(\frac{\ln x}{b}\right)^{-2a}\right)^{-p}\right), \tag{8}$$

$$\mathrm{r}_F(x)=\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]^{-1}, \tag{9}$$

$$m(x)=\frac{\mathrm{S}(x)}{f(x)}=\frac{bx}{2ap}\left(\frac{\ln x}{b}\right)^{2a+1}\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]^{\left(p+1\right)}\left(1-\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]^{-p}\right), \tag{10}$$

and

$$e\left(x\right)=\frac{d\ln \mathrm{F}(x)}{d\ln x}=\frac{2ap}{b}\left(\frac{\ln x}{b}\right)^{-2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]^{-1}. \tag{11}$$

The mode of the GLBIII distribution is obtained by solving $\frac{d}{dx}\left(\ln\left(f\left(x\right)\right)\right)=0$, *i.e.*,

$$-b\left(\frac{\ln x}{b}\right)\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]-\left(2a+1\right)\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]+2a\left(p+1\right)\left(\frac{\ln x}{b}\right)^{\left(-2a-1\right)}\left(\frac{\ln x}{b}\right)=0.$$

Let $\frac{\ln X}{b}=Y$, we have $bY^{-2a+1}-(2ap-1)Y^{-2a}+bY+(2a+1)=0.$

For a=1/2, we obtain $\hat{x}=\exp\left(-\left(b2^{-1}+1\right)+2^{-1}\sqrt{b^2+4+4bp}\right).$

The quantile function of the GLBIII distribution is $x_q=\exp\left[b\left(q^{-\frac{1}{p}}-1\right)^{-\frac{1}{2a}}\right]$ and its random number generator is $x=\exp\left[b\left(z^{-\frac{1}{p}}-1\right)^{-\frac{1}{2a}}\right]$ where the random variable Z has the uniform distribution on $(0,1)$.

2.3 Sub Models of the GLBIII Distribution

The GLBIII distribution has the following sub models.

S.No.	X	$2a$	b	p	**Model**
1	X	$2a$	b	1	new double log-logistic distribution
2	X	p	b	p	new log Para-log-logistic distribution
3	X	1	b	p	new extended generalized inverse log Weibull distribution
4	X	1	b	1	new extended generalized inverse log exponential distribution
5	X	1	b	2	new extended generalized inverse log Rayleigh distribution
6	X	$2a$	$\beta p^{-\frac{1}{2a}}$	$p\to\infty$	generalized inverse log Weibull distribution
7	X	1	$\beta p^{-\frac{1}{2a}}$	$p\to\infty$	generalized inverse log exponential distribution
8	$\ln X=Y$	$2a$	1	p	generalized BIII distribution
9	$\ln X=Y$	$2a$	b	p	BIII distribution
10	$\ln X=Y$	1	b	p	Lomax distribution

2.4 Shapes of the GLBIII Density and Hazard Rate Functions

Figure 1 shows that shapes of the GLBIII density are exponential, arc, J, reverse-J, bimodal, left-skewed, right-skewed and symmetrical. The GLBIII distribution has constant, increasing, decreasing, increasing-decreasing, upside-down bathtub and modified bathtub hazard rate function (Figure 2).

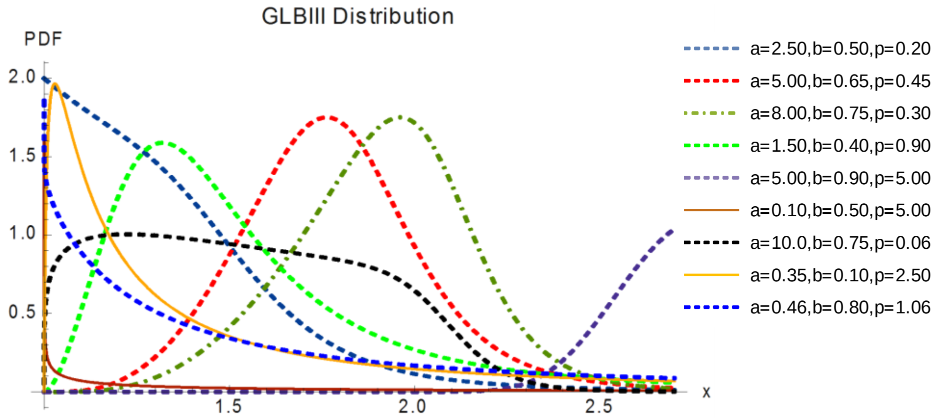

Fig. 1: Plots of pdf of the GLBIII Distribution

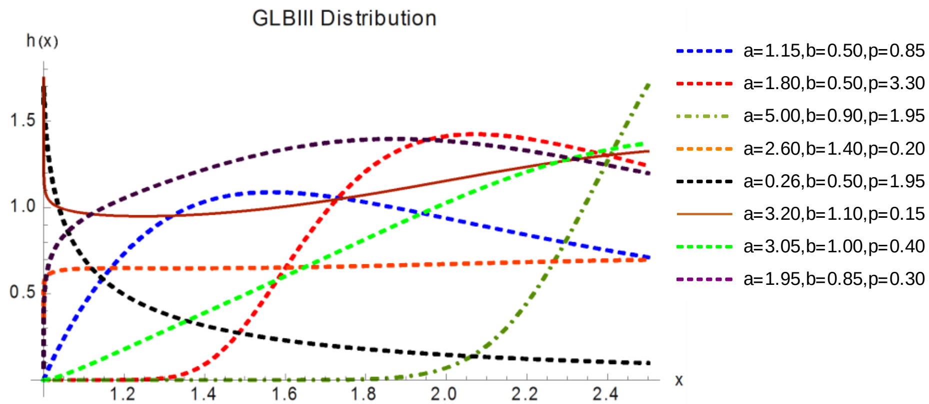

Fig. 2: Plots of hrf of the GLBIII Distribution

2.5 Stochastic Ordering

The idea of stochastic ordering is often used to display the ordering tool in life time models (Shaked et al., 1995). Stochastic orders are also used to study location and magnitude of random variables.

Definition 2.5.1:

A random variable is said to be smaller than a random variable Y in the (i) stochastic order $X \leq_{st} Y$ if $F_x(x) \geq F_y(x)$ for all x, (ii) hazard rate order $X \leq_{hr} Y$ if

$h_x(x) \geq h_y(x)$ for all x, (iii) mean residual life order $X \leq_{mrl} Y$ if $m_x(x) \geq m_y(x)$ for all x, and (iv) likelihood ratio order $X \leq_{lr} Y$, if $\frac{f_x(x)}{f_y(x)}$ decreases in x.

Theorem 2.5.1:

Let the random variables X and Y have the GLBIII distributions. Then, X is said to be smaller than Y in likelihood ratio order $X \leq_{lr} Y$, for (i) $b_1 = b_2, p_1 = p_2, a_1 > a_2$ and (ii) $a_1 = a_2, b_1 = b_2, p_1 < p_2$, if $\frac{d}{dx}\left[\ln \frac{f_x(x)}{f_y(x)}\right] < 0$.

Proof:

For $X \sim GLBIII(a_1, b_1, p_1)$ and $Y \sim GLBIII(a_2, b_2, p_2)$, we have

$$\ln \frac{f_x(x)}{f_y(x)} = -2(a_1 - a_2)\ln(\ln x) + \ln\left(\frac{a_1 p_1 b_1^{2a_1}}{a_2 p_2 b_2^{2a_2}}\right)$$
$$-(p_1+1)\ln\left[1+\left(\frac{\ln x}{b_1}\right)^{-2a_1}\right] + (p_2+1)\ln\left[1+\left(\frac{\ln x}{b_2}\right)^{-2a_2}\right],$$

$$\frac{d}{dx}\left[\ln \frac{f_x(x)}{f_y(x)}\right] = \left[\frac{2(a_2 - a_1)}{x \ln x} + (p_1+1)\frac{\frac{2a_1}{xb_1}\left(\frac{\ln x}{b_1}\right)^{-2a_1-1}}{\left[1+\left(\frac{\ln x}{b_1}\right)^{-2a_1}\right]} - (p_2+1)\frac{\frac{2a_2}{xb_2}\left(\frac{\ln x}{b_2}\right)^{-2a_2-1}}{\left[1+\left(\frac{\ln x}{b_2}\right)^{-2a_2}\right]}\right],$$

Case (i) $b_1 = b_2, p_1 = p_2, a_1 > a_2$, we have $\frac{d}{dx}\left[\ln \frac{f_x(x)}{f_y(x)}\right] < 0.$

Case (ii) $a_1 = a_2, b_1 = b_2, p_1 < p_2$,

we have $\frac{d}{dx}\left[\ln \frac{f_x(x)}{f_y(x)}\right] = [p_1 - p_2]\frac{\frac{2a_1}{xb_1}\left(\frac{\ln x}{b_1}\right)^{-2a_1-1}}{\left[1+\left(\frac{\ln x}{b_1}\right)^{-2a_1}\right]}$ that is $\frac{d}{dx}\left[\ln \frac{f_x(x)}{f_y(x)}\right] < 0$.

3. MOMENTS

Some descriptive measures for the GLBIII distribution such as ordinary and incomplete moments and other related properties are derived in this section.

3.1 Moments about Origin

The rth moment about origin of X with the GLBIII distribution is

$$\mu_r' = E(X^r) = \int_1^\infty x^r f(x)dx,$$

$$\mu_r' = \int_1^\infty x^r \frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{(-2a-1)}\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]^{-(p+1)} dx,$$

Let $\left(\frac{\ln x}{b}\right)^{-2a} = w,\ \frac{-2a}{bx}\left(\frac{\ln x}{b}\right)^{(-2a-1)} = dw, \quad \ln x = bw^{-\frac{1}{2a}}, \quad x = e^{bw^{-\frac{1}{2a}}},$

we arrive at

$$\mu_r' = p\int_0^\infty e^{rbw^{-\frac{1}{2a}}}\left[1+w\right]^{-(p+1)} dw,$$

$$\mu_r' = p\sum_{k=0}^\infty \frac{(rb)^k}{\text{k!}} B\left(1-\frac{k}{2a}, p+\frac{k}{2a}\right), \ \text{r} = 1,2,3,\ldots \ . \tag{12}$$

Mean and variance of the GLBIII distribution are

$$E(X) = p\sum_{k=0}^\infty \frac{(b)^k}{\text{k!}} B\left(1-\frac{k}{2a}, p+\frac{k}{2a}\right), \tag{13}$$

$$\text{Var}(X) = p\sum_{k=0}^\infty \frac{(2b)^k}{\text{k!}} B\left(1-\frac{k}{2a}, p+\frac{k}{2a}\right) - \left(p\sum_{k=0}^\infty \frac{(b)^k}{\text{k!}} B\left(1-\frac{k}{2a}, p+\frac{k}{2a}\right)\right)^2. \tag{14}$$

The factorial moments for the GLBIII distribution are given by

$$E[X]_n = \sum_{r=1}^n \alpha_r \mu_r' = p\sum_{r=1}^n\left[\alpha_r \sum_{k=0}^\infty \frac{(rb)^k}{\text{k!}} B\left(1-\frac{k}{2a}, p+\frac{k}{2a}\right)\right], \tag{15}$$

where $[X]_i = X(X+1)(X+2)......(X+i-1)$ and α_r is Stirling number of the first kind.

The Mellin transform is used to obtain moments of a probability distribution. By definition, the Mellin transform is

$$\text{M}\{f(x);\text{s}\} = f^*(s) = \int_0^\infty f(x)\, x^{s-1}dx. \tag{16}$$

The Mellin transform with the GLBIII distribution is written as

$$M_X(s) = E\left[X^{s-1}\right] = p\sum_{k=0}^{\infty}\frac{\left((s-1)b\right)^k}{\text{k!}}B\left(1-\frac{k}{2a}, p+\frac{k}{2a}\right). \tag{17}$$

The rth moment about origin of ln(X) with the GLBIII distribution is

$$E\left(\left(\ln(\text{X})\right)^r\right) = \int_1^{\infty}(\ln x)^r\,\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{-2a-1}\left(1+\left(\frac{\ln x}{b}\right)^{-2a}\right)^{-(p+1)}dx,$$

$$E\left(\left(\ln(\text{X})\right)^r\right) = pb^r B\left(1-\frac{r}{2a}, p+\frac{r}{2a}\right)\ r=1,2,3,\ldots \tag{18}$$

Negative moments help in harmonic mean and many other measures. The r[th] negative moment about origin of *X* with the GLBIII distribution is

$$\mu'_{-r} = E\left(X^{-r}\right) = \int_1^{\infty}x^{-r}\,\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{-2a-1}\left(1+\left(\frac{\ln x}{b}\right)^{-2a}\right)^{-(p+1)}dx,$$

Let

$$\left(\frac{\ln x}{b}\right)^{-2a} = w,\ \frac{-2a}{bx}\left(\frac{\ln x}{b}\right)^{(-2a-1)} = dw,\quad \ln x = bw^{-\frac{1}{2a}},\quad x = e^{bw^{-\frac{1}{2a}}},$$

$$\mu'_{-r} = p\int_0^{\infty}e^{-rbw^{-\frac{1}{2a}}}\left[1+w\right]^{-(p+1)}dw$$

$$\mu'_{-r} = E\left(X^{-r}\right) = p\sum_{i=0}^{\infty}\frac{(-rb)^i}{i!}B\left(1-\frac{i}{2a}, p+\frac{i}{2a}\right). \tag{19}$$

The qth moment about mean, Pearson's measure for skewness, kurtosis, moment generating function and cumulants of X with the GLBIII distribution are obtained from the relations

$$\mu_q = \sum_{v=1}^{q}\binom{q}{v}(-1)^v\,\mu'_v\,\mu'_{v-q},\ \gamma_1 = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}},\ \beta_2 = \frac{\mu_4}{(\mu_2)^2},$$

$$M_X(t) = E\left[e^{tX}\right] = \sum_{r=1}^{\infty}\frac{t^r}{\text{r!}}E(X)^r,$$

and

$$k_r = \mu'_r - \sum_{c=1}^{r-1}\binom{r-1}{c-1}k_c\ \mu'_{r-c}.$$

Table 2 displays the numerical measures of the median, mean, standard deviation, skewness and kurtosis of the GLBIII distribution for selected parameter values to describe their effect on these measures.

Table 2
Median, Mean, Standard Deviation, Skewness and Kurtosis of the GLBIII Distribution

Parameters a, b, p	Median	Mean	Standard Deviation	Skewness	Kurtosis
5,0.5,1.5	1.69412	1.72092	0.169446	4.96677	305.388
5,0.3,1.5	1.37204	1.38354	0.0784278	2.22326	39.2645
5,0.1,1.5	1.11119	1.11391	0.0204519	1.49773	12.332
5,0.1,5	1.12859	1.13259	0.0211289	2.04685	17.5119
5,0.05,1.25	1.05285	1.05396	1.25735	1.25735	9.65814
5,0.05,0.2	1.03607	1.03551	0.0144314	0.122825	3.34257
5,0.05,0.1	1.02527	1.02622	0.016278	0.37709	2.069378
5,0.05,0.05	1.01254	1.01732	0.0162025	0.899206	3.14658
10,0.1,0.1	1.07321	1.07005	0.0262677	-0.342413	2.41511
10,0.5,0.3	1.56462	1.55504	0.112433	-0.266478	3.73011
10,0.5,0.1	1.42371	1.41125	0.170044	-0.149073	2.36218
10,0.5,0.05	1.28349	1.3004	0.193087	0.29747	2.10463
10,0.1,0.05	1.05118	1.05209	0.0311248	0.126999	1.95597
10,0.5,0.5	1.60507	1.60289	0.0921806	0.070709	4.54126
10,0.3,0.5	1.3283	1.32668	0.0458387	-0.0463268	4.32342
10,3,0.5	17.0989	17.8197	6.66389	11.3433	1479.27
6,3,1.5	22.9938	26.7182	2550.81	-440.561	586931
10,2,5	9.02216	9.37429	1.70529	5.97514	372.343
5,0.5,5	1.83093	1.87046	0.193718	10.1907	1124.27
5,3,5	37.6723	53.551	3308.388	-212.388	230.315

3.2 Moments of Order Statistics

Moments of order statistics are used to predict failure of future items based on the times of a few early failures. The pdf of *mth* order statistic $X_{\mathrm{m}:n}$ is given by

$$f\left(x_{\mathrm{m}:n}\right)=\frac{1}{B\left(\mathrm{m},n-m+1\right)}\left[F\left(x\right)\right]^{m-1}\left[1-F\left(x\right)\right]^{n-m} f\left(x\right).$$

The pdf of *mth* order statistic $X_{\mathrm{m}:n}$ for the GLBIII distribution is

$$f\left(x_{m:n}\right)=\frac{\sum_{j=0}^{n-m}(-1)^{j}\binom{n-m}{j}}{B(m,n-m+1)}\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{-2a-1}\left(1+\left(\frac{\ln x}{b}\right)^{-2a}\right)^{-(pm+pj+1)}. \tag{20}$$

Moments about origin of mth order statistic $X_{m:n}$ for the GLBIII distribution are given by

$$E\left(X^{r}{}_{m:n}\right)=\int_{1}^{\infty}x^{r}f\left(x_{m:n}\right)dx$$

$$=\int_{1}^{\infty}x^{r}\frac{\sum_{j=0}^{n-m}(-1)^{j}\binom{n-m}{j}}{B(m,n-m+1)}\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{-2a-1}\left(1+\left(\frac{\ln x}{b}\right)^{-2a}\right)^{-(pm+pj+1)}dx,$$

Taking $\left(\frac{\ln x}{b}\right)^{-2a}=w,\ \frac{-2a}{bx}\left(\frac{\ln x}{b}\right)^{(-2a-1)}=dw,\quad \ln x=bw^{-\frac{1}{2a}},\quad x=e^{bw^{-\frac{1}{2a}}}$, we have

$$E\left(X^{r}{}_{m:n}\right)=\frac{\sum_{j=0}^{n-m}(-1)^{j}\binom{n-m}{j}}{B(m,n-m+1)}\sum_{i=0}^{\infty}\frac{(rb)^{i}}{i!}B\left(1-\frac{i}{2a},\,pm+pj+\frac{i}{2a}\right). \tag{21}$$

Mean of mth order statistic $X_{m:n}$ for the GLBIII distribution is

$$E\left(X_{m:n}\right)=\frac{\sum_{j=0}^{n-m}(-1)^{j}\binom{n-m}{j}}{B(m,n-m+1)}\sum_{i=0}^{\infty}\frac{(b)^{i}}{i!}B\left(1-\frac{i}{2a},\,pm+pj+\frac{i}{2a}\right).$$

4. RELIABILITY AND UNCERTAINTY MEASURES

Different reliability and uncertainty measures are studied.

4.1 Stress-strength Reliability for The GLBIII Distribution

If $X_1\sim GLBIII\left(a,b,p_1\right),X_2\sim GLBIII\left(a,b,p_2\right)$ such that X_1 represents "strength" and X_2 represents "stress" and X_1,X_2 follow a joint pdf $f\left(x_1,x_2\right)$, therefore $R=\Pr\left(X_2<X_1\right)=\int_{1}^{\infty}f_{x_1}(x)F_{x_2}(x)dx$ is reliability parameter R. Then reliability of the component is

$$R=\int_{1}^{\infty}\frac{2ap_1}{bx}\left(\frac{\ln x}{b}\right)^{-2a-1}\left(1+\left(\frac{\ln x}{b}\right)^{-2a}\right)^{-(p_1+p_2+1)}dx=\frac{p_1}{\left(p_1+p_2\right)}. \tag{22}$$

Therefore, R is independent of *a* and *b*.

4.2 Estimation of Multicomponent Stress-Strength System Reliability for the GLBIII Distribution

Suppose a machine has at least "*s*" components working out of "*k*" components. The strengths of all components of system are $X_1, X_2, ...X_k$ and stress *Y* is applied on the system. Both strengths $X_1, X_2, ...X_k$ and stress Y are i.i.d. and independent. G is cdf of *Y* and *F* is cdf of *X*. The reliability of a machine is the probability that the machine functions properly.

If $X \sim GLBIII\left(a, b, p_1\right)$, and $Y \sim GLBIII\left(a, b, p_2\right)$ with unknown shape parameters p_1, p_2 and common scale parameter b, where X and Y and are independently distributed. The reliability in multicomponent stress- strength for GLBIII distribution using:

$$R_{s,k} = P\left(strengths > stress\right) = P[at\,least\, "s"\, of\; (\mathrm{X}_1, \mathrm{X}_2, ...\mathrm{X}_k)\, exceed\, Y], \tag{23}$$

$$R_{s,k} = \sum_{l=s}^{k} \binom{k}{l} \int_{-\infty}^{\infty} [1 - G(y)]^l [G(y)]^{k-l}\, dF(y), \tag{24}$$

(Bhattacharyya and Johnson; 1974)

$$R_{s,k} = \sum_{l=s}^{k} \binom{k}{l} \int_{1}^{\infty} \left(1 - \left[1 + \left(\frac{\ln y}{b}\right)^{-2a}\right]^{-p_1}\right)^{l} \left(\left[1 + \left(\frac{\ln y}{b}\right)^{-2a}\right]^{-p_1}\right)^{(k-l)} d\left[1 + \left(\frac{\ln y}{b}\right)^{-2a}\right]^{-p_2},$$

Let $t = \left[1 + \left(\frac{\ln y}{b}\right)^{-2a}\right]^{-p_2}$, we obtain $R_{s,k} = \sum_{\ell=s}^{k} \binom{k}{\ell} \int_{0}^{1} \left(1 - t^{\nu}\right)^{\ell} t^{\nu(k-\ell)}\, dt.$

Let $z = t^{\nu}$, $t = z^{\frac{1}{\nu}}$ $\quad dt = \frac{1}{\nu} z^{\frac{1}{\nu}-1} dz$,

$$R_{s,k} = \sum_{\ell=s}^{k} \binom{k}{\ell} \int_{0}^{1} (z)^{\ell} \left(1 - z\right)^{(k-\ell)} \frac{1}{\nu} z^{\frac{1}{\nu}-1} dz$$

$$R_{s,k} = \frac{1}{\nu} \sum_{\ell=s}^{k} \binom{k}{\ell} B\left(\ell + \frac{1}{\nu}, k - \ell + 1\right), \; where \quad \nu = \frac{p_1}{p_2}.$$

Finally we obtain as

$$R_{s,k} = \frac{1}{\nu} \sum_{\ell=s}^{k} \frac{k!}{(k-\ell)!} \left(\prod_{j=0}^{\ell} \left(k + \nu - j\right)\right)^{-1} \tag{25}$$

The probability in (25) is called reliability in a multicomponent stress-strength model.

4.3 Shannon Entropy

Claude Shannon (1948) introduced "Shannon entropy" to measure expected information in a message. Shannon entropy for random variable X having pdf (4) is given by

$$h(X)=E\left(-\ln f(X)\right)=-\int_{1}^{\infty}\ln f(x)\,f(x)\,dx \tag{26}$$

$$h(X)=-\int_{1}^{\infty}\ln\left\{\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{-2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]^{-(p+1)}\right\}$$
$$\left(\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+1)}\right)dx,$$

$$h(X)=pbB\left(1-\frac{1}{2a},\ p+\frac{1}{2a}\right)+\frac{(2a+1)}{2a}\left(\psi(1)-\psi(p)\right)+\frac{(p+1)}{p}-\ln\left(\frac{2ap}{b}\right), \tag{27}$$

where $\psi(x)$ is digamma function, $\psi(p)=\frac{d}{dp}\left[\ln\Gamma(p)\right]$.

Awad (1987) provided the extension of Shannon entropy as $A(X)=-\int_{1}^{\infty}f(x)\ln\frac{f(x)}{\delta}dx.$

If random variable X has the GLBIII distribution, then Awad entropy is given by

$$A(X)=\ln\delta+pbB\left(1-\frac{1}{2a},\ p+\frac{1}{2a}\right)+\frac{(2a+1)}{2a}\left(\psi(1)-\psi(p)\right)+\frac{(p+1)}{p}-\ln\left(\frac{2ap}{b}\right). \tag{28}$$

4.4 Renyi Entropy, Q-Entropy, Havrda and Chavrat Entropy and Tsallis-Entropy

Renyi entropy (1970) is an extension of Shannon entropy. Renyi entropy for the GLBIII distribution is given as

$$I_R(v)=\frac{1}{1-v}\log\left(\int_{-\infty}^{\infty}(f(x))^{v}dx\right)\quad v\neq 1, v>0,$$

$$\int_{1}^{\infty}\left[f(x)\right]^{\nu}dx=\int_{1}^{\infty}\left[\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+1)}\right]^{\nu}dx,$$

$$\int_{1}^{\infty}\left[f(x)\right]^{\nu}dx=\sum_{k=0}^{\infty}\frac{(-1)^{k}\,\Gamma\left(\nu\left(p+1\right)+k\right)}{\Gamma\left(\nu\left(p+1\right)\right)}\frac{2^{\nu}a^{\nu}p^{\nu}}{b^{2v-2ak+2av}}\frac{1}{k!}\frac{\Gamma\left(1-2ak+2av+v\right)}{\left(1-v\right)^{1-2ak+2av+v}},$$

$$I_{R}=\frac{1}{1-\nu}\log\left\{\sum_{k=0}^{\infty}\frac{(-1)^{k}\,\Gamma\left(\nu\left(p+1\right)+k\right)}{\Gamma\left(\nu\left(p+1\right)\right)}\frac{2^{\nu}a^{\nu}p^{\nu}}{b^{2v-2ak+2av}}\frac{1}{k!}\frac{\Gamma\left(1-2ak+2av+v\right)}{\left(1-v\right)^{1-2ak+2av+v}}\right\}. \quad (29)$$

The Q-entropy for the GLBIII distribution is

$$H_{q}\left(f\right)=\frac{1}{1-q}\log\left\{1-\sum_{k=0}^{\infty}\frac{(-1)^{k}\,\Gamma\left(q\left(p+1\right)+k\right)}{\Gamma\left(q\left(p+1\right)\right)}\frac{2^{q}a^{q}p^{q}}{b^{2q-2ak+2aq}}\frac{1}{k!}\frac{\Gamma\left(1-2ak+2aq+q\right)}{\left(1-q\right)^{1-2ak+2aq+q}}\right\}. \quad (30)$$

The Havrda and Chavrat entropy (1967) for the GLBIII distribution is

$$I_{R}=\frac{1}{\nu-1}\log\left\{\sum_{k=0}^{\infty}\frac{(-1)^{k}\,\Gamma\left(\nu\left(p+1\right)+k\right)}{\Gamma\left(\nu\left(p+1\right)\right)}\frac{2^{\nu}a^{\nu}p^{\nu}}{b^{2v-2ak+2av}}\frac{1}{k!}\frac{\Gamma\left(1-2ak+2av+v\right)}{\left(1-v\right)^{1-2ak+2av+v}}\right\}. \quad (31)$$

The Tsallis-entropy (1988) for the GLBIII distribution is

$$S_{q}\left(f\left(x\right)\right)=\frac{1}{q-1}\left\{1-\sum_{k=0}^{\infty}\frac{(-1)^{k}\,\Gamma\left(q\left(p+1\right)+k\right)}{\Gamma\left(q\left(p+1\right)\right)}\frac{2^{q}a^{q}p^{q}}{b^{2q-2ak+2aq}}\frac{1}{k!}\frac{\Gamma\left(1-2ak+2aq+q\right)}{\left(1-q\right)^{1-2ak+2aq+q}}\right\}. \quad (32)$$

Shannon entropy, collision entropy, Hartley entropy and Min entropy can be obtained from Renyi entropy. Renyi entropy tends to Shannon entropy as $v\rightarrow 1$. Renyi entropy tends to quadratic entropy $as\, v\rightarrow 2$.

Entropies are applied to study heart beat intervals (cardiac autonomic neuropathy (CAN), DNA sequences, anomalous diffusion, daily temperature fluctuations (climatic), and information content signals.

5. CHARACTERIZATIONS

In this section, we present characterizations of the GLBIII distribution based on; (i) conditional expectation; (ii) ratio of truncated moments; (iii) reverse hazard function and (iv) elasticity function.

5.1 Characterization based on Conditional Expectation

The conditional expectation is employed to characterize the GLBIII distribution.

Proposition 5.1.1

Let $X:\Omega\rightarrow\left(1,\infty\right)$ be a continuous random variable with cdf F($0<F\left(x\right)<1$ with $x>1$). Then for p>1, X has cdf (5) if and only if

$$E\left(\left(\frac{\ln x}{b}\right)^{-2a} < t\right) = \frac{1+p\left(\frac{\ln t}{b}\right)^{-2a}}{(p-1)}. \tag{33}$$

Proof:

For random variable X with pdf (4), it is easy to show that

$$E\left(\left(\frac{\ln x}{b}\right)^{-2a} < t\right) = \frac{1+p\left(\frac{\ln t}{b}\right)^{-2a}}{(p-1)} \quad for \;\; p>1.$$

Conversely if (33) holds, then

$$\frac{1}{F(t)}\int_1^t \left(\frac{\ln x}{b}\right)^{-2a} f(x)dx = \frac{1+p\left(\frac{\ln t}{b}\right)^{-2a}}{(p-1)}. \tag{34}$$

Differentiating both sides of (34) with respect to t, we obtain

$$(p-1)\left(\frac{\ln t}{b}\right)^{-2a} f(t) = f(t) + p\left(\frac{\ln t}{b}\right)^{-2a} f(t) - \frac{2ap}{\ln t}\left(\frac{\ln t}{b}\right)^{-2a-1} F(t).$$

After simplification and integration from 1 to ∞, we obtain

$$F(t) = \left[1+\left(\frac{\ln t}{b}\right)^{-2a}\right]^{-p}$$

is the cdf of the GLBIII distribution.

5.2 Characterization Based on the Ratio of Two Truncated Moments

The GLBIII distribution is characterized using Theorem 1 (Glanzel; 1990) on the basis of a simple relationship between two truncated moments of X. Theorem 1 is given in Appendix B.

Proposition 5.2.1:

Let $X:\Omega \to (1,\infty)$ be a continuous random variable and let

$$g_1(x) = p^{-1}\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]^{p+1} \quad \text{and} \quad g_2(x) = 2p^{-1}\left(\frac{\ln x}{b}\right)^{-2a}\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]^{(p+1)} \quad x>1.$$

The random variable *X* has pdf (4) if and only if the function $q(x)$ defined in Theorem 1, has the form $q(x) = \left(\frac{\ln x}{b}\right)^{2a}, \quad x>1.$

Proof:

If X has pdf (4), then

$$\left(1-F(x)\right)E\left(g_1(x)/X\ge x\right)=\left(\frac{\text{lnx}}{b}\right)^{-2a},x>1,$$

and

$$\left(1-F(x)\right)E\left(g_2(x)/X\ge x\right)=\left(\frac{\text{lnx}}{b}\right)^{-4a},\ x>1,$$

$$\frac{E[\mathrm{g}_1(X)/\mathrm{X}\ge \mathrm{x}]}{E[\mathrm{g}_2(X)/\mathrm{X}\ge \mathrm{x}]}=q(x)=\left(\frac{\ln x}{b}\right)^{2a}\ x>1.$$

Also $q(x)g_2(x)-g_1(x)\ne 0$, for $x>1$.

The differential equation $s'(x)=\dfrac{q'(x)g_2(x)}{q(x)g_2(x)-g_1(x)}=\dfrac{4a}{bx}\left(\dfrac{\ln x}{b}\right)^{-1}$ has solution $s(x)=\ln\left(\dfrac{\ln x}{b}\right)^{4a}$.

Therefore in the light of Theorem 5.2.1, X has pdf (4).

Corollary 5.2.1:

Let $X:\Omega\to(1,\infty)$ be a continuous random variable and let

$$g_2(x)=2p^{-1}\left(\frac{\ln x}{b}\right)^{-2a}\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]^{(p+1)},\ x>1.$$

The pdf of X is (4) if and only if there exist functions q and g_1 (defined in Theorem 1), satisfying the differential equation

$$\frac{q'(x)g_2(x)}{q(x)g_2(x)-g_1(x)}=\frac{4a}{bx}\left(\frac{\ln x}{b}\right)^{-1}.$$

Remark 6.1.1:

The general solution of above the differential equation is

$$q(x)=\left(\frac{\ln x}{b}\right)^{4a}\left[\int\left(-\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{-2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{2a}\right]^{-(p+1)}\right)g_1(x)dx+D\right],$$

where D is a constant.

5.3 Characterization Based on Reverse Hazard Function

Definition 5.3.1:

Let $X:\Omega \to (1,\infty)$ be a continuous random variable having absolutely continuous cdf $F(x)$ and pdf $f(x)$ provided the reverse hazard function r_F is twice differentiable function satisfying differential equation

$$\frac{d}{dx}\left[\ln f(x)\right] = \frac{r_F'(x)}{r_F(x)} + r_F(x). \tag{35}$$

Proposition 5.3.1

Let X: $\Omega \to (1,\infty)$ be continuous random variable .The pdf of X is (4) if and only if its reverse hazard function, r_F satisfies the first order differential equation

$$\mathrm{x r}_F'(x) + \mathrm{r}_F(x) = \frac{2ap\left(\frac{\ln x}{b}\right)^{-2a-2}\left[-(2a+1) - \left(\frac{\ln x}{b}\right)^{-2a}\right]}{b^2 x\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]^2}. \tag{36}$$

Proof:

If X has pdf (4), then (36) holds. Now if (36) holds, then

$$\frac{d}{dx}\left\{\mathrm{x r}_F(x)\right\} = \frac{2ap}{b}\frac{d}{dx}\left\{\frac{\left(\frac{\ln x}{b}\right)^{-2a-1}}{\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]}\right\},$$

or

$$\mathrm{r}_F(x) = \frac{2ap\left(\frac{\ln x}{b}\right)^{-2a-1}}{bx\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]},$$

is reverse hazard function of the GLBIII distribution.

5.4 Characterization Based On Elasticity Function

Definition 5.4.1:

Let $X:\Omega \to (1,\infty)$ be a continuous random variable having absolutely continuous $F(x)$ and pdf $f(x)$ provided the elasticity function $e_F(x)$ is twice differentiable function satisfying differential equation

$$\frac{d}{dx}\left[\ln f\left(x\right)\right]=\frac{\mathrm{e}'(x)}{e\left(x\right)}+\frac{e\left(x\right)}{x}-\frac{1}{x}. \tag{37}$$

Proposition 5.4.1:

Let $X:\Omega\rightarrow\left(1,\infty\right)$ be continuous random variable .The pdf of X is (4) provided that its elasticity function, $e_F\left(x\right)$ satisfies the first order differential equation

$$\mathrm{e}'_F(x)=\frac{2ap\left(\frac{\ln x}{b}\right)^{-2a-2}\left[-\left(2a+1\right)-\left(\frac{\ln x}{b}\right)^{-2a}\right]}{b^2x\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]^2}. \tag{38}$$

Proof:

If X has pdf (4), then (38) holds. Now if (38) holds, then

$$\frac{d}{dx}\left\{e_F\left(x\right)\right\}=\frac{2ap}{b}\frac{d}{dx}\left\{\frac{\left(\frac{\ln x}{b}\right)^{-2a-1}}{\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]}\right\},$$

or

$$\mathrm{e}_F(x)=\frac{2ap\left(\frac{\ln x}{b}\right)^{-2a-1}}{b\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]},$$

is the elasticity function of the GLBIII distribution.

6. MAXIMUM LIKELIHOOD ESTIMATION

In this section, parameters estimates are derived using maximum likelihood method. The log likelihood function for the GLBIII distribution with the vector of parameters $\Phi=\left(a,b,p\right)$ is

$$\ln L\left(x_i,\Phi\right)=n\ln 2+n\,\mathrm{lna}+n\,\mathrm{lnp}-2na\,\mathrm{lnb}$$
$$-\sum_{i=1}^{n}\ln x_i-\left(2a+1\right)\sum_{i=1}^{n}\ln\left(\ln x_i\right)-\left(p+1\right)\sum_{i=1}^{n}\ln\left[1+\left(\frac{\ln x_i}{b}\right)^{-2a}\right]. \tag{39}$$

In order to compute the estimates of the parameters of the GLBIII distribution, the following nonlinear equations must be solved simultaneously:

$$\frac{\partial \ln L}{\partial p} = \frac{n}{p} - \sum_{i=1}^{n} \ln\left[1+\left(\frac{\ln x_i}{b}\right)^{-2a}\right] = 0\,, \tag{40}$$

$$\frac{\partial \ln L}{\partial b} = \left\{-\frac{2na}{b} - \frac{2a(p+1)}{b}\sum_{i=1}^{n}\left(\frac{\ln x_i}{b}\right)^{-2a}\left[1+\left(\frac{\ln x_i}{b}\right)^{-2a}\right]^{-1}\right\} = 0\,, \tag{41}$$

$$\frac{\partial \ln L}{\partial a} = \left\{\begin{matrix} \frac{n}{a} - 2n\ln \mathrm{b} - 2\sum_{i=1}^{n}\ln(\ln x_i) + 2(p+1) \\ \sum_{i=1}^{n}\left(\frac{\ln x_i}{b}\right)^{1-2a}\left[1+\left(\frac{\ln x_i}{b}\right)^{-2a}\right]^{-1} \end{matrix}\right\} = 0. \tag{42}$$

The above equations 40-41 can be solved either directly or using the R (optim and maxLik functions), SAS (PROC NLMIXED) and Ox program (sub-routine Max BFGS) or employing non-linear optimization approaches such as the quasi-Newton algorithm.

7. SIMULATION STUDY

In this section, a simulation study to assess the performance of the MLEs of the GLBIII parameters with respect to sample size n is carried out. This performance is done based on the following simulation study:

Step 1: Generate 10000 samples of size *n* from the GLBIII distribution based on the inverse cdf method.

Step 2: Compute the MLEs for 10000 samples, say $(\hat{a}, \hat{b}, \hat{p})$ for i =1, 2,..., 10000 based on non-linear optimization algorithm with constraint matching to range of parameters. (1.5, 0.75, 0.5), (2, 1, 1) and (2.5, 2, 1.5) are taken as the true parameter values (a, b, p) .

Step 3: Compute the means, biases and mean squared errors of MLEs.

For this purpose, we have selected different arbitrarily parameters and n = 50, 100, 200, 300, 500 sample sizes. All codes are written in R and the results are summarized in Table 3. The results clearly show that when the sample size increases, the MSE of estimated parameters decrease and biases drop to zero. As shape parameter increases, MSE of estimated parameters increases. This shows the consistency of MLE estimators.

Table 3
Means, Bias and MSEs of the GLBIII distribution (1.25, 0.9, 0.5), (1.75, 1.25, 0.75) and (2, 1.5, 1)

Sample	Statistics	$a=1.25$	$b=0.9$	$p=0.5$	$a=1.75$	$b=1.25$	$p=0.75$	$a=2.0$	$b=1.5$	$p=1.0$
n=50	Means	1.5325	0.8991	0.6659	1.9566	1.2161	1.5758	2.1376	1.4294	3.4596
	Bias	0.2825	-9e-04	0.1659	0.2066	-0.0339	0.8258	0.1376	-0.0706	2.4596
	MSE	9.6972	0.0826	4.2327	6.7392	0.1035	45.8896	0.9189	0.1547	190.6051
n=100	Means	1.3083	0.8981	0.5354	1.8088	1.2364	0.8603	2.0545	1.4727	1.2989
	Bias	0.0583	-0.0019	0.0354	0.0588	-0.0136	0.1103	0.0545	-0.0273	0.2989
	MSE	0.0753	0.0364	0.0668	0.1084	0.0426	1.2666	0.1273	0.0606	7.0456
n=200	Means	1.2774	0.8979	0.5146	1.7748	1.2427	0.7869	2.0273	1.4865	1.0762
	Bias	0.0274	-0.0021	0.0146	0.0248	-0.0073	0.0369	0.0273	-0.0135	0.0762
	MSE	0.0304	0.017	0.0145	0.0454	0.0192	0.044	0.0559	0.0267	0.1473
n=300	Means	1.2679	0.8987	0.5092	1.7686	1.2451	0.7723	2.017	1.4908	1.045
	Bias	0.0179	-0.0013	0.0092	0.0186	-0.0049	0.0223	0.017	-0.0092	0.045
	MSE	0.0191	0.0111	0.0088	0.0288	0.0122	0.0252	0.0349	0.0167	0.0664
n=500	Means	1.2598	0.899	0.5054	1.759	1.2471	0.7636	2.0118	1.4962	1.0225
	Bias	0.0098	-0.001	0.0054	0.009	-0.0029	0.0136	0.0118	-0.0038	0.0225
	MSE	0.0108	0.0068	0.0052	0.0169	0.0074	0.0139	0.0198	0.0096	0.0322

8. APPLICATIONS

The GLBIII distribution is compared with LBIII, GBIII, BIII and Log-log distributions. Different goodness fit measures like Cramer-von Mises (W), Anderson Darling (A), Kolmogorov-Smirnov statistics with p-values, Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and likelihood ratio statistics are computed for are computed for river peak flows series and annual maximum precipitation records using R-Package.

The better fit corresponds to smaller W, A, K-S, AIC, CAIC, BIC, HQIC and $-\ell$ value. The maximum likelihood estimates (MLEs) of unknown parameters and values of goodness of fit measures are computed for the GLBIII distribution and its sub-models. The MLEs**,** their standard errors (in parentheses) and goodness-of-fit statistics like W, A, K-S (p-value) are given in table 4 and 6. Table 5 and 7 displays goodness-of-fit values.

8.1 River Peak Flows Series:

The data for 47 years of Styx River (Jeogla) about annual maximum flood peaks series are analyzed (Kuczera and Frank; 2015). The values of data are: 878, 541, 521, 513, 436, 411, 405, 315, 309, 300, 294, 258, 255, 235, 221, 220, 206, 196, 194, 190, 186, 177, 164, 126, 117, 111, 108, 105, 92.2, 88.6, 79.9, 74, 71.9, 62.6, 61.2, 60.3, 58, 53.5, 39.1, 26.7, 26.1, 23.8, 22.4, 22.1, 18.6, 13, 8.18.

Table 4
MLEs and their Standard Errors (in parentheses) for Data Set I

Model	a	b	p	**W**	**A**	**K-S (p-value)**
GLBIII	11.4601050 (4.80067147)	6.0126550 (0.23905762)	0.1592211 (0.08340439)	**0.02869574**	**0.1995404**	**0.0627 (0.9872)**
LBIII	1.658518 (0.1604954)	1	101.995263 (42.5143551)	0.4409367	2.511915	0.1827 (0.0761)
GBIII	2.781859 (1.0735258)	287.895085 (100.38211)	0.338965 (0.1979333)	0.04942164	0.3241298	0.0728 (0.9492)
BIII	43.5108866 (15.35191549)	1	0.8943899 (0.09025759)	0.2313807	1.397193	0.1255 (0.4152)
Log-log	0.3203919 (0.03630092)	1	1	0.09965742	0.62833	0.6759 (5.551e-16)

Table 5
Goodness-of-Fit Statistics for Data Set I

Model	**AIC**	**CAIC**	**BIC**	**HQIC**	$-\ell$
GLBIII	**591.0896**	**591.6477**	**596.64**	**593.1782**	**292.5448**
LBIII	619.6599	619.9326	623.3602	621.0523	307.8299
GBIII	594.0128	594.5709	599.5632	596.1015	294.0064
BIII	603.4944	603.7672	607.1947	604.8869	299.7472
Log-log	738.459	738.5481	740.3093	739.1554	368.2296

The GLBIII distribution is best fitted than LBIII, GBIII, BIII and Log-log distributions because the values of all criteria of goodness of fit are significantly smaller for the GLBIII.

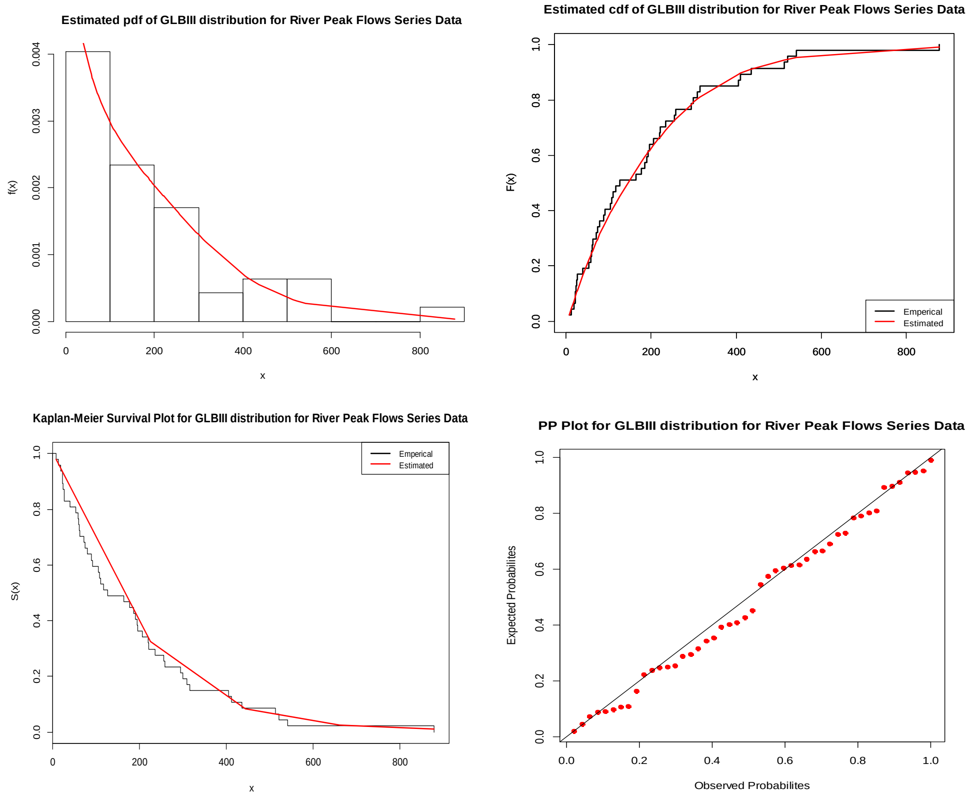

Fig. 3: Fitted pdf, cdf, Survival and PP Plots of the GLBIII Distribution for River Peak Flows Series

We can also observe that the GLBIII distribution is best fitted to empirical data (Figure 3).

8.2 Annual Maximum Precipitation Data:

The dataset comprise on 59 annual maximum precipitation data of the Karachi city, Pakistan for the years 1950–2009(1987 is missing). The precipitation records are necessary for water management studies and flood defense systems. The precipitation data are used to predict the flood and drought. The precipitation data also help to minimize the risk of large hydraulic structures. The values of data are: 117.6, 157.7, 148.6, 11.4, 5.6, 63.6, 62.4, 11.8, 6.5, 54.9,39.9, 16.8, 30.2, 38.4, 76.9, 73.4, 85, 256.3, 24.9, 148.6, 160.5, 131.3, 77, 155.2, 217.2, 105.5, 166.8, 157.9, 73.6, 291.4, 210.3, 315.7, 107.7, 33.3, 302.6, 159.1, 78.7, 33.2, 52.2, 92.7,150.4, 43.7, 68.3, 20.8, 179.4, 245.7, 19.5, 30, 270.4, 160, 96.3, 185.7, 429.3, 184.9, 262.5, 80.6, 138.2, 28, 39.3.

Table 6
MLEs and their Standard Errors (in parentheses) for Data Set II

Model	*a*	*b*	*p*	**W**	**A**	**K-S (p-value)**
GLBIII	16.1115744 (7.14847882)	5.5874251 (0.16009004)	0.1112545 (0.05820963)	**0.03142441**	**0.1881365**	**0.0599 (0.984)**
LBIII	1.488039 (0.1218159)	48.805267 (14.4169169)		0.672348	3.974308	0.2168 (0.007787)
GBIII	4.2809076 (1.5556558)	216.3395534 (41.4512185)	0.2198697 (0.1067724)	0.0603646	0.3639023	0.081 (0.8335)
BIII	36.2259705 (10.45670494)	1	0.9234018 (0.08016585)	0.345271	2.113796	0.146 (0.1616)
Log-log	0.3475763 (0.03513562)	1	1	0.1660651	1.019286	0.6658 (< 2.2e-16)

Table 7
Goodness-of-Fit Statistics for Data Set II

Model	**AIC**	**CAIC**	**BIC**	**HQIC**	$-\ell$
GLBIII	**680.9666**	**681.4029**	**687.1992**	**683.3995**	**337.4833**
LBIII	733.1876	733.4019	737.3427	734.8096	364.5938
GBIII	684.4543	684.8907	690.6869	686.8873	339.2272
BIII	706.385	706.5992	710.54	708.00693	351.1925
Log-log	872.7224	872.7926	874.8	873.5334	435.3612

The GLBIII distribution is best fitted than LBIII, GBIII, BIII and Log-log distributions because the values of all criteria of goodness of fit are significantly smaller for GLBIII.

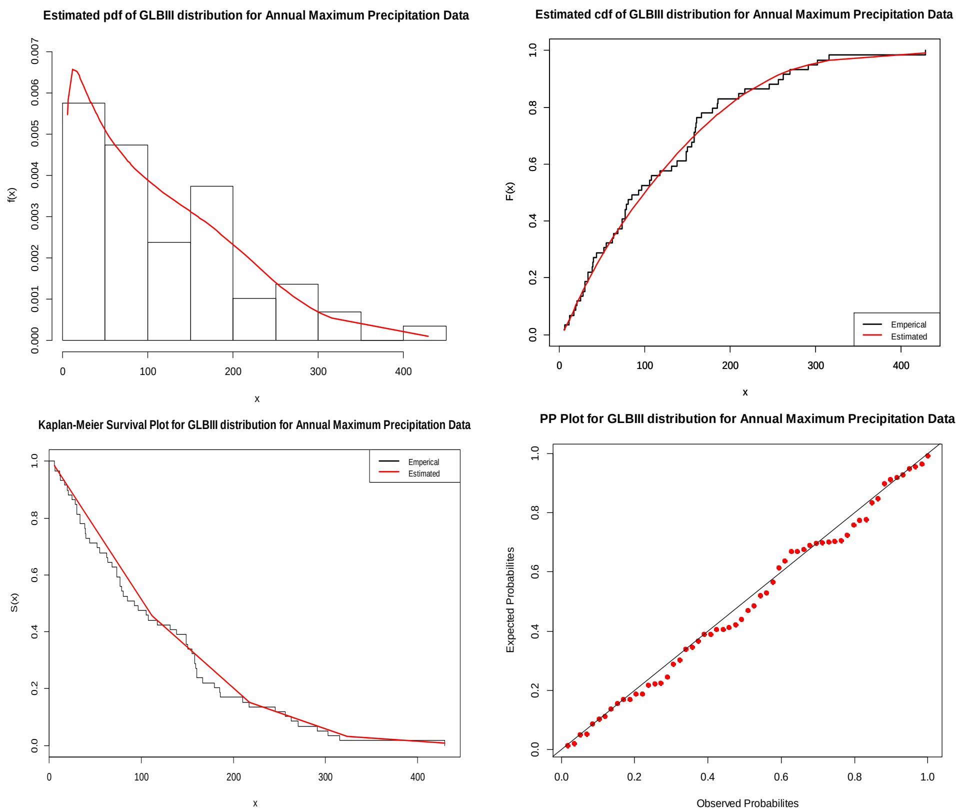

Fig. 4: Fitted pdf, cdf, Survival and PP Plots of the GLBIII Distribution for River Peak Flows Series

We can also perceive that the GLBIII distribution is best fitted to empirical data (Figure 4).

9. CONCLUDING REMARKS

We have proposed the GLBIII distribution, developed on the basis of the generalized log Pearson differential equation. The GLBIII distribution is also derived from transformation and compounding mixture of distributions. The GLBIII distribution is very flexible due to its density and hazard rate function accommodating various shapes. We have studied certain properties including descriptive measures, sub-models, ordinary moments, factorial moments and moments of order statistics, incomplete moments reliability and uncertainty measures. Two important characterizations of the GLBIII distribution are studied. Maximum likelihood estimates of the parameters for the GLBIII distribution are computed. The simulation study has performed for the GLBIII distribution to assess and illustrate the performance of the MLEs. Goodness of fit has shown that the GLBIII distribution is better fit. Applications of the GLBIII model to river peak flows series and annual maximum precipitation data are illustrated to show the significance and flexibility of the GLBIII distribution. We have shown that the GLBIII distribution is empirically better for hydrological and climatic applications.

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APPENDIX A

Lemma(ii):

Let $w(x;a,b,\theta)=\frac{2a\theta}{bx}\left(\frac{\ln x}{b}\right)^{-2a-1}e^{-\theta\left(\frac{\ln x}{b}\right)^{-2a}}$, $x>1$ be pdf of generalized inverse log-Weibull distribution and let θ have gamma distribution with pdf $g(\theta,p)=\frac{1}{\Gamma(p)}\theta^{p-1}e^{-\theta}\quad \theta>0$. Then X has GLBIII distribution.

Proof:

For compounding $f(x,a,b,p)=\int_0^\infty w(x;a,b,\theta)g(\theta;p)d\theta$,

$$f(x;a,b,p)=\int_0^\infty \frac{2a\theta}{bx}\left(\frac{\ln x}{b}\right)^{-2a-1}e^{-\theta\left(\frac{\ln x}{b}\right)^{-2a}}\frac{1}{\Gamma(p)}\theta^{p-1}e^{-\theta}d\theta,$$

$$f(x,a,b,p)=\frac{2ap}{bx}\left(\frac{\ln x}{b}\right)^{-2a-1}\left[1+\left(\frac{\ln x}{b}\right)^{-2a}\right]^{-(p+1)},\quad x>1.$$

APPENDIX B

Theorem 1:

Let $(\Omega,\mathrm{F},\mathrm{P})$ be a given probability space and let $[a_1,a_2]$ be an interval with $a_1<a_2$ ($a_1=-\infty, a_2=\infty$), Let $X:\Omega\to[a_1,a_2]$ be a continuous random variable with the distribution function F and let g_1 and g_2 be two real functions deed on $[a_1,a_2]$ such that $\frac{E[\mathrm{g}_1(X)|\mathrm{X}\ge \mathrm{x}]}{E[\mathrm{g}_2(X)|\mathrm{X}\ge \mathrm{x}]}=q(x)$ is defined with some real function $q(x)$. Assume that $\mathrm{g}_1,\mathrm{g}_2\,\varepsilon\, C([a_1,a_2])$, $\lambda\,\varepsilon\, C^2([a_1,a_2])$ and F is twice continuously differentiable and strictly monotone function on the set $[a_1,a_2]$: Finally, assume that the equation $g_2q(x)=g_1$ has no real solution in the interior of $[a_1,a_2]$.Then F is obtained from the functions $\mathrm{g}_1,\mathrm{g}_2$ and $q(x)$ as $F(x)=\int_a^x k\left|\frac{q'(t)}{q(t)g_2(t)-g_1(t)}\right|\exp(-s(t))dt$, where $s(t)$ is the solution of equation $s'(t)=\frac{q'(t)g_2(t)}{q(t)g_2(t)-g_1(t)}$ and k is a constant, chosen to make $\int_{a_1}^{a_2} dF=1$.