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ESTIMATION OF FECUNDABILITY IN BAYESIAN APPROACH USING DIFFERENT LOSS FUNCTIONS

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ABSTRACT

In this paper, we present a Bayesian approach to find the estimators of the mean value of fecundability for heterogeneous cohort of women under different symmetric and asymmetric loss functions such as squared error loss function (SELF), quadratic loss function (QLF), scaled loss function, 0-1 type of loss function and modified linear-exponential (MLINEX) loss function etc. Most of the estimators are derived on the basis of a stated theorem. In addition, a study among the obtained Bayes' estimators of fecundability parameter along with the properties viz. arithmetic mean, harmonic mean, mode and second raw moment about origin of the posterior fecundability distribution is also provided.

1. INTRODUCTION

The most widely applicable technique is to estimate fecundability usually called mean value of fecundability from geometric model directly for homogeneous cohort of women where fecundability is considered to be constant. When it varies among women, then it is typically estimated from beta distribution of first kind which is considered as a useful model.

It is assumed that women's fecundability has the constant value during the period of observation. Suppose X is the random month of waiting to conception, then it has the conditional geometric distribution with fecundability parameter θ. The probability mass function of X is defined as;

$$P(X = x;\, \theta) = \theta(1-\theta)^{x-1}; \quad x = 1, 2, 3, \cdots, \propto, \quad 0 \le \theta \le 1 , \tag{1.1}$$

The model (1.1) is known as the conditional distribution of conception delay. Now if θ varies among couples, then θ has the following probability density function;

$$f(\theta;\, \alpha, \beta) = \frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}\ ; \ 0 \le \theta \le 1, \ \ \alpha, \beta > 0 , \tag{1.2}$$

where α and β are two shape quantities.

One of the underlying assumptions in a model of homogeneous fecundability women population is that the fecundability of each couple remains constant from month to month until pregnancy. The estimation of the mean value of fecundability for such a homogeneous group of women is reciprocal of the arithmetic mean of the model (1.1).

Chowdhury and Umbach (2012) mentioned homogeneous fecundability in a couple of women may not be always realistic in many reasons, such as it may be governed by many socio-economic and demographic variables, if the couples are temporarily separated, the spouses intentionally change the timing of intercourse and a miscarriage is not reported etc. As a results, the assumption due to constant fecundability has taken earlier may also be violated. Evidently, the mean value of fecundability θ for heterogeneous cohort of women may be viewed as a random variable taking non-negative value which ranges from 0 to 1.

However, the estimation of mean value of fecundability was studied by different authors such as Bongaarts (1975), Balakrishnan (1979), Bendel and Hua (1978), Chowdhury and Umbach (2012), Goldman et al. (1985), Islam and Yadava (1997), Islam et al. (2009), Islam et al. (2015) and James (1963) etc.

Now the purpose of this paper is to find the estimators of fecundability parameter θ for heterogeneous women population in a Bayesian approach using different symmetric and asymmetric loss functions called squared error loss function (SELF), quadratic loss function (QLF), scaled loss function, 0-1 type of loss function, modified linear-exponential (MLINEX) loss function etc., considering the conditional geometric distribution in (1.1) as a fecundability model distribution and beta distribution of first kind in (1.2) as a prior density for θ.

While conducting the study it has found that the obtained Bayes' estimators of θ in **section 3** are directly related to some properties viz. arithmetic mean, harmonic mean, mode, second raw moment about origin of the posterior fecundability model. It is essentially an updated version of our prior knowledge about θ in light of the sample data. The posterior distribution plays an important role in Bayesian approach where the parameter is assumed to be a random variable. It also tells the whole information about the parameter.

In **section 3**, a theorem on Bayes' estimator of fecundability parameter for particular loss function has been stated. The theorem is also proved and most of the Bayes' estimators under the different loss functions mentioned in **section 2,** hence derived. Finally, for different values of shape characteristic gamma of MLINEX loss function, a detailed discussion is made on Bayes' estimator for MLINEX loss function with above mentioned properties of posterior fecundability model and those of Bayes' estimators under squared error, quadratic and scaled loss functions etc.

2. PRELIMINARIES

Suppose the distribution of a random variable X depends on a single parameter θ and let Ω denotes the parameter space of possible values of θ. Now consider the general problem of estimating the unknown parameter θ, from the results of a random sample of n observations, by the method of Bayes' estimation.

Denoting the sample observations $x_1, x_2, ..., x_n$ by $\underline{x}$, let $\hat{\theta}$ be an estimator of θ and also let $L(\hat{\theta}, \theta)$ be a loss function, the loss incurred by taking the value of θ to be $\hat{\theta}$.

If $l(\theta \mid \underline{x})$ is the likelihood function of θ given the sample observations $\underline{x}$ and $\pi(\theta)$ is the prior density of θ, then using Bayes' theorem the posterior density of θ given the sample observations $\underline{x}$ is

$$\pi(\theta \mid \underline{x}) = \frac{l(\theta \mid \underline{x})\pi(\theta)}{p(x)}, \tag{2.1}$$

where $p(x) = \int_{\Omega} l(\theta \mid \underline{x})\pi(\theta)d\theta$.

Hence, Bayes' estimator $\hat{\theta}$ of θ is that estimator which minimizes the posterior risk and will be a solution of the equation

$$\int_{\Omega} \frac{\partial L}{\partial \hat{\theta}} l(\theta \mid \underline{x})\pi(\theta)d\theta = 0, \tag{2.2}$$

where L stands for loss function.

Assuming the existence of (2.2) and that sufficient regularity conditions prevail to permit differentiation under the sign of integral. Also the validity and the desirability that the equation should lead to a unique solution necessarily impose restrictions of one's choice of loss function $L(\hat{\theta}, \theta)$ and prior density $\pi(\theta)$.

Here, we consider the following loss functions to find the estimators of fecundability parameter θ.

i) $L_1(\hat{\theta}, \theta) = c(\hat{\theta} - \theta)^2; \; c > 0$.

ii) $L_2(\hat{\theta}, \theta) = c(\hat{\theta} - \theta)^2 \big/ \theta^2 \; ; c > 0$

iii) $L_3(\hat{\theta}, \theta) = c(\hat{\theta} - \theta)^2 \big/ \theta \; ; c > 0$.

iv) $L_4(\hat{\theta}, \theta) = c\left(\sqrt{\hat{\theta}} - \sqrt{\theta}\right)^2; \; c > 0$.

v) $L_5(\hat{\theta}, \theta) = c\left(\sqrt{\hat{\theta}} - \sqrt{\theta}\right)^2 \big/ \theta \; ; \; c > 0$.

vi) $L_6(\hat{\theta}, \theta) = \begin{cases} 0 & if \;\; |\hat{\theta} - \theta| < \delta \\ 1 & otherwise \end{cases}$,

where δ is a small known quantity.

vii) $L_7\left(\hat{\theta},\theta\right)=\varpi\left[\left(\frac{\hat{\theta}}{\theta}\right)^{\gamma}-\gamma\ln\left(\frac{\hat{\theta}}{\theta}\right)-1\right];\ \gamma\neq 0,\ \ \varpi>0,$

where c and ϖ are scale characteristics and γ is a shape characteristic of the above loss functions.

These L_1, L_2, L_3 and L_6 are called the squared error loss function (SELF), quadratic loss function (QLF), scaled loss function, 0-1 type of loss function respectively which are symmetric and modified linear-exponential (MLINEX) loss function L_7 is an asymmetric.

3. BAYES' ESTIMATION

The Bayes' estimators of fecundability parameter θ under the first five loss functions L_1, L_2, L_3, L_4 and L_5 primarily depend on the following theorem. The theorem is also proved.

Theorem 3.1:

Suppose $X_1, X_2, \cdots, X_n$ be a random sample of size n drawn from the population having probability mass function in (1.1). If we consider a conjugate prior density for fecundability parameter as $\theta \sim Beta\left(p,q\right)$, then the Bayes' estimator of parameter θ for the particular loss function $L\left(\hat{\theta},\theta\right)=c\theta^{a}\left(\hat{\theta}^{b}-\theta^{b}\right)^{2}$ is $\hat{\theta}=\left[\frac{\Gamma\left(n+p+a+b\right)\Gamma\left(y+p+q+a\right)}{\Gamma\left(n+p+a\right)\Gamma\left(y+p+q+a+b\right)}\right]^{\frac{1}{b}}$; if exists, where $y=\sum_{i=1}^{n}x_i$, $c>0$, a and b are two known quantities.

Proof:

The likelihood function of θ in (1.1) for the given observed sample $\underline{x}=\left(x_1, x_2, \cdots, x_n\right)$ is given by

$$
\begin{aligned}
l\left(\theta \mid \underline{x}\right) &= \theta^{n}\left(1-\theta\right)^{\sum_{i=1}^{n}x_i-n} \\
&= \theta^{n}\left(1-\theta\right)^{y-n}, \qquad (3.1)
\end{aligned}
$$

where $y=\sum_{i=1}^{n}x_i$.

The maximum likelihood estimator of θ is $\frac{n}{y}$. It is noted that the part of the likelihood function which is relevant to a Bayesian inference on the unknown parameter θ is $\theta^{n}\left(1-\theta\right)^{y-n}$.

For the problem under consideration the conjugate prior density for θ as

$$\pi(\theta)=\frac{1}{B(p,q)}\theta^{p-1}(1-\theta)^{q-1};\ 0\le\theta\le 1,\quad p>0,\ q>0, \tag{3.2}$$

where p and q, the estimates of α and β in (1.2) may be obtained from the sample data in any classical approach like method of maximum likelihood and method of moments if unknown.

Now by combining Equations (3.1) and (3.2), we obtain the posterior density of θ as

$$\pi\left(\theta\,|\,\underline{x}\right)\propto l\left(\theta\,|\,\underline{x}\right)\pi(\theta)$$

$$\Rightarrow\pi\left(\theta\,|\,\underline{x}\right)\propto\theta^{n+p-1}(1-\theta)^{y-n+q-1}.$$

This implies that the posterior density of θ for the given sample $\underline{x}=(x_1,\,x_2,\,\cdots,\,x_n)$ is

$$\pi(\theta\,|\,\underline{x})=\frac{1}{B(n+p,\ y-n+q)}\theta^{n+p-1}(1-\theta)^{y-n+q-1};\quad 0\le\theta\le 1,\ p>0,\ q>0, \tag{3.3}$$

which follows that $\theta\sim Beta(n+p,\ y-n+q)$. It has noted that the beta posterior fecundability model is essentially an updated version of our prior knowledge about θ in light of the sample data. Usually, the mean value of fecundability for heterogeneous women population is modeled on two parameters also suggested by Jain (1969).

The posterior mean and variance in (3.3) are $\dfrac{n+p}{y+p+q}$ and $\dfrac{(n+p)(y-n+q)}{(y+p+q)^2(y+p+q+1)}$ respectively. For $(n+p)>1$ and $(y+p+q)>1$, the posterior harmonic mean is $\dfrac{n+p-1}{y+p+q-1}$.

Generally, the mode of the posterior density in (3.3) is $M_0=\dfrac{n+p-1}{y+p+q-2}$, if $(n+p)>1$ and $(y+p+q)>2$. Meanwhile Podder (2011) also discussed the existence of mode for different shapes of a beta distribution.

From Equation (2.2) we can express the Bayes' estimator of θ for the particular loss function as

$$\hat{\theta}=\left[\frac{E_\theta\left(\theta^{a+b}\,|\,\underline{x}\right)}{E_\theta\left(\theta^{a}\,|\,\underline{x}\right)}\right]^{\frac{1}{b}}, \tag{3.4}$$

where E_θ stands for posterior expectation.

Here,

$$E_\theta\left(\theta^{a+b} \mid \underline{x}\right)=\int_0^1 \theta^{a+b}\pi\left(\theta \mid \underline{x}\right)d\theta$$

$$=\frac{1}{B\left(n+p, y-n+q\right)}\int_0^1 \theta^{n+p+a+b-1}\left(1-\theta\right)^{y-n+q-1}d\theta$$

$$=\frac{B\left(n+p+a+b,\ y-n+q\right)}{B\left(n+p,\ y-n+q\right)}.$$

Similarly, $E_\theta\left(\theta^a \mid \underset{-}{x}\right)=\dfrac{B\left(n+p+a\ \ y-n+q\right)}{B\left(n+p,\ y-n+q\right)}$.

Using the above relations in (3.4), we obtain the Bayes' estimator of fecundability parameter θ for the particular loss function $L\left(\hat{\theta},\theta\right)=c\theta^a\left(\hat{\theta}^b-\theta^b\right)^2$ as

$$\hat{\theta}=\left[\frac{\Gamma\left(n+p+a+b\right)\Gamma\left(y+p+q+a\right)}{\Gamma\left(n+p+a\right)\Gamma\left(y+p+q+a+b\right)}\right]^{\frac{1}{b}}; \text{ if exists} \tag{3.5}$$

and hence proved.

The above **Theorem 3.1** gives the following Bayes' estimators of the fecundability parameter θ under the loss functions L_1, L_2, L_3, L_4 and L_5 for different values of a and b respectively.

a) Substituting $a=0$ and $b=1$ in (3.5), the Bayes' estimator of θ for SELF L_1 is given by

$$\hat{\theta}_1=\frac{n+p}{y+p+q}$$, which is the mean of the posterior density in (3.3).

b) Substituting $a=-2$ and $b=1$ in (3.5), the Bayes' estimator of θ for QLF L_2 becomes

$$\hat{\theta}_2=\frac{n+p-2}{y+p+q-2},$$

exists when $\left(n+p\right)$ and $\left(y+p+q\right)$ both are greater than two.

c) Substituting $a=-1$ and $b=1$ in (3.5), the Bayes' estimator of θ for scaled loss function L_3 becomes $\hat{\theta}_3=\dfrac{n+p-1}{y+p+q-1}$, exists when $\left(n+p\right)$ and $\left(y+p+q\right)$ both are greater than one. It also same as harmonic mean of the posterior fecundability model.

d) Substituting $a=0$ and $b=\frac{1}{2}$, in (3.5) the Bayes' estimator of θ under the loss function L_4 as

$$\hat{\theta}_4=\left[\frac{\Gamma\left(n+p+\frac{1}{2}\right)\Gamma\left(y+p+q\right)}{\Gamma\left(n+p\right)\Gamma\left(y+p+q+\frac{1}{2}\right)}\right]^2 .$$

e) Substituting $a=-1$ and $b=\frac{1}{2}$, in (3.5), the Bayes' estimator of θ under the loss function L_5 provides

$$\hat{\theta}_5=\left[\frac{\Gamma\left(n+p-\frac{1}{2}\right)\Gamma\left(y+p+q-1\right)}{\Gamma\left(n+p-1\right)\Gamma\left(y+p+q-\frac{1}{2}\right)}\right]^2 ,$$

exists when $\left(n+p\right)$ and $\left(y+p+q\right)$ both are greater than one.

f) For a symmetrical posterior fecundability model, the Bayes' estimator of θ for 0-1 type of loss function L_6 is the mode of the posterior density. The posterior density in (3.3) will be symmetrical, if $\left(n+p\right)=\left(y-n+q\right)\geq 2$. Therefore, $\hat{\theta}_6=M_0=0.5$ is the mode in the symmetrical family of posterior densities in (3.3).

Otherwise, for triangular shaped and general cases of beta distribution in (3.3) when $\left(n+p\right)$ and $\left(y-n+q\right)$ both are greater than or equal to one, the Bayes' estimator of θ is M_0, the midpoint of the interval $\mathbf{I}$ of length 2δ which maximizes the posterior density $P\left(\theta\in I\mid x\right)$ in (3.3), Lehmann (1983).

g) Similarly, the Bayes' estimator of θ for MLINEX loss function is given by

$$\hat{\theta}_7=\left[E_\theta\left(\theta^{-\gamma}\mid \underline{x}\right)\right]^{-\frac{1}{\gamma}} . \qquad (3.6)$$

Using $E_\theta\left(\theta^{-\gamma}\mid \underline{x}\right)=\frac{\Gamma\left(n+p-\gamma\right)\Gamma\left(y+p+q\right)}{\Gamma\left(n+p\right)\Gamma\left(y+p+q-\gamma\right)}$, we obtained the Bayes' estimator of θ as

$$\hat{\theta}_7=\left[\frac{\Gamma\left(n+p-\gamma\right)\Gamma\left(y+p+q\right)}{\Gamma\left(n+p\right)\Gamma\left(y+p+q-\gamma\right)}\right]^{-\frac{1}{\gamma}} \text{, if exists.}$$

4. DISCUSSION

In this section, we further present some relations among the Bayes' estimators of fecundability parameter θ for different loss functions along with the properties of posterior fecundability model obtained in **section 3** are as follows:

i) It has been seen that when $\gamma = -1$, the Bayes' estimator of θ for MLINEX loss function is same as the Bayes' estimator for SELF and mean of the posterior density in (3.3).

ii) When $\gamma = 1$, the Bayes' estimator of θ for MLINEX and scaled loss functions and also posterior harmonic mean are identical.

iii) When $\gamma = -2$, the Bayes' estimator of θ for MLINEX loss function is equal to the second raw moment about origin of the posterior density in (3.3).

iv) When $\gamma = 2$, the Bayes' estimator of θ for MLINEX loss function is the geometric mean of the Bayes' estimator of θ for QLF and posterior harmonic mean.

v) When $\gamma = -\frac{1}{2}$, the Bayes' estimator of θ for MLINEX loss function is equal to the Bayes' estimator under the loss function L_4.

vi) The Bayes' estimator of θ for scaled loss function L_3 is same as posterior harmonic mean.

vii) For 0-1 type of loss function the Bayes' estimator is the mode of the posterior fecundability distribution if it is symmetrical.

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