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TOPP-LEONE MUKHERJEE-ISLAM DISTRIBUTION: PROPERTIES AND APPLICATIONS

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ABSTRACT

In this paper, Topp-Leone Mukherjee-Islam (TLMI) distribution is suggested. The probability density function and the distribution function of the TLMI are provided. The *rth* moment, mean, variance, coefficient of skewness, kurtosis, and coefficient of variation are derived. The order statistics of the TLMI distribution random variable are introduced. Reliability analysis including the hazard rate function reliability function are studied. The maximum likelihood estimators of the TLMI distribution parameters and the Rényi entropy as a measure of the uncertainty in the model are obtained. The usefulness of the TLMI distribution is illustrated using real lifetime data set from medical science.

KEYWORDS

Topp-Leone distribution; Mukherjee-Islam distribution; Reliability function; Hazard function; Order statistics; Rényi entropy.

MSC 2010: 62H10, 62H12.

1. INTRODUCTION

A family of univariate distributions is proposed by Topp and Leone (1955) with a cumulative distribution function (CDF) given by

$$F_{TL}(x;b,\alpha)=\begin{cases}0 & \text{if } x<0,\\ \left(\frac{x}{b}\right)^{\alpha}\left(2-\frac{x}{b}\right)^{\alpha} & \text{if } 0\le x\le b<\infty,\\ 1 & \text{if } x>b,\end{cases} \tag{1}$$

where $0<\alpha<1$, with corresponding probability density function (PDF) given by

$$f_{TL}(x;b,\alpha)=\frac{2\alpha}{b}\left(\frac{x}{b}\right)^{\alpha-1}\left(1-\frac{x}{b}\right)^{\alpha}\left(2-\frac{x}{b}\right)^{\alpha-1}. \tag{2}$$

These distributions are known as the J-shaped distributions since $f(x) > 0$, $f'(x) < 0$, and $f''(x) > 0$ for all $0 < x < b$, where f' and f'' are the first and second derivatives of f, respectively.

When the scale parameter $b = 1$ in (2), the standard Topp-Leone distribution will reduces to

$$f_{TL}(x; b=1, \alpha) = 2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1},\ 0 \le x \le 1, \tag{3}$$

See Al-Shomrani et al. (2016) and Genç (2012). Recently, Al-Shomrani et al. (2016) considered the standard CDF of the Topp-Leone distribution by letting $b = 1$ in (1) as

$$F_{TL}(x; b=1, \alpha) = x^{\alpha}(2-x)^{\alpha},\ 0 \le x \le 1, \alpha > 0, \tag{4}$$

to suggest the Topp–Leone family of distributions in the form

$$F_{TL-G}(x; \alpha) = \left[G(x)\right]^{\alpha}\left[2 - G(x)\right]^{\alpha}, x \in R, \alpha > 0, \tag{5}$$

with PDF defined as

$$f_{TL-G}(x; \alpha) = 2\alpha g(x)\left[1 - G(x)\right]\left[G(x)\right]^{\alpha-1}\left[2 - G(x)\right]^{\alpha-1}, x \in R, \alpha > 0, \tag{6}$$

where $g(x) = G'(x)$. To obtain the base distribution from Topp–Leone family of distributions let

$$\alpha = \frac{\ln\left[G(x)\right]}{\ln\left[G(x)\right]\left[2 - G(x)\right]}.$$

Several authors studied different properties of the Topp_Leone distribution as Nadarajah and Kotz (2003), Al-Shomrani et al. (2016), Ghitany et al. (2005), MirMostafaee (2014), MirMostafaee and Genç (2012).

Mukherjee-Islam (1983) suggested a failure distribution for reliability and Bayesian analysis known as Mukherjee-Islam distribution (MI) with CDF given by

$$F_{MI}(x; p, \theta) = \left(\frac{x}{\theta}\right)^{p}, 0 < x \le \theta;\ \theta, p > 0, \tag{7}$$

where θ and p are the scale and shape parameters of the distribution and the corresponding PDF is

$$f_{MI}(x; p, \theta) = \frac{p}{\theta^{p}} x^{p-1}, 0 < x \le \theta;\ \theta, p > 0. \tag{8}$$

Modification of size-biased Mukherjee-Islam distribution is proposed by Siddiqui et al. (2016). Khan (2016) studied the reliability analysis of Mukherjee–Islam distribution under three different prior distributions. Various types of distributions are suggested in the literature, for example see Haq et al. (2017) for The Marshall-Olkin length-biased exponential distribution, Khaleel, et al. (2018) for Burr type X distribution, Al-Omari et al. (2017) for transmuted Janardan distribution. Al-Omari et al. (2018), proposed size-biased Ishita distribution, Al-khazaleh et al. (2016), suggested transmuted two-parameter Lindley distribution, Mdlongwa et al. (2017) introduced the Burr XII modified Weibull, and Foya et al. (2017) suggested gamma log-logistic Weibull distribution.

In this paper, we substitute the CDF of the MI distribution given in (7) in the Topp-Leone family given in (6) to introduce a new continuous distribution called as the Topp-Leone Mukherjee-Islam (TLMI) distribution, and we will denote to this distribution as $F_{TLMI}(x; p, \theta, \alpha)$.

The rest of this paper is organized as follows: In Section 2, we demonstrated the Topp-Leone Mukherjee-Islam distribution. The statistical properties including the *r*th moment, moment generating function, variance, skewness and kurtosis are discussed in Section 3. The distributions of order statistics are given in Section 4. The reliability analysis is provided in Section 5. The quantiles and the maximum likelihood estimates are investigated in Section 6. The Rényi entropy for the TLMI distribution is defined in Section 7. In Section 8, a real data set illustration is discussed in details. Finally, some conclusions are provided in Section 9.

2. TOPP-LEONE MUKHERJEE-ISLAM DISTRIBUTION

A random variable X is said to have Topp-Leone Mukherjee-Islam distribution if its cumulative distribution function is

$$\begin{aligned} F_{TLMI}(x; p, \theta, \alpha) &= \left[\left(\frac{x}{\theta}\right)^p\right]^\alpha \left[2 - \left(\frac{x}{\theta}\right)^p\right]^\alpha \\ &= \left[2\left(\frac{x}{\theta}\right)^p - \left(\frac{x}{\theta}\right)^{2p}\right]^\alpha, \quad 0 < x \le \theta, \alpha > 0 \end{aligned} \tag{9}$$

where $\theta > 0$ and $p > 0$ are the scale and shape parameters of the distribution. The PDF corresponding to (9) becomes

$$\begin{aligned} f_{TLMI}(x; p, \theta, \alpha) &= \alpha \left[2\left(\frac{x}{\theta}\right)^p - \left(\frac{x}{\theta}\right)^{2p}\right]^{\alpha-1} \left[\frac{2p}{\theta^p} x^{p-1} - \frac{2p}{\theta^{2p}} x^{2p-1}\right] \\ &= \frac{2\alpha p}{\theta^{p\alpha}} x^{p\alpha-1} \left[1 - \left(\frac{x}{\theta}\right)^p\right] \left[2 - \left(\frac{x}{\theta}\right)^p\right]^{\alpha-1}. \end{aligned} \tag{10}$$

The following figures illustrate the PDFs and CDFs shapes of the TLMI distribution for different values of the distribution parameters.

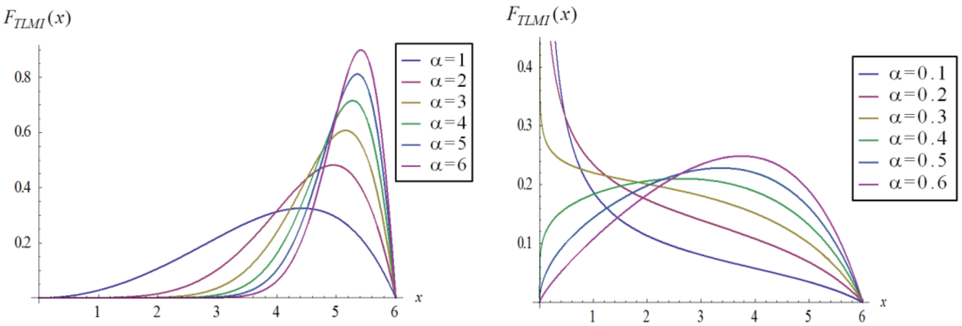

Figure 1: The PDF's of the TLMI Distribution with $\theta = 6, p = 3$ and $\alpha = 1, 2, 3, 4, 5, 6$

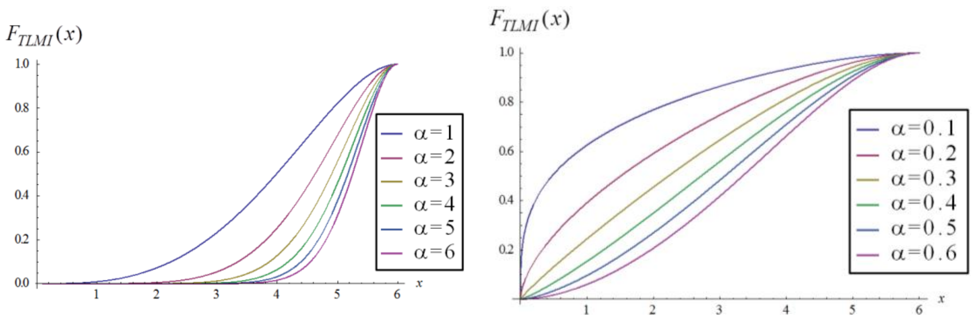

Figure 2: The CDF's Function of the TLMI Distribution with $\theta = 6, p = 3$ and $\alpha = 1, 2, 3, 4, 5, 6$

It is clear from Figure (1) that the TMLI distribution is skewed and the shape of the distribution is based on the distribution parameters values.

3. MOMENTS OF THE $F_{TLMI}(x; p, \theta, \alpha)$

In this section, the rth moment of the TLMI random variable is derived. Also, we obtained the mean, variance, coefficient of kurtosis, coefficient of skewness, and coefficient of variation.

Theorem 1:

Let $X \sim F_{TLMI}(x; p, \theta, \alpha)$, then the rth moment of X is given by

$$E\left(X^r\right) = \alpha\, \theta^r 2^{\alpha+\omega(r)} \left[B\left(\frac{1}{2}; \omega(r), \alpha\right) - 2B\left(\frac{1}{2}; \omega(r)+1, \alpha\right) \right], \quad \Re\left(\omega(r)\right) > 0, \quad (11)$$

where $\omega(r) = \alpha + \dfrac{r}{p}$ and $B(x; a, b)$ is the incomplete beta function defined as $B(x; a, b) = \int_o^x t^{a-1}(1-t)^{b-1}\, dt$

Proof:

$$E\left(X^r\right)=\int_0^\infty x^r f(x)dx=\int_0^\theta x^r\alpha\left[2\left(\frac{x}{\theta}\right)^p-\left(\frac{x}{\theta}\right)^{2p}\right]^{\alpha-1}\left[\frac{2p}{\theta^p}x^{p-1}-\frac{2p}{\theta^{2p}}x^{2p-1}\right]dx$$

Let $y=\left(\frac{x}{\theta}\right)^p$, then $x=\theta y^{\frac{1}{p}}$ and $dx=\frac{\theta}{p}y^{\frac{1}{p}-1}dy$. Therefore,

$$E\left(X^r\right)=\int_0^1\alpha\left(\theta y^{\frac{1}{p}}\right)^r\left(2y-y^2\right)^{\alpha-1}\left[\frac{2p}{\theta^p}\left(\theta y^{\frac{1}{p}}\right)^{p-1}-\frac{2p}{\theta^{2p}}\left(\theta y^{\frac{1}{p}}\right)^{2p-1}\right]\frac{\theta}{p}y^{\frac{1}{p}-1}dy$$

$$=\int_0^1\alpha\frac{\theta^{r+1}}{p}y^{\frac{r}{p}+\frac{r}{p}-1}\left(2y-y^2\right)^{\alpha-1}\left[\frac{2p}{\theta}y^{1-\frac{1}{p}}-\frac{2p}{\theta}y^{2-\frac{1}{p}}\right]dy$$

$$=\int_0^1\alpha\frac{\theta^{r+1}}{p}y^{\frac{r}{p}+\frac{r}{p}-1}\left(2y-y^2\right)^{\alpha-1}\left[\frac{2p}{\theta}y^{1-\frac{1}{p}}(1-y)\right]dy$$

$$=\int_0^1 2\alpha\theta^r y^{\frac{r}{p}+\alpha-1}\left(2-y^2\right)^{\alpha-1}(1-y)dy$$

$$=\int_0^1 2\alpha\theta^r y^{\frac{r}{p}+\alpha-1}\left[2\left(1-\frac{y}{2}\right)\right]^{\alpha-1}dy-\int_0^1 2\alpha\theta^r y^{\frac{r}{p}+\alpha}\left[2\left(1-\frac{y}{2}\right)\right]^{\alpha-1}dy$$

Let $z=\frac{y}{2}$, then $dz=\frac{1}{2}dy$.

$$E\left(X^r\right)=2\alpha\theta^r\left\{\int_0^{\frac{1}{2}}(2z)^{\frac{r}{p}+\alpha-1}2^{\alpha-1}(1-z)^{\alpha-1}zdz-\int_0^{\frac{1}{2}}(2z)^{\frac{r}{p}+\alpha}2^{\alpha-1}(1-z)^{\alpha-1}zdz\right\}$$

$$=2\alpha\theta^r\left\{2^{\frac{r}{p}+2\alpha-1}\int_0^{\frac{1}{2}}z^{\frac{r}{p}+\alpha-1}(1-z)^{\alpha-1}zdz-2^{\frac{r}{p}+2\alpha}\int_0^{\frac{1}{2}}z^{\frac{r}{p}+\alpha}(1-z)^{\alpha-1}zdz\right\}$$

$$=2^{\frac{r}{p}+2\alpha}\alpha\theta^r\left\{B\left(\frac{1}{2};\frac{r}{p}+\alpha,\alpha\right)-2B\left(\frac{1}{2};\frac{r}{p}+\alpha+1,\alpha\right)\right\}$$

$$=\alpha\,\theta^r 2^{\alpha+\omega(r)}\left[B\left(\frac{1}{2};\omega(r),\alpha\right)-2B\left(\frac{1}{2};\omega(r)+1,\alpha\right)\right],$$

where $B(x;a,b)$ and $\omega(r)$ are defined in Theorem 1 above. □

Therefore, based on (11) we can have the first moment (mean) as

$$E(X) = \alpha\,\theta\,2^{\alpha+\omega(1)}\left(B\left[\frac{1}{2};\omega(1),\alpha\right] - 2B\left[\frac{1}{2};\omega(1)+1,\alpha\right]\right), \quad \Re(\omega(1)) > 0 . \quad (12)$$

The second moment will be

$$E\left(X^2\right) = \alpha\theta^2 2^{\alpha+\omega(2)}\left(B\left[\frac{1}{2};\omega(2),\alpha\right] - 2B\left[\frac{1}{2};\omega(2)+1,\alpha\right]\right), \quad \Re(\omega(2)) > 0. \quad (13)$$

Thus, from (12) and (13) the variance of the TLMI distribution can be written as

$$\begin{aligned} Var(X) &= E\left(X^2\right) - \left[E(X)\right]^2 \\ &= \alpha\theta^2 2^{\alpha+\omega(2)}\left\{-4^{\alpha}\alpha\left(B\left[\frac{1}{2};\omega(1),\alpha\right] - 2B\left[\frac{1}{2};\omega(1)+1,\alpha\right]\right)^2\right. \\ &\qquad \left. + B\left[\frac{1}{2};\omega(2),\alpha\right] - 2B\left[\frac{1}{2};\omega(2)+1,\alpha\right]\right\} \quad (14) \end{aligned}$$

The third and fourth moments of the TLMI distribution are

$$E\left(X^3\right) = \alpha\theta^3 2^{\alpha+\omega(3)}\left(B\left[\frac{1}{2};\omega(3),\alpha\right] - 2B\left[\frac{1}{2};\omega(3)+1,\alpha\right]\right), \quad \Re(\omega(3)) > 0, \quad (15)$$

and

$$E\left(X^4\right) = \alpha\theta^4 2^{\alpha+\omega(4)}\left(B\left[\frac{1}{2};\omega(4),\alpha\right] - 2B\left[\frac{1}{2};\omega(4)+1,\alpha\right]\right), \quad \Re(\omega(4)) > 0. \quad (16)$$

Now, the coefficient of variation of the TLMI distribution is defined as

$$\begin{aligned} \mathbb{C}_1 &= \frac{\sigma}{\mu} \\ &= \frac{\sqrt{-4^{\alpha}\alpha\left\langle B\left[\frac{1}{2};\omega(1),\alpha\right] - 2B\left[\frac{1}{2};\omega(1)+1,\alpha\right]\right\rangle^2 + B\left[\frac{1}{2};\omega(2),\alpha\right] - 2B\left[\frac{1}{2};\omega(2)+1,\alpha\right]}}{2^{\alpha}\alpha^{\frac{1}{2}}\left\{B\left[\frac{1}{2};\omega(1),\alpha\right] - 2B\left[\frac{1}{2};\omega(1)+1,\alpha\right]\right\}}, \end{aligned} \quad (17)$$

where $\Re[\omega(1)], \Re[\omega(2)] > 0$

and the coefficient of skewness is

$$\mathbb{C}_2=\frac{E\left(X^3\right)-3\mu\sigma^2-\mu^3}{\sigma^3}$$

$$=-2^{-\alpha}\alpha^{-\frac{1}{2}}\frac{\left\{\begin{array}{l}-2^{1+4\alpha}\alpha^2\left(B\left[\frac{1}{2};\omega(1),\alpha\right]-2B\left[\frac{1}{2};\omega(1)+1,\alpha\right]\right)^3\\ \qquad -B\left[\frac{1}{2};\omega(3),\alpha\right]+2B\left[\frac{1}{2};\omega(3)+1,\alpha\right]\\ \qquad +34^{\alpha}\alpha\left(B\left[\frac{1}{2};\omega(1),\alpha\right]-2B\left[\frac{1}{2};\omega(1)+1,\alpha\right]\right)\\ \qquad \times\left(B\left[\frac{1}{2};\omega(2),\alpha\right]-2B\left[\frac{1}{2};\omega(2)+1,\alpha\right]\right)\end{array}\right\}}{\left\{\begin{array}{l}-4^{\alpha}\alpha\left(B\left[\frac{1}{2};\omega(1),\alpha\right]-2B\left[\frac{1}{2};\omega(1)+1,\alpha\right]\right)^2\\ \qquad +B\left[\frac{1}{2};\omega(2),\alpha\right]-2B\left[\frac{1}{2};\omega(2)+1,\alpha\right]\end{array}\right\}^{3/2}}, \tag{18}$$

where $\Re\left[\omega(1)\right],\Re\left[\omega(3)\right]>0,$ and the coefficient of kurtosis is given by

$$\mathbb{C}_3=\frac{E\left(X^4\right)-4\mu E\left(X^3\right)+6E\left(X^2\right)\sigma^2+3E\left(X^4\right)}{\sigma^8}$$

$$\mathbb{C}_3=\frac{\begin{array}{l}32^{\frac{4}{p}+8\alpha}\alpha^3\theta^2\left(B\left[\frac{1}{2};\omega(1),\alpha\right]-2B\left[\frac{1}{2};\omega(1)+1,\alpha\right]\right)^4\\ \quad +32^{1+\frac{2}{p}+4\alpha}\alpha\left(B\left[\frac{1}{2};\omega(1),\alpha\right]-2B\left[\frac{1}{2};\omega(1)+1,\alpha\right]\right)^2\\ \quad \times\left\{\begin{array}{l}-2^{\frac{2}{p}+4\alpha}\alpha\left(B\left[\frac{1}{2};\omega(1),\alpha\right]-2B\left[\frac{1}{2};\omega(1)+1,\alpha\right]\right)^2\\ \qquad +2^{2(\frac{1}{p}+\alpha)}\left(B\left[\frac{1}{2};\omega(2),\alpha\right]-2B\left[\frac{1}{2};\omega(2)+1,\alpha\right]\right)\end{array}\right\}\\ \quad -2^{2+\frac{4}{p}+4\alpha}\alpha\theta^2\left(B\left[\frac{1}{2};\omega(1),\alpha\right]-2B\left[\frac{1}{2};\omega(1)+1,\alpha\right]\right)\\ \quad \times\left(B\left[\frac{1}{2};\omega(3),\alpha\right]-2B\left[\frac{1}{2};\omega(3)+1,\alpha\right]\right)\\ \quad +2^{\frac{4}{p}+2\alpha}\theta^2\left(B\left[\frac{1}{2};\omega(4),\alpha\right]-2B\left[\frac{1}{2};\omega(4)+1,\alpha\right]\right)\end{array}}{\theta^6\alpha^3\left\{\begin{array}{l}-2^{\frac{2}{p}+4\alpha}\alpha\left(B\left[\frac{1}{2};\omega(1),\alpha\right]-2B\left[\frac{1}{2};\omega(1)+1,\alpha\right]\right)^2\\ \qquad +2^{2(\frac{1}{p}+\alpha)}\left(B\left[\frac{1}{2};\omega(2),\alpha\right]-2B\left[\frac{1}{2};\omega(2)+1,\alpha\right]\right)\end{array}\right\}^4}. \tag{19}$$

In Table (1), we presented some values of mean, variance, the coefficient of variation, coefficients of skewness and kurtosis of the TLMI distribution for some parameter values.

Table 1
Mean, Variance, $\mathbb{C}_1$, $\mathbb{C}_2$, and $\mathbb{C}_3$ of the TLMI Distribution with $\alpha = 1,2,3,4,5,6,7,8,9,10$, $p = 3,5$ and $\theta = 7,10$

α	Mean	Variance	$\mathbb{C}_1$	$\mathbb{C}_2$	$\mathbb{C}_3$
	$P = 3$ and $\theta = 7$				
1	4.5	1.8	0.29814	-0.452045	2.55787
2	5.26154	0.95258	0.18550	-0.676228	3.13091
3	5.60496	0.62347	0.14088	-0.744511	3.36828
4	5.80796	0.45586	0.11625	-0.771771	3.48066
5	5.94495	0.35610	0.10038	-0.783699	3.53902
6	6.04497	0.29057	0.08917	-0.788765	3.57083
7	6.12196	0.24452	0.08077	-0.790414	3.58833
8	6.18349	0.21053	0.07421	-0.790245	3.59760
9	6.23406	0.18449	0.06889	-0.789075	3.60190
10	6.27656	0.16395	0.06451	-0.787346	3.60312
α	$P = 5$ and $\theta = 10$				
1	7.57576	2.13171	0.19273	-0.792742	3.32482
2	8.38745	0.95200	0.11633	-0.915178	3.84332
3	8.72917	0.58034	0.08727	-0.930553	3.96211
4	8.92503	0.40769	0.07154	-0.925901	3.98336
5	9.05480	0.31028	0.06152	-0.916480	3.97461
6	9.14841	0.24854	0.05449	-0.906212	3.95587
7	9.21983	0.20625	0.04926	-0.896273	3.93417
8	9.27651	0.17565	0.04518	-0.887011	3.91222
9	9.32285	0.15256	0.04190	-0.878495	3.89109
10	9.36162	0.13458	0.03919	-0.870692	3.87118

Based on Table (1), it can be seen that as the value of α increases, the mean increases while the variance decreases. Also, the TLMI distribution is skewed.

4. ORDER STATISTICS

This section deals with deriving order statistics of the unknown parameters of the TLMI distribution. The order statistics have many applications. Assume $X_1, X_2, ..., X_n$ is a random sample of size *n* from a distribution with PDF $f(x)$ and CDF $F(x)$. Let $X_{(l)}$ be the *l*th order statistic. The PDF of $X_{(l)}$ is defined as

$$f_{(l)}(x) = \frac{n!}{(l-1)!(n-l)!}[F(x)]^{l-1}[1-F(x)]^{n-l} f(x) \text{ , for } l = 1,2,...,n \text{ .} \tag{20}$$

See David and Nagaraja (2003). If the sample is selected from the TLMI distribution, then the PDF of $X_{(l)}$ is

$$f_{TLMI(l)}(x;\alpha,p,\theta) = l\binom{n}{l}\frac{2\alpha p}{\theta^{l\alpha p}}x^{l\alpha p-1}\left[1-\left(\frac{x}{\theta}\right)^{p}\right]\left[2-\left(\frac{x}{\theta}\right)^{p}\right]^{l\alpha-1} \times\left\{1-\left[2-\left(\frac{x}{\theta}\right)^{p}\right]^{\alpha}\left(\frac{x}{\theta}\right)^{\alpha p}\right\}^{n-l}. \tag{21}$$

If we take $l=1$ in (21), we get the PDF of the minimum order statistic as

$$f_{TLMI(1)}(x;\alpha,p,\theta) = \frac{2\alpha pn}{\theta^{\alpha p}}x^{\alpha p-1}\left[1-\left(\frac{x}{\theta}\right)^{p}\right]\left[2-\left(\frac{x}{\theta}\right)^{p}\right]^{\alpha-1} \times\left[1-\left(2-\left(\frac{x}{\theta}\right)^{p}\right)^{\alpha}\left(\frac{x}{\theta}\right)^{\alpha p}\right]^{n-1}, \tag{22}$$

and for $l=n$ in (21), we get the PDF of the maximum order statistic

$$f_{TLMI(n)}(x;\alpha,p,\theta) = \frac{2\alpha pn}{\theta^{\alpha np}}x^{\alpha np-1}\left[1-\left(\frac{x}{\theta}\right)^{p}\right]\left[2-\left(\frac{x}{\theta}\right)^{p}\right]^{\alpha n-1}. \tag{23}$$

5. RELIABILITY ANALYSIS

The reliability function is the probability of an item not failing prior to a time *t*. The reliability function of the TLMI distribution is given by

$$R_{TLMI}(t) = 1 - F_{TLMI}(t) = 1-\left[2\left(\frac{t}{\theta}\right)^{p}-\left(\frac{t}{\theta}\right)^{2p}\right]^{\alpha}. \tag{24}$$

Figure (3) shows the reliability function of the TLMI distribution for $\theta=6, p=3$ and $\alpha=1,2,3,4,5,6$.

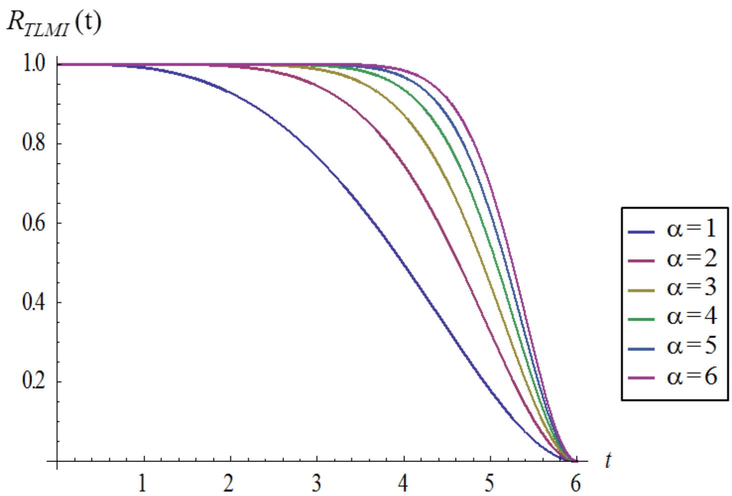

Figure 3: The Reliability Function of the TLMI Distribution with $\theta = 6, p = 3$ and $\alpha = 1, 2, 3, 4, 5, 6$

The hazard rate function of TLMI distribution is defined as

$$H_{TLMI}(t) = \frac{\frac{2\alpha p}{\theta^{p\alpha}} t^{p\alpha-1}\left[\left(\frac{t}{\theta}\right)^{p} - 1\right]\left[2 - \left(\frac{t}{\theta}\right)^{p}\right]^{\alpha-1}}{1 - \left[2\left(\frac{t}{\theta}\right)^{p} - \left(\frac{t}{\theta}\right)^{2p}\right]^{\alpha}}, \qquad (25)$$

which is known as instantaneous failure rate which is used in characterizing life phenomenon. Note that if $\alpha = p = 1$, then $H_{TLMI}(t) = \frac{2}{t-\theta}$, and $H_{TLMI}(t) \to 0$ as $\theta \to \infty$ or $t \to \infty$.

Figure (4) shows the hazard rate function of the TLMI distribution for $\theta = 6, p = 3$ and $\alpha = 1, 2, 3, 4, 5, 6$.

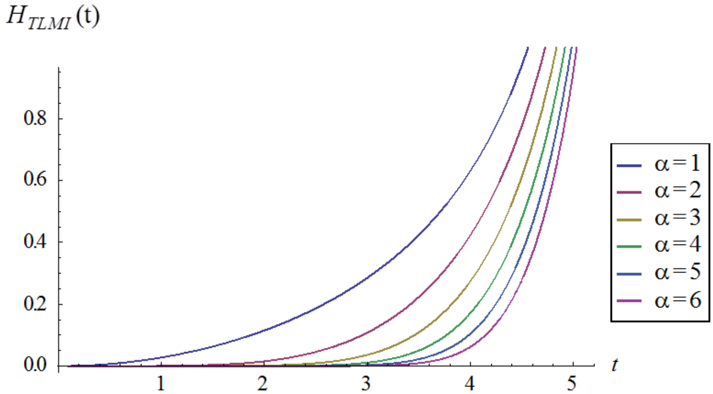

Figure 4: The Hazard Function of the TLMI Distribution with $\theta=6, p=3$ and $\alpha=1,2,3,4,5,6$

6. QUANTILE AND ESTIMATION

The Quantile of the TLMI distribution is the real solution of the equation $F\left(x_q\right)=q$, then by inverting (9) we have

$$x_q=\theta\left(1-\sqrt{1-q^{1/\alpha}}\right)^{1/p}, p>0, \quad \theta>0, \quad 0<q^{1/\alpha}<1. \tag{26}$$

As $p\to\infty$ or $\alpha\to\infty$, then $x_q=\theta$. Simulating a TLMI random variable is directly. Let U be a uniform variate on the interval (0, 1), then the random variable $x_q=q$ follows (10). The median of the TLMI distribution is $x_{0.5}=\theta\left(1-\sqrt{1-0.5^{1/\alpha}}\right)^{1/p}$.

Now, let $X_1, X_2,..., X_n$ be a random sample of size n from the TLMI distribution with parameters α, θ and p then the likelihood function is given by

$$\begin{aligned}L_{TLMI}(\alpha,\theta,p) &= \prod_{i=1}^{n}\left\{\frac{2\alpha p}{\theta^{p\alpha}}x_i^{p\alpha-1}\left[\left(\frac{x_i}{\theta}\right)^p-1\right]\left[2-\left(\frac{x_i}{\theta}\right)^p\right]^{\alpha-1}\right\}\\ &=\left(\frac{2\alpha p}{\theta^{p\alpha}}\right)^n\prod_{i=1}^{n}x_i^{p\alpha-1}\prod_{i=1}^{n}\left[\left(\frac{x_i}{\theta}\right)^p-1\right]\prod_{i=1}^{n}\left[2-\left(\frac{x_i}{\theta}\right)^p\right]^{\alpha-1}\end{aligned} \tag{27}$$

Hence, the log likelihood function $\mathbb{F}=\ln\left(L_{TLMI}(\alpha,\theta,p)\right)$ will be

$$\mathbb{F}=\ln\left\{\left(\frac{2\alpha p}{\theta^{p\alpha}}\right)^{n}\prod_{i=1}^{n}x_i^{p\alpha-1}\prod_{i=1}^{n}\left[\left(\frac{x_i}{\theta}\right)^{p}-1\right]\prod_{i=1}^{n}\left[2-\left(\frac{x_i}{\theta}\right)^{p}\right]^{\alpha-1}\right\}$$

$$=n\ln\frac{2\alpha p}{\theta^{p\alpha}}+\sum_{i=1}^{n}\ln x_i^{p\alpha-1}+\sum_{i=1}^{n}\ln\left[\left(\frac{x_i}{\theta}\right)^{p}-1\right]+\sum_{i=1}^{n}\ln\left[2-\left(\frac{x_i}{\theta}\right)^{p}\right]^{\alpha-1}$$

Differentiating Equation (27) with respect to θ,α and p results in

$$\frac{\partial\mathbb{F}}{\partial\theta}=\frac{-n\alpha p}{\theta}+\frac{(\alpha-1)p}{\theta}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\theta}\right)^{p}}{2-\left(\frac{x_i}{\theta}\right)^{p}}+\frac{p}{\theta}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\theta}\right)^{p}}{1-\left(\frac{x_i}{\theta}\right)^{p}}, \tag{28}$$

$$\frac{\partial\mathbb{F}}{\partial\alpha}=\frac{n}{\alpha}-np\,ln\theta+p\sum_{i=1}^{n}\ln x_i+\sum_{i=1}^{n}\ln\left(2-\left(\frac{x_i}{\theta}\right)^{p}\right), \tag{29}$$

$$\frac{\partial\mathbb{F}}{\partial p}=\frac{n}{p}-n\alpha\ln\theta+\alpha\sum_{i=1}^{n}\ln x_i-(\alpha-1)\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\theta}\right)^{p}\ln\left(\frac{x_i}{\theta}\right)}{2-\left(\frac{x_i}{\theta}\right)^{p}}-\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\theta}\right)^{p}\ln\left(\frac{x_i}{\theta}\right)}{1-\left(\frac{x_i}{\theta}\right)^{p}}. \tag{30}$$

The maximum likelihood estimators $\hat{\alpha},\hat{\theta},\hat{p}$ of α,θ,p can be obtained by equating the above nonlinear system to zero such that $\frac{\partial\mathbb{F}}{\partial\theta}=0$, $\frac{\partial\mathbb{F}}{\partial\alpha}=0$, $\frac{\partial\mathbb{F}}{\partial p}=0$ and solving these equations simultaneously.

7. RÉNYI ENTROPY

The entropy of a random variable X is a measure of variation of uncertainty. A large entropy value indicates greater uncertainty in the data. The Rényi (1961) entropy is defined as

$$I_R(\beta)=\frac{1}{1-\beta}\log\left(\int_0^{\infty}f(x)^{\beta}dx\right),\ \text{where }\beta>0\text{ and }\beta\neq1. \tag{31}$$

Also, the Generalized Maximum Entropy (GME) can be obtained for the TLMI distribution. For more details about the GME see Ciavolino and Al-Nasser (2009), Ciavolino and Dahlgaard (2009), Carpita and Ciavolino (2017). Ciavolino and Carpita (2015) studied the GME estimator for the regression model with a composite Indicator as explanatory variable.

The Rényi entropy of the TLMI random variable X is given in the following theorem.

Theorem 1:

The Rényi entropy of the TLMI distribution is defined as

$$I_R(\beta)=\frac{1}{1-\beta}\log\left\{\left(2^{\alpha}\alpha\right)^{\beta}\left(\frac{p}{\theta}\right)^{\beta-1}\left[\sum_{i=1}^{\infty}\binom{(\alpha-1)\beta}{i}\left(-\frac{1}{2}\right)^{i}\right]B\left(\alpha\beta-\frac{\beta}{p}+\frac{1}{p}+i\,,\beta-1\right)\right\}, \tag{32}$$

where $B(\alpha,\beta)=\int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

Proof:

$$I_R(\beta)=\frac{1}{1-\beta}\log\left(\int_0^{\infty}\left(f(x)\right)^{\beta}dx\right)$$

$$=\frac{1}{1-\beta}\log\left(\int_0^{\theta}\alpha^{\beta}\left[2\left(\frac{x}{\theta}\right)^{p}-\left(\frac{x}{\theta}\right)^{2p}\right]^{(\alpha-1)\beta}\left[\frac{2p}{\theta^{p}}x^{p-1}-\frac{2p}{\theta^{2p}}x^{2p-1}\right]^{\beta}dx\right)$$

Let $y=\left(\frac{x}{\theta}\right)^{p}$, then $x=\theta y^{\frac{1}{p}}$ and $dx=\frac{\theta}{p}y^{\frac{1}{p}-1}dy$. Therefore,

$$I_R(\beta)=\frac{1}{1-\beta}\log\left(\int_0^{1}\alpha^{\beta}\left(2y-y^{2}\right)^{(\alpha-1)\beta}\left[\frac{2p}{\theta^{p}}\left(\theta y^{\frac{1}{p}}\right)^{p-1}-\frac{2p}{\theta^{2p}}\left(\theta y^{\frac{1}{p}}\right)^{2p-1}\right]^{\beta}\frac{\theta}{p}y^{\frac{1}{p}-1}dy\right)$$

$$=\frac{1}{1-\beta}\log\left(\left(2\alpha\right)^{\beta}\left(\frac{p}{\theta}\right)^{\beta-1}\int_0^{1}y^{\alpha\beta-\beta+\frac{1}{p}-1}\left(2-y\right)^{(\alpha-1)\beta}\left[y^{1-\frac{1}{p}}\left(1-y\right)\right]^{\beta}dy\right)$$

$$=\frac{1}{1-\beta}\log\left(\left(2\alpha\right)^{\beta}\left(\frac{p}{\theta}\right)^{\beta-1}\int_0^{1}y^{\alpha\beta-\frac{\beta}{p}+\frac{1}{p}-1}\left(2\left(1-\frac{y}{2}\right)\right)^{(\alpha-1)\beta}\left(1-y\right)^{\beta}dy\right)$$

$$=\frac{1}{1-\beta}\log\left(\left(2^{\alpha}\alpha\right)^{\beta}\left(\frac{p}{\theta}\right)^{\beta-1}\int_0^{1}y^{\alpha\beta-\frac{\beta}{p}+\frac{1}{p}-1}\left(1-\frac{y}{2}\right)^{(\alpha-1)\beta}\left(1-y\right)^{\beta}dy\right)$$

Using binomial series, we can write

$$I_R(\beta)=\frac{1}{1-\beta}\log\left(\left(2^{\alpha}\alpha\right)^{\beta}\left(\frac{p}{\theta}\right)^{\beta-1}\int_0^{1}y^{\alpha\beta-\frac{\beta}{p}+\frac{1}{p}-1}\left(\sum_{i=1}^{\infty}\binom{(\alpha-1)\beta}{i}\left(-\frac{y}{2}\right)^{i}\right)\left(1-y\right)^{\beta}dy\right)$$

$$=\frac{1}{1-\beta}\log\left(\left(2^{\alpha}\alpha\right)^{\beta}\left(\frac{p}{\theta}\right)^{\beta-1}\left(\sum_{i=1}^{\infty}\binom{(\alpha-1)\beta}{i}\left(-\frac{1}{2}\right)^{i}\right)\int_0^{1}y^{\alpha\beta-\frac{\beta}{p}+\frac{1}{p}+i-1}\left(1-y\right)^{\beta}dy\right)$$

$$=\frac{1}{1-\beta}\log\left(\left(2^{\alpha}\alpha\right)^{\beta}\left(\frac{p}{\theta}\right)^{\beta-1}\left(\sum_{i=1}^{\infty}\binom{(\alpha-1)\beta}{i}\left(-\frac{1}{2}\right)^{i}\right)B\left(\alpha\beta-\frac{\beta}{p}+\frac{1}{p}+i\,,\beta-1\right)\right).\ \square$$

8. APPLICATION

In this section, we demonstrate the applicability of the proposed TLMI distribution for a real data set. The data listed below represent the lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross and Clark (1975, P. 105). The data are as follows:

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.

These data have been widely used by many authors. For example, Shanker (2015) reported that Shanker distribution fits these data better than exponential and Lindley (Lindley, 1958) distributions.

We use the proposed TLMI distribution to fit these data in addition to the Lindley, exponential and Shanker distributions, respectively, are

$$f_{LD}(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x} \quad ; x>0,\ \theta>0, \tag{33}$$

$$f_{ED}(x;\theta)=\theta e^{-\theta x} \quad ; x>0,\ \theta>0, \tag{34}$$

$$f_{SD}(x;\theta)=\frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x} \quad ; x>0,\ \theta>0. \tag{35}$$

The MLEs of the parameters, the Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), the maximized log likelihood (MLL), the Kolmogorov–Smirnov Statistics (K–S) with its respective *P*-value, for the above distributions as well as our proposed model are given in Table 2, where

$$\text{AIC}=-2MLL+2\delta,\ \text{CAIC}=-2MLL+\frac{2\delta n}{n-\delta-1},\ \text{BIC}=-2MLL+\delta Log(n)$$

where δ is the number of parameters and n is the sample size.

Table 2
The Statistics AIC, CAIC, BIC, K-S and -2MLL for of the Fitted Distributions of Data

Model	AIC	CAIC	BIC	-2MLL	K-S	P-value	Parameter estimate		
Expo-nential	67.7	67.9	68.7	65.7	0.3895	0.000	0.5253	-------	-------
Lindley	62.5	62.7	63.5	60.5	0.3410	0.002	0.8161	-------	-------
Shanker	61.8	62.0	62.8	59.7	0.3151	0.005	0.8039	-------	-------
TLMI	2.76	3.01	10.6	0.28	0.0569	0.9028	$\hat{p}=10.5114$	$\hat{\theta}=1.0144$	$\hat{\alpha}=0.0852$

The results presented in Table (2) indicate that the proposed TLMI distribution fits the data better than the other distributions considered in this study. Hence, the TLMI distribution is preferred to Shanker, exponential and Lindley distributions for modeling lifetime data set.

9. CONCLUSIONS

In this paper a new continuous distribution is proposed and is called TLMI distribution. Some of the important mathematical properties including the moments, the coefficients of variation, skewness and kurtosis are studied. Also, distribution of order statistics, hazard rate and reliability functions, quartile and generation of random numbers are investigated. Estimation of the TLMI distribution parameters using the maximum likelihood estimation are obtained and the Rényi entropy is provided. An application of the proposed distribution to a real data set is provided and compared with other distributions considered in this study.

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