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EXPONENTIATED GENERALIZED INVERSE RAYLEIGH DISTRIBUTION WITH APPLICATIONS IN MEDICAL SCIENCES

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ABSTRACT

In this paper, we introduce Exponentiated generalized inverse Rayleigh distribution and study its different structural properties. The parameters of the proposed distribution have been estimated through the method of maximum likelihood estimation. Further, two types of data sets are considered for making the comparison among special cases of EGIR distribution in terms of model fitting.

KEYWORDS

Exponentiated distribution, Inverse Rayleigh distribution, Reliability analysis, Maximum likelihood estimation, Real life data, AIC, BIC.

1. INTRODUCTION

The inverse Rayleigh distribution is one of the most flexible distributions among the inverted scale family. It has been considered as a suitable model in life testing and reliability theory. The inverse Rayleigh distribution has many applications in the area of reliability studies. Trayer (1964) introduced the inverse Rayleigh distribution and discussed its application in reliability theory. Voda (1972) mentioned that the distribution of lifetimes of several types of experimental units can be approximated by the inverse Rayleigh distribution. The density function of the generalized inverse Rayleigh distribution (GIRD) is given by:

$$g(x)=\frac{2\alpha}{\lambda^2 x^3}e^{-(\lambda x)^{-2}}\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha-1} \quad ;x>0,\alpha,\lambda>0. \tag{1}$$

The CDF of IRD is given by

$$G(x)=1-\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha} \quad ;x>0,\alpha,\lambda>0. \tag{2}$$

where α and λ are shape and scale parameters respectively.

The generalized inverted Rayleigh (GIR) distribution is a general form of the inverse Rayleigh distribution (IRD). A generalized inverted scale family of distributions was introduced by Potdar and Shirke (2013). They generalized the scale family by introducing a shape parameter α to obtain a generalized scale family of distributions. Gharraph (1993) derived different measures of IRD. He obtained the like closed-form expressions for the

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mean, harmonic mean, geometric mean, mode and the median of this distribution.

The Exponentiated family of distribution is derived by powering a positive real number to the cumulative distribution function (CDF) of an arbitrary parent distribution by a shape parameter, say $\gamma > 0$. Its PDF is given by:

$$f(x) = \gamma\, g(x)\left[G(x)\right]^{\gamma-1}. \tag{3}$$

The corresponding cumulative distribution function (CDF) is given by

$$F(x) = \left[G(x)\right]^{\gamma}, \gamma > 0. \tag{4}$$

Gupta et al. (1998) first proposed a generalization of the standard exponential distribution, called the exponentiated exponential (EE) distribution, defined by the cumulative distribution function, $G(x) = \left(1 - e^{-\lambda x}\right)^{\alpha}; x > 0, \alpha, \lambda > 0.$ This equation is simply the power of the standard exponential cumulative distribution. The extensions of inverse Rayleigh distribution exist in literature and these models are applied in many areas, including reliability, life tests and survival analysis. Exponentiated inverse Rayleigh distribution has been studied by Rehman et al. (2015). Many other generalizations of the gamma, Fréchet and Gumbel distributions have been proposed by Nadarajah and Kotz (2006) although the way they defined the CDF of the last two distributions is slightly different. Oguntunde et al. (2014) have introduced the Exponentiated Generalized Inverted Exponential distribution. They provided another generalization of the inverted exponential distribution which serves as a competitive model and an alternative to both the generalized inverse exponential distribution and the inverse exponential distribution.

The rest of the paper is organized as follows. In Section 2, we define the EGIR distribution, provide its cumulative distribution function, the probability density function and the reliability analysis; we also discuss the special sub-models of this distribution. A formula for generating EGIR random samples from the EGIR distribution is given in Section 3. Section 4 discusses some important statistical properties of the EGIR distribution such as the moments, harmonic mean, generating functions and the distribution of the order statistics. Maximum likelihood estimates of the parameters to the distribution are presented in Section 5. Applications and simulated data sets are performed in Section 6. Results and conclusion based on the data sets as well as simulation study is given in last section 7.

2. EXPONENTIATED GENERALIZED INVERSE RAYLEIGH DISTRIBUTION

With this understanding, we put Equations (1) and (2) into Equation (3) to form the PDF and CDF of the Exponentiated generalized inverse Rayleigh (EGIR) distribution as

$$f(x) = \frac{2\alpha\gamma}{\lambda^2 x^3} e^{-(\lambda x)^{-2}} \left(1 - e^{-(\lambda x)^{-2}}\right)^{\alpha-1} \left[1 - \left(1 - e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]^{\gamma-1}; \; x > 0; \alpha, \gamma, \lambda > 0.$$

(5)

and

$$F(x)=\left[1-\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]^{\gamma} \quad ; 0<x<\infty; \alpha,\lambda, \gamma>0, \tag{6}$$

where α and γ are shape parameters and λ is a scale parameter.

The graphs of density function and cumulative distribution function plotted for different values of parameters α,λ and γ are given in Figure 1 and 2 respectively.

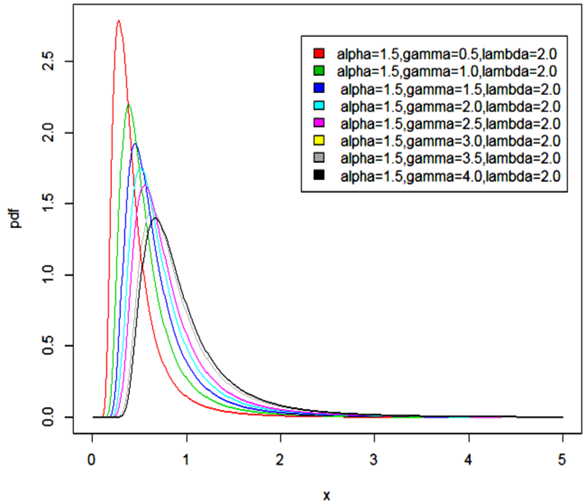

Figure 1: The Graph of Density Function

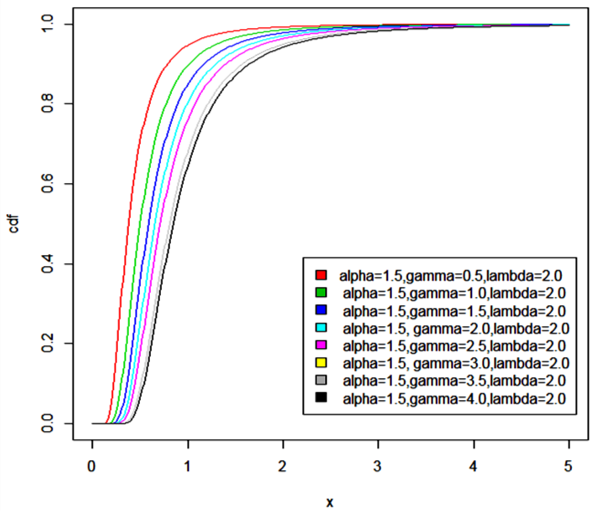

Figure 2: The Graph of Distribution Function

Figure 1 gives the description of some of the possible shapes of EGIR distribution for different values of the parameters α, λ and γ. Figure 1 illustrates that the density function of EGIR distribution is positively skewed, for fixed α and λ, it becomes more and more flatter as the value of γ is increased. Figure 2 shows the graph of distribution function which is an increasing function.

2.1 Special Cases

Let *X* denote a non-negative random variable with the PDF in Equation (5), some other well-known theoretical distributions are found to be sub-models of the proposed *EGIR* distribution such as;

1) For $\gamma = 1$, Equation (5) reduces to give the two parameter generalized Inverse Rayleigh (GIR) distribution with probability density function as:

$$f(x) = \frac{2\alpha}{\lambda^2 x^3} e^{-(\lambda x)^{-2}} \left(1 - e^{-(\lambda x)^{-2}}\right)^{\alpha-1} \quad ; x > 0, \alpha, \lambda > 0.$$

2) For $\alpha = 1$ and $\gamma = 1$, Equation (5) reduces to give the one parameter Inverse Rayleigh (IR) distribution with probability density function as:

$$f(x) = \frac{2}{\lambda^2 x^3} e^{-(\lambda x)^{-2}} \; ; x > 0, \lambda > 0 \; .$$

3) For $\gamma = 1$ and $\lambda = 1$, Equation (5) reduces to give the one parameter generalized standard Inverse Rayleigh (GSIR) distribution with probability density function as:

$$f(x) = \frac{2\alpha}{x^3} e^{-(x)^{-2}} \left(1 - e^{-(x)^{-2}}\right)^{\alpha-1} \quad ; x > 0, \alpha > 0.$$

4) For $\alpha = 1$, Equation (5) reduces to give the two parameter Inverse Rayleigh (EIR) distribution with probability density function as:

$$f(x) = \frac{2\gamma}{\lambda^2 x^3} e^{-\left\{\frac{\gamma}{(\lambda x)^2}\right\}} \quad ; x > 0; \; \gamma, \lambda > 0.$$

5) For $\alpha = 1$ and $\lambda = 1$, Equation (5) reduces to give the one parameter exponentiated standard Inverse Rayleigh (ESIR) distribution with probability density function as:

$$f(x) = \frac{2\gamma}{x^3} e^{-(x)^{-2}} \left[1 - \left(1 - e^{-(x)^{-2}}\right)\right]^{\gamma-1} \; ; x > 0, \gamma > 0.$$

2.2 Reliability Analysis

The reliability (survival) function of EGIR distribution is given by

$$R(x) = 1 - F(x) = 1 - \left[1 - \left(1 - e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]^{\gamma} \; ; x > 0; \; \alpha, \lambda, \gamma > 0.$$

The hazard function (failure rate) is given by

$$h(x)=\frac{f(x)}{R(x)}=\frac{2\alpha\gamma}{\lambda^2x^3}\frac{e^{-(\lambda x)^{-2}}\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha-1}\left[1-\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]^{\gamma-1}}{1-\left[1-\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]^{\gamma}}.$$

The Reverse Hazard function of the EGIR distribution is obtained by

$$\varphi(x)=\frac{f(x)}{F(x)}=\frac{2\alpha\gamma}{\lambda^2x^3}\frac{e^{-(\lambda x)^{-2}}\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha-1}\left[1-\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]^{\gamma-1}}{1-\left[1-\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]^{\gamma}}.$$

The graphical representation of the reliability function and hazard function for the EGIR distribution is shown in figure 3 and 4 respectively.

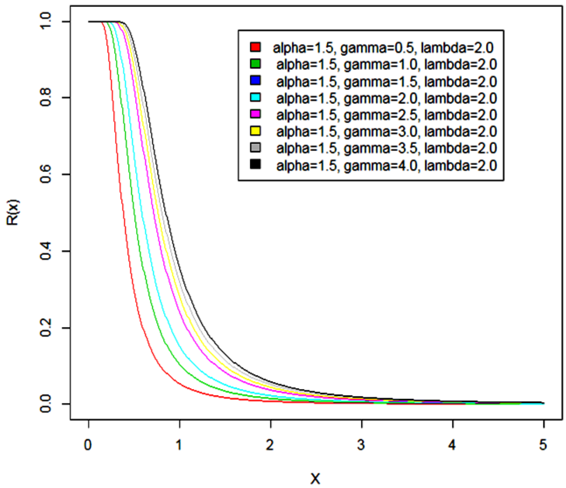

Figure 3: The Graph of Reliability Function

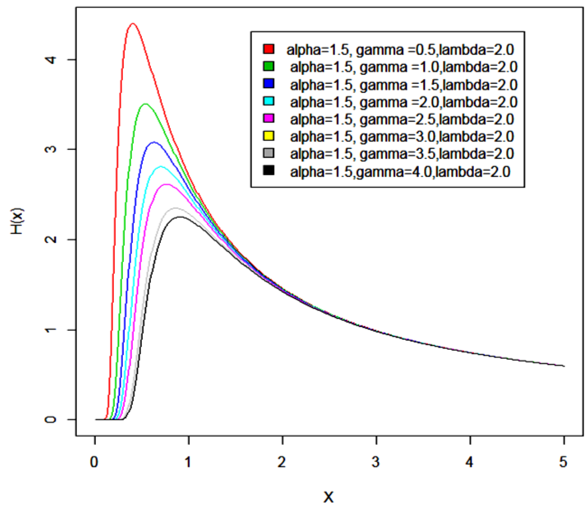

Figure 3: The Graph of Hazard Function

Figure 3 and 4 gives the reliability and hazard patterns of EGIR distribution for different values of the parameters γ. The figures show that the reliability function of EGIR distribution is downward skewed for fixed α and λ, it becomes more and more flatter as the value of γ is increased. The behavior of instantaneous failure rate of the EGIIR distribution has upside-down bathtub shape curve.

3. QUANTILE AND RANDOM NUMBER GENERATION

The Inverse CDF method is used for generating random numbers from a particular distribution. In this method, random numbers from a particular distribution are generated by solving the equation obtained on equating CDF of a distribution to a number *u*. The number *u* is itself being generated from *U*~(0,1). In case of EGIR distribution we define it by the following equation:

$$F(x) = u, \text{ where } u \sim U(0,1).$$

$$\Rightarrow \quad \left[1-\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]^{\gamma} = u. \tag{7}$$

On solving equation (7) for *x*, at fixed values of parameters (α, γ, λ), we will obtain the random number from the EGIR distribution as:

$$x = \frac{1}{\lambda}\left[-\log\left(1-\left(1-u^{1/\gamma}\right)^{1/\alpha}\right)\right]^{-1/2}. \tag{8}$$

For *u*=1/4, 1/2 and ¾ in equation (8) we get the resulting first quartile, second quartile (Median) and third quartile respectively.

4. STATISTICAL PROPERTIES OF THE EGIR DISTRIBUTION

This section provides some basic statistical properties of the exponentiated generalized Inverse Rayleigh (EGIR) distribution.

4.1 Moments of the EGIR Distribution

Theorem 4.1:

If $X \sim \text{EGIR}(\alpha,\lambda,\gamma)$, then r^{th} moment of a continuous random variable X is given as follow:

$$\mu_r = E(X^r) = A_{jk}\,\frac{\alpha\gamma\,\Gamma(1-r/2)}{\lambda^r (k+1)^{(1-r/2)}}.$$

Proof:

Let X is an absolutely continuous non-negative random variable with PDF, $f(x)$, then the r^{th} moment of X can be obtained by:

$$\mu_r = E(X^r) = \int_0^\infty x^r f(x)\,dx.$$

From the PDF of the TGIR distribution in (5), then shows that $E(X^r)$ can be written as:

$$E(X^r) = \int_0^\infty x^r \frac{2\alpha\gamma}{\lambda^2 x^3} e^{-(\lambda x)^{-2}} \left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha-1} \left[1-\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]^{\gamma-1} dx.$$

Making the substitution, $y = \frac{1}{\lambda^2 x^2}$; $\frac{-2}{\lambda^2 x^3} dx = dy$, so that $x = \frac{1}{\lambda y^{1/2}}$, we obtain

$$E(X^r) = \frac{\alpha\gamma}{\lambda^r} \int_0^\infty y^{1-r/2-1} e^{-y} \left(1-e^{-y}\right)^{\alpha-1} \left[1-\left(1-e^{-y}\right)^{\alpha}\right]^{\gamma-1} dy. \tag{9}$$

Using the expansion of $\left[1-\left(1-e^{-y}\right)^{\alpha}\right]^{\gamma-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\gamma)}{\Gamma(\gamma-j)\,j!} \left(1-e^{-y}\right)^{-\alpha j}.$

Expression (9) takes the following form:

$$E(X^r) = \frac{\alpha\gamma}{\lambda^r} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\gamma)}{\Gamma(\gamma-j)\,j!} \int_0^\infty y^{1-r/2-1} e^{-y} \left(1-e^{-y}\right)^{-\alpha(j+1)-1} dy. \tag{10}$$

Also,

$$\left(1-e^{-y}\right)^{-\alpha(j+1)-1} = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\alpha(j+1))}{\Gamma(\alpha(j+1)-k)\,k!} e^{-yk}.$$

Expression (8) takes the following form:

$$E(X^r) = \frac{\alpha\gamma}{\lambda^r} A_{jk} \int_0^\infty y^{1-r/2-1} e^{-y(k+1)} dy.$$

After some calculations,

$$\mu_r = E(X^r) = A_{jk} \frac{\alpha\gamma\,\Gamma(1-r/2)}{\lambda^r (k+1)^{(1-r/2)}}, \tag{11}$$

where, $A_{jk} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{j+k}\,\Gamma(\gamma)\Gamma(\alpha(j+1))}{\Gamma(\gamma-j)\,j!\,\Gamma(\alpha(j+1)-k)k!}$.

We observe that equation (11) only exists when r < 1. This implies that the second moment and other higher order moments of the distribution do not exist.

4.2 Harmonic Mean

The harmonic mean (*H*) is given by:

$$\frac{1}{H} = E\left(\frac{1}{X}\right) = \int_0^{\infty} \frac{2\alpha\gamma}{\lambda^2 x^4} e^{-(\lambda x)^{-2}} \left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha-1} \left[1-\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]^{\gamma-1} dx..$$

Making the substitution; $y = \frac{1}{\lambda^2 x^2}$, $\frac{-2}{\lambda^2 x^3} dx = dy$, so that $x = \frac{1}{\lambda y^{1/2}}$,we obtain

$$\frac{1}{H} = \int_0^{\infty} y^{(3/2)-1} e^{-y} \left(1-e^{-y}\right)^{\alpha-1} \left[1-\left(1-e^{-y}\right)^{\alpha}\right]^{\gamma-1} dy\,. \tag{12}$$

Using the expansion of $\left[1-\left(1-e^{-y}\right)^{\alpha}\right]^{\gamma-1} = \sum_{j=0}^{\infty} \frac{(-1)^j\,\Gamma(\gamma)}{\Gamma(\gamma-j)\,j!} \left(1-e^{-y}\right)^{-\alpha j}$.

Expression (12) takes the following form:

$$\frac{1}{H} = \alpha\lambda\gamma \sum_{j=0}^{\infty} \frac{(-1)^j\,\Gamma(\gamma)}{\Gamma(\gamma-j)\,j!} \int_0^{\infty} y^{3/2-1} e^{-y} \left(1-e^{-y}\right)^{-\alpha(j+1)-1} dy. \tag{13}$$

Also, $\left(1-e^{-y}\right)^{-\alpha(j+1)-1} = \sum_{k=0}^{\infty} \frac{(-1)^k\,\Gamma(\alpha(j+1))}{\Gamma(\alpha(j+1)-k)\,k!} e^{-yk}$.

Expression (11) takes the following form:

$$\frac{1}{H} = \frac{\alpha\gamma}{\lambda^r} A_{jk} \int_0^{\infty} y^{3/2-1} e^{-y(k+1)} dy.$$

After some calculations,

$$\frac{1}{H} = A_{jk} \frac{\alpha\gamma\,\Gamma(3/2)}{\lambda^r (k+1)^{3/2}}, \tag{14}$$

where, $A_{jk} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{j+k}\,\Gamma(\gamma)\Gamma(\alpha(j+1))}{\Gamma(\beta-j)\,j!\,\Gamma(\alpha(j+1)-k)\,k!}$.

4.3 Moment Generating Function (MGF)

In this sub section, we derive the moment generating function of EGIR distribution.

Theorem 4.2:

Let X have a EGIR distribution. Then moment generating function of X denoted by $M_X(t)$ is given by:

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} A_{jk} \frac{\alpha\gamma\,\Gamma(1-r/2)}{\lambda^r (k+1)^{1-r/2}}. \tag{15}$$

Proof:

By definition

$$M_X(t) = E\left(e^{tx}\right) = \int_0^{\infty} e^{tx} f(x)\,dx.$$

Using Taylor series

$$M_X(t) = \int_0^{\infty} \left(1 + tx + \frac{(tx)^2}{2!} + \cdots\right) f(x)\,dx$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x)\,dx. = \sum_{i=0}^{\infty} \frac{t^r}{r!} E\left(X^r\right)$$

$$\Rightarrow M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} A_{jk} \frac{\alpha\gamma\,\Gamma(1-r/2)}{\lambda^r (k+1)^{1-r/2}}.$$

This completes the proof.

4.4 Characteristic Function

In this sub section, we derive the Characteristic function of EGIR distribution.

Theorem 4.3:

Let X have a EGIR distribution. Then characteristic function of X denoted by $\varphi_X(t)$ is given by:

$$\varphi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} A_{jk} \frac{\alpha\gamma\,\Gamma(1-r/2)}{\lambda^r (k+1)^{1-r/2}}. \tag{16}$$

Proof:

By definition

$$\varphi_X(t) = E\left(e^{itx}\right) = \int_0^{\infty} e^{itx} f(x)\,dx..$$

Using Taylor series

$$\varphi_X(t)=\int_0^\infty\left(1+itx+\frac{(itx)^2}{2!}+\cdots\right)f(x)\,dx.$$

$$=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\int_0^\infty x^r f(x)\,dx=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}E(X^r)$$

$$\Rightarrow \varphi_X(t)=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}A_{jk}\,\frac{\alpha\gamma\,\Gamma(1-r/2)}{\lambda^r(k+1)^{1-r/2}}.$$

This completes the proof.

4.5 Order Statistics

Order statistics extensively used in the fields of reliability and life testing. Let $X_{(1)},X_{(2)},X_{(3)}....,X_{(n)}$ be the ordered statistics of the random sample $X_1,X_2,...X_n$ drawn from the continuous distribution with CDF and PDF given respectively by (5) and (6), PDF of r^{th} order statistics (i.e., $X_{(r)}$) of EGIR distribution is given by:

$$f_{(r)}(x)=\frac{n!}{(r-1)!(n-r)!}\frac{2\alpha\gamma}{\lambda^2x^3}e^{-(\lambda x)^{-2}}\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha-1}$$
$$\left[1-\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]^{r\gamma-1}\left[1-\left[1-\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]^{\gamma}\right]^{n-r}. \tag{17}$$

The PDF of first order statistics, (i.e., $X_{(1)}$) is given by:

$$f_{(1)}(x)=n\frac{2\alpha\gamma}{\lambda^2x^3}e^{-(\lambda x)^{-2}}\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha-1}$$
$$\left[1-\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]^{\gamma-1}\left[1-\left[1-\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]^{\gamma}\right]^{n-1}. \tag{18}$$

and the PDF of nth order (i.e., $X_{(n)}$) is given by:

$$f_{(n)}(x)=\frac{n!}{(r-1)!(n-r)!}\frac{2\alpha\gamma}{\lambda^2x^3}e^{-(\lambda x)^{-2}}\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha-1}\left[1-\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]^{n\gamma-1}. \tag{19}$$

5. ESTIMATION OF PARAMETERS

We estimate the parameters of the proposed model using the method of maximum likelihood estimation (MLE) as follows; Let $X_1,X_2,...X_n$ be a random sample of size n from the *EGIR* distribution. The likelihood function is given by:

$$L(x)=\left(\frac{2\alpha\gamma}{\lambda^2}\right)^n \frac{e^{-\lambda^{-2}\sum_{n=0}^{\infty}x_i^{-2}}}{\prod_{n=0}^{\infty}x_i^3}\prod_{n=0}^{\infty}\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha-1}\prod_{n=0}^{\infty}\left[1-\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]^{\gamma-1}. \tag{20}$$

And the log- likelihood function is given by

$$\begin{aligned} l=\log L = n\ln 2+n\ln\alpha+n\ln\gamma-2n\ln\lambda-\frac{\sum_{n=0}^{\infty}x_i^{-2}}{\lambda^2}-3\sum_{n=0}^{\infty}\ln x_i \\ +(\alpha-1)\sum_{n=0}^{\infty}\ln\left(1-e^{-(\lambda x)^{-2}}\right)+(\gamma-1)\sum_{n=0}^{\infty}\ln\left[1-\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]. \end{aligned} \tag{21}$$

Differentiating Equation (21) with respect to each of the parameters α,λ and γ, gives;

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}+\sum_{n=0}^{\infty}\ln\left(1-e^{-(\lambda x)^{-2}}\right)-(\gamma-1)\sum_{n=0}^{\infty}\frac{\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha}\ln\left(1-e^{-(\lambda x)^{-2}}\right)}{\left[1-\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]}. \tag{22}$$

$$\begin{aligned} \frac{\partial l}{\partial\lambda}=\frac{-2n}{\lambda}+\frac{2\sum_{n=0}^{\infty}x_i^{-2}}{\lambda^3}-\frac{(\alpha-1)}{\lambda^2}\sum_{n=0}^{\infty}\frac{e^{-(\lambda x)^{-2}}}{\left(1-e^{-(\lambda x)^{-2}}\right)}\frac{1}{x_i^2} \\ +\alpha(\gamma-1)\sum_{n=0}^{\infty}\frac{\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha-1}e^{-(\lambda x)^{-2}}}{\left[1-\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]}\frac{1}{x_i^2}. \end{aligned} \tag{23}$$

and

$$\frac{\partial l}{\partial\gamma}=\frac{n}{\gamma}+\sum_{n=0}^{\infty}\ln\left[1-\left(1-e^{-(\lambda x)^{-2}}\right)^{\alpha}\right]. \tag{24}$$

Now, solving the resulting non-linear system of equations $\frac{\partial l}{\partial\alpha}=0, \frac{\partial l}{\partial\lambda}=0$ and $\frac{\partial l}{\partial\gamma}=0$ gives the maximum likelihood estimate of the parameters α,λ and γ respectively. Also, all the second order derivatives exist. Thus we have the 3 x 3 inverse dispersion matrix given by

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\gamma}\\ \hat{\lambda}\end{pmatrix}\sim N\left[\begin{pmatrix}\alpha\\ \gamma\\ \lambda\end{pmatrix},\begin{pmatrix}\hat{V}_{\alpha\alpha} & \hat{V}_{\alpha\gamma} & \hat{V}_{\alpha\lambda}\\ \hat{V}_{\gamma\alpha} & \hat{V}_{\gamma\gamma} & \hat{V}_{\alpha\gamma}\\ \hat{V}_{\lambda\alpha} & \hat{V}_{\lambda\alpha} & \hat{V}_{\lambda\lambda}\end{pmatrix}\right]$$

$$V^{-1} = -E\begin{pmatrix} V_{\alpha\alpha} & V_{\alpha\gamma} & V_{\alpha\lambda} \\ V_{\gamma\alpha} & V_{\gamma\gamma} & V_{\alpha\gamma} \\ V_{\lambda\alpha} & V_{\lambda\alpha} & V_{\lambda\lambda} \end{pmatrix}, \tag{25}$$

where

$$V_{\alpha\alpha} = \frac{\partial^2 l}{\partial\alpha^2}, V_{\gamma\gamma} = \frac{\partial^2 l}{\partial\gamma^2}, V_{\lambda\lambda} = \frac{\partial^2 l}{\partial\lambda^2}$$

and

$$V_{\alpha\gamma} = V_{\gamma\alpha} = \frac{\partial^2 l}{\partial\alpha\,\partial\gamma}, V_{\alpha\lambda} = V_{\lambda\alpha} = \frac{\partial^2 l}{\partial\alpha\,\partial\lambda}, V_{\gamma\lambda} = V_{\lambda\gamma} = \frac{\partial^2 l}{\partial\gamma\,\partial\lambda}.$$

The solution of the above inverse dispersion matrix will yield the asymptotic variance and covariance of the maximum likelihood estimators $\hat{\alpha}, \hat{\gamma}, \hat{\lambda}$. Hence, the approximate 100 (1- θ) % confidence intervals for α, γ, λ are given respectively by

$$\hat{\alpha} \pm Z_{\frac{\theta}{2}}\sqrt{\hat{V}_{\alpha\alpha}}, \hat{\gamma} \pm Z_{\frac{\theta}{2}}\sqrt{\hat{V}_{\gamma\gamma}}, \hat{\lambda} \pm Z_{\frac{\theta}{2}}\sqrt{\hat{V}_{\lambda\lambda}},$$

where $Z_{\frac{\theta}{2}}$ is the θ-th percentile of the standard normal distribution.

6. SIMULATION AND REAL DATA APPLICATION

In this section, we consider both a real life and two simulated data sets to compare the flexibility of the EGIR distribution over the existing sub models. The real data set represent the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938 (Lee, 1992). This data set has recently been studied by Ramos et al. (2013). The simulated data sets are of sizes 70 and 100, simulated from EGIR distribution with parameters $(\alpha, \lambda, \gamma) =$ (0.3, 2.8, 2.3). The data set is simulated by using the inverse CDF method discussed in section 3. The analysis involved in this study has been performed with the help of R software.

Table 1

(Real Data Set) Survival times of 121 patients with breast cancer
0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0.

Simulated Data with n=70
2.1857909, 3.2657407, 7.5405588, 123.0365878, 1.7099486, 103.2784646, 292.0878697, 11.8597624, 9.9788068, 0.8851332, 1.7391335, 1.5457906, 13.8231335, 3.4187854, 24.4192233, 5.3905946, 16.7660105, 7379.8861247, 3.3659148, 25.9541610, 220.3676203, 1.7817000, 11.2699028, 1.2407629, 2.1999588, 3.4452597, 0.5643854, 3.3963745, 67.0814598, 2.8954614, 5.0482513, 8.5813914, 5.2966300, 1.6076277, 40.9134565, 12.3901938, 29.9025979, 1.1415482, 17.4600079, 3.7974909, 38.3416200, 10.9874156, 27.1534713, 6.8765778, 6.1918899, 28.6661032, 0.6458494, 4.9476128, 18.5139951, 14.3120606, 4.9556017, 60.0833794, 4.2209389, 2.0207120, 0.9350786, 1.0944277, 2.6449693, 5.8979040, 11.9410877, 3.7323206, 134.5802740, 2.4294021, 4.5924390, 2.8100624, 11.2199821, 2.1247539, 4.9746497, 23.7522422, 1.0103047, 72.4485218.

Simulated Data with n=100
2.1857909, 3.2657407, 7.5405588, 123.0365878, 1.7099486, 103.2784646, 292.0878697, 11.8597624, 9.9788068, 0.8851332, 1.7391335, 1.5457906, 13.8231335, 3.4187854, 24.4192233, 5.3905946, 16.7660105, 7379.8861247, 3.3659148, 25.9541610, 220.3676203, 1.7817000, 11.2699028, 1.2407629, 2.1999588, 3.4452597, 0.5643854, 3.3963745 67.0814598, 2.8954614, 5.0482513, 8.5813914, 5.2966300, 1.6076277, 40.9134565, 12.3901938, 29.9025979, 1.1415482, 17.4600079, 3.7974909, 38.3416200, 10.9874156, 27.1534713, 6.8765778, 6.1918899, 28.6661032, 0.6458494, 4.9476128, 18.5139951, 14.3120606, 4.9556017, 60.0833794, 4.2209389, 2.0207120, 0.9350786, 1.0944277, 2.6449693, 5.8979040, 11.9410877, 3.7323206, 134.5802740, 2.4294021, 4.5924390, 2.8100624, 11.2199821, 2.1247539, 4.9746497, 23.7522422, 1.0103047, 72.4485218, 2.8815776, 46.5105438, 2.9654522, 2.8246874, 4.9296499, 93.2454965, 62.5289551, 3.4969611, 25.9278741, 519.2786367, 4.1636004, 16.2152646, 3.6346269, 2.7366697, 22.1327641, 1.7167859, 16.0694929, 1.2188081, 2.0260411, 1.3434023, 1.9812447, 0.8689279, 10.7056688, 73.4172075, 26.2674055, 30.7212172, 4.5223684, 3.7799050, 34.7775319, 8.8140785.

For the comparison of the EGIR distribution and its sub models we have taken into consideration the criteria like AIC (Akaike information criterion) and BIC (Bayesian information criterion). The distribution which has lesser values of AIC and BIC is considered to be better.

$$\text{AIC} = 2k - 2\log L \quad \text{and BIC}=k\log n\text{-}2\log L,$$

where k is the number of parameters in the statistical model, n is the sample size and -2 log L is the maximized value of the log-likelihood function under the considered model.

The summary of the data is given in Table 1. The result of the estimates of the parameters, log-likelihood, Akaike Information Criteria (AIC) and Bayesian Information Criteria (BIC) for the above data is presented in Tables 2, 3 and 4.

Table 2
MLEs of the Model Parameters using Real Data, the resulting SEs in Braces and Criteria for Comparison

Distribution	*A*	*γ*	*λ*	log-lik	AIC	BIC
EGIRD	0.33281426 (0.01834291)	474.48745714 (151.89636490)	591.52674133 (1.36264638)	636.9514	1279.903	1288.29
IRD	–	–	0.43281559 (0.01967307)	1092.548	2187.096	2189.891
GSIRD	0.14340833 (0.01303649)	–	–	709.1095	1420.219	1423.015
GIRD	0.13971018 (0.01355667)	–	1.09856413 (0.13590789)	708.7998	1421.60	1427.191
EIRD	**–**	**5.3382295**	**–**	**1092.548**	**2187.096**	**2189.891**

Table 3
MLEs of the Model Parameters using Generated Data of Size 70, the resulting SEs in braces and Criteria for Comparison with Generated Data Sets

Distribution	*α*	*γ*	*λ*	log-lik	AIC	BIC
EGIRD	0.36966081 (0.06666444)	8.11275329 (19.47762395)	5.05188129 (12.81115063)	272.1127	550.2254	556.9709
IRD	–	–	0.4710067 (0.0281476)	350.5502	703.1005	705.349
GSIRD	0.23606422 (0.02821456)	–	–	276.0082	554.0164	556.2649
GIRD	0.26138353 (0.03454498)	–	0.79551050 (0.09762153)	274.5823	553.1646	557.6616
EIRD	–	1.1971208 (29.5314667)	0.5153426 (6.3563552)	350.5502	705.1005	709.5975

Table 4
MLEs of the Model Parameters using Generated Data of Size 100, the resulting SEs in Braces and Criteria for Comparison with Generated Data Sets

Distribution	*α*	*γ*	*λ*	log-lik	AIC	BIC
EGIRD	0.3698557 (0.05535488)	6.7198127 (11.42607933)	3.3578727 (5.84148261)	403.9983	813.9966	821.8121
IRD	–	–	0.38716806 (0.01935798)	511.2278	1024.456	1027.061
GSIRD	0.22036591 (0.02203614)	–	–	413.4504	828.9008	831.506
GIRD	0.26680996 (0.02956033)	–	0.64746238 (0.06605219)	407.1506	818.3011	823.5115
EIRD	–	1.2301135 (23.1165323)	0.4294075 (4.0346583)	511.2278	1026.456	1031.666

7. RESULTS AND CONCLUSIONS

This research paper introduces the EGIR distribution as an extension of the GIR distribution. The new parameter (γ) provides more flexibility in modeling reliability data. Some of its properties are discussed illustrating the usefulness of the EGIR distribution to real data using MLE. From the results in Tables 2, 3 and 4 it can be seen that parameter log L, AIC and BIC statistics for the data set and simulated data consecutively. From the above results, it is evident that the EGIR distribution is the best distribution for fitting these data sets compared to other distributions (sub-models).

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