Pak. J. Statist.
2018 Vol. 34(5), 359-375

ESTIMATION OF PARAMETERS OF TWO PARAMETER EXPONENTIATED GAMMA DISTRIBUTION USING R SOFTWARE

Afaq Ahmad[1§], **S.P Ahmad**[2] and **A. Ahmed**[3]
[1] Department of Mathematical Sciences, Islamic University of Science & Technology, Awantipoora, Kashmir, India
[2] Department of Statistics, University of Kashmir, Srinagar, India
[3] Department of Statistics and Operations Research, Aligarh Muslim University, India
[§] Corresponding author Email: baderaafaq@gmail.com

ABSTRACT

In this paper, we study the Bayes estimators of the parameter of the exponentiated gamma distribution. The prior distribution here used is non informative extension of Jeffery's prior and informative Gamma prior. We have used three different loss functions viz SELF, Entropy and LINEX loss function. Finally Simulation Study is done and Posterior risks are obtained to compare the performance of the Bayes estimates. Results are also examined using real life data.

KEYWORDS

Exponentiated Gamma distribution, Priors, Squared Error loss function, Entropy loss function, LINEX loss function, R Software

1. INTRODUCTION

Gupta et al. (1998) proposed exponentiated gamma distribution as an alternative to gamma and Weibull distributions. This model is flexible enough to accommodate both monotonic as well as non-monotonic failure rates. Shawky and Bakoban (2008) discussed the exponentiated gamma distribution as an important model of life time models and derived estimators of the shape parameter, reliability and failure rate functions in the case of complete and type-II censored samples. The probability density function (p.d.f.) of the exponentiated gamma (EG) distribution is given below

$$f(x;\theta,\sigma)=\theta\sigma^2 x e^{-\sigma x}\left[1-e^{-\sigma x}(\sigma x+1)\right]^{\theta-1}, \qquad x;\theta,\sigma>0 \tag{1}$$

and the cumulative distribution function (CDF) of the distribution is:

$$F(x;\theta,\sigma)=\left[1-e^{-\sigma x}(\sigma x+1)\right]^{\theta} \tag{2}$$

The corresponding reliability function is given by

$$R(t)=1-\left[1-e^{-\sigma t}(\sigma t+1)\right]^{\theta} \tag{3}$$

and the hazard function is

$$h(t)=\frac{\theta\sigma^2 te^{-\sigma t}\left[1-e^{-\sigma t}(\sigma t+1)\right]^{\theta-1}}{1-\left[1-e^{-\sigma t}(\sigma t+1)\right]^{\theta}} \tag{4}$$

Here θ and σ are shape and scale parameters respectively. The two parameter exponentiated gamma distribution will be denoted by EG (θ , σ).

2. MATERIALS AND METHODS

The Bayesian paradigm comprises proper choice of prior(s) for the parameter(s). From the Bayesian perspective, there is no clear cut way from which one can accomplish that one prior is better than the other. However, very often priors are preferred according to one's subjective knowledge and beliefs. However, if one has sufficient information about the parameter(s), it is better to select informative prior(s); otherwise, it is preferable to use non-informative prior(s). Navid and Aslam (2012) derived the Bayesian estimator of the one parameter exponentiated gamma distribution using different prior under Entropy loss function and Quadratic loss function. Afaq et al. (2015) study the Bayesian estimation of shape parameter for Lomax distribution, Raqab and Madi (2009) study Bayesian analysis for exponentiated Rayleigh distribution, Mudasir et al. (2015) study parameter estimation for inverse Weibull distribution, Dar et al. (2017) studied the Bayesian estimation of Maxwell boltzman distribution. In this paper we consider both types of priors: the extended Jefferys' prior and the natural conjugate prior.

The extended Jefferys prior proposed by Al-Kutubi (2005) is given as

$$g_1(\theta)\propto\left[I(\theta)\right]^c \quad , \qquad c\,\varepsilon\, R^+$$

where $\left[I(\theta)\right]=-nE\left[\frac{\partial^2 \log f(x;\theta,\alpha)}{\partial\theta^2}\right]$ is the Fisher's information matrix. For the model (1),

$$g_1(\theta)=\frac{1}{\theta^{2c}} \tag{5}$$

The conjugate prior in this case will be the gamma prior, and the probability density function is taken as

$$g_2(\theta)=\frac{a^b}{\Gamma b}e^{-a\theta}\theta^{b-1} \ , \qquad a,b,\theta>0 \tag{6}$$

With the above priors, we use three different loss functions viz squared error loss function, LINEX loss function and Entropy loss function for the model (1).

3. MAXIMUM LIKELIHOOD ESTIMATION

Let us consider a random sample $\underline{x} = (x_1, x_2, ..., x_n)$ of size n from the exponentiated gamma distribution. Then the log-likelihood function for the given sample observation is

$$\ln L(\theta,\sigma) = n\ln\theta + 2n\ln\sigma + \sum_{i=1}^{n}\ln x_i - \sigma\sum_{i=1}^{n} x_i + (\theta-1)\sum_{i=1}^{n}\ln\left(1-e^{-\sigma x}(\sigma x+1)\right)$$

As the scale parameter σ is supposed to be known, the ML estimator of shape θ is obtained by differentiating above equation with respect to θ

$$\frac{\partial \ln L(\theta,\sigma)}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\ln\left(1-e^{-\sigma x_i}(\sigma x_i+1)\right) = 0$$

$$\hat{\theta} = \frac{-n}{\sum_{i=1}^{n}\ln\left(1-e^{-\sigma x_i}(\sigma x_i+1)\right)} \tag{7}$$

4. BAYESIAN ESTIMATION OF θ AND *R* UNDER THE POSTULATION OF EXTENDED JEFFERY'S PRIOR

4.1 Bayes Estimator of θ

Merging the prior distribution (5) with the likelihood function, the posterior density of θ is derived as follows:

$$\pi_1(\theta\,|\,\underline{x}) \propto \theta^n\sigma^{2n}\prod_{i=1}^{n}x_i\; e^{-\sigma\sum_{i=1}^{n}x_i}\;\prod_{i=1}^{n}\left(1-e^{-\sigma x_i}(\sigma x_i+1)\right)^{\theta-1}\frac{1}{\theta^{2c}}$$

$$\pi_1(\theta\,|\,\underline{x}) = S\theta^{n-2c}e^{\theta\sum_{i=1}^{n}\ln\left(1-e^{-\sigma x_i}(\sigma x_i+1)\right)}$$

$$\pi_1(\theta\,|\,\underline{x}) = S\theta^{n-2c}e^{-\theta T}$$

where

$$T = \sum_{i=1}^{n}\ln\left(1-e^{-\sigma x_i}(\sigma x_i+1)\right)^{-1}$$

and

$$S^{-1} = \int_0^{\infty}\theta^{n-2c}\,e^{-T\theta}\,d\theta$$

$$\Rightarrow S^{-1} = \frac{\Gamma(n-2c+1)}{T^{n-2c+1}}$$

Hence the posterior density of θ is given as

$$\pi_1(\theta\,|\,\underline{x}) = \frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\theta^{n-2c}\;e^{-\theta T} \tag{8}$$

which is density function of gamma distribution with scale parameter $T=\sum_{i=1}^{n}\ln\left(1-e^{-\sigma x_i}(\sigma x_i+1)\right)^{-1}$ and shape parameter $(n-2c+1)$.

4.1.1 Estimation under Squared Error Loss Function

The risk function under SELF $L(\hat{\theta},\theta)=c_1(\hat{\theta}-\theta)^2$ for some constant c_1 is given by

$$R(\hat{\theta},\theta)=\int_0^{\infty}c_1(\hat{\theta}-\theta)^2\,\frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\theta^{n-2c}\;e^{-\theta T}\,d\theta$$

$$=\frac{c_1T^{n-2c+1}}{\Gamma(n-2c+1)}\left[\hat{\theta}^2\int_0^{\infty}\theta^{n-2c}e^{-\theta T}d\theta-2\hat{\theta}\int_0^{\infty}\theta^{n-2c+1}e^{-\theta T}\,d\theta+\int_0^{\infty}\theta^{n-2c+2}e^{-\theta T}\,d\theta\right]$$

$$=\frac{c_1T^{n-2c+1}}{\Gamma(n-2c+1)}\left[\hat{\theta}^2\,\frac{\Gamma(n-2c+1)}{T^{n-2c+1}}-2\hat{\theta}\frac{\Gamma(n-2c+2)}{T^{n-2c+2}}+\frac{\Gamma(n-2c+3)}{T^{n-2c+3}}\right]$$

$$=c_1\hat{\theta}^2-2c_1\hat{\theta}\frac{(n-2c+1)}{T}+c_1\,\frac{(n-2c+2)(n-2c+1)}{T^2}$$

Now solving $\dfrac{\partial R(\hat{\theta},\theta)}{\partial\hat{\theta}}=0$, we obtain the Bayes estimator as

$$\hat{\theta}=\frac{(n-2c+1)}{T} \tag{9}$$

4.1.2 Estimation under LINEX Loss Function

The risk function under LINEX loss function $l(\theta,\hat{\theta})=\exp\left\{b_1\left(\hat{\theta}-\theta\right)\right\}-b_1\left(\hat{\theta}-\theta\right)-1$ for some constant b_1 is given by

$$R(\hat{\theta},\theta)=\int_0^{\infty}\left(\exp\left\{b_1\left(\hat{\theta}-\theta\right)\right\}-b_1\left(\hat{\theta}-\theta\right)-1\right)\frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\theta^{n-2c}\;e^{-\theta T}\,d\theta$$

$$=\frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\left[\begin{array}{l}e^{b_1\hat{\theta}}\int_0^{\infty}\theta^{n-2c}e^{-\theta(b_1+T)}d\theta-b_1\hat{\theta}\int_0^{\infty}\theta^{n-2c}e^{-\theta T}\,d\theta\\ \qquad\qquad+b_1\int_0^{\infty}\theta^{n-2c+1}e^{-\theta T}\,d\theta-\int_0^{\infty}\theta^{n-2c}e^{-\theta T}\,d\theta\end{array}\right]$$

$$=\frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\left[\begin{array}{l}e^{b_1\hat{\theta}}\,\dfrac{\Gamma(n-2c+1)}{(b_1+T)^{n-2c+1}}-b_1\hat{\theta}\dfrac{\Gamma(n-2c+1)}{T^{n-2c+1}}\\ \qquad\qquad+b_1\,\dfrac{\Gamma(n-2c+2)}{T^{n-2c+2}}-\dfrac{\Gamma(n-2c+1)}{T^{n-2c+1}}\end{array}\right]$$

$$= e^{b_1 \hat{\theta}} \left(\frac{T}{b_1 + T} \right)^{n-2c+1} - b_1 \hat{\theta} + b_1 \frac{(n-2c+1)}{T} - 1$$

Now solving $\frac{\partial R(\hat{\theta}, \theta)}{\partial \hat{\theta}} = 0$, we obtain the Bayes estimator as

$$\hat{\theta} = \frac{1}{b_1} \log \left(\frac{b_1 + T}{T} \right)^{n-2c+1} \tag{10}$$

4.1.3 Estimation under Entropy Loss Function

The risk function under entropy loss function $L(\delta) = b[\delta - \log\delta - 1]$ for some constant b is given by

$$R(\hat{\theta}, \theta) = \int_0^\infty b\left(\delta - \log(\delta) - 1\right) \pi_1(\theta \mid \underline{x})\, d\theta$$

$$= \int_0^\infty b \left(\frac{\hat{\theta}}{\theta} - \log\left(\frac{\hat{\theta}}{\theta}\right) - 1 \right) \frac{T^{n-2c+1}}{\Gamma(n-2c+1)} \theta^{n-2c}\, e^{-\theta T}\, d\theta$$

$$= b \frac{T^{n-2c+1}}{\Gamma(n-2c+1)} \int_0^\infty \left(\frac{\hat{\theta}}{\theta} - \log\left(\frac{\hat{\theta}}{\theta}\right) - 1 \right) \theta^{n-2c}\, e^{-\theta T}\, d\theta$$

$$= b \frac{T^{n-2c+1}}{\Gamma(n-2c+1)} \left[\hat{\theta} \int_0^\infty \theta^{n-2c-1} e^{-\theta T} d\theta - \log(\hat{\theta}) \int_0^\infty \theta^{n-2c} e^{-\theta T} d\theta + \int_0^\infty \log(\theta)\, \theta^{n-2c} e^{-\theta T}\, d\theta - \int_0^\infty \theta^{n-2c} e^{-\theta T}\, d\theta \right]$$

$$= b \frac{T^{n-2c+1}}{\Gamma(n-2c+1)} \left[\hat{\theta} \frac{\Gamma(n-2c)}{T^{n-2c}} - \log(\hat{\theta}) \frac{\Gamma(n-2c+1)}{T^{n-2c+1}} + \frac{\Gamma'(n-2c+1)}{T^{n-2c+1}} - \frac{\Gamma(n-2c+1)}{T^{n-2c+1}} \right]$$

$$= b \left[\frac{\hat{\theta} T}{n-2c} - \log(\hat{\theta}) + \frac{\Gamma'(n-2c+1)}{\Gamma(n-2c+1)} - 1 \right]$$

Now solving $\frac{\partial R(\hat{\theta}, \theta)}{\partial \hat{\theta}} = 0$, we obtain the Bayes estimator as

$$\hat{\theta} = \frac{(n-2c)}{T} \tag{11}$$

4.2 Bayes Estimator of Reliability Function *R*(*t*)

For given t, from the reliability function (3), we get,

$$\theta = \frac{\log(1-R)}{\log(1-e^{-\sigma x}(\sigma x+1))}$$

Using this transformation, the posterior pdf *R* given *x*, can be obtained from the posterior pdf (8). After simplification, it reduces to

$$\pi_2(R\,|\,\underline{x}) = \frac{T^{n-c+1}}{\Gamma(n-c+1)}\left[\frac{\log(1-R)}{\log(1-e^{-\sigma x}(\sigma x+1))}\right]^{n-c} e^{-\frac{\log(1-R)}{\log(1-e^{-\sigma x}(\sigma x+1))}T} \tag{12}$$

Now, the Bayes estimator of *R* under SELF relative to the posterior (8) is obtained as

$$\hat{R}_S = \int_0^\infty N_1\, R\; dR$$

where $N_1 = \frac{T^{n-c+1}}{\Gamma(n-c+1)}\left[\frac{\log(1-R)}{\log(1-e^{-\sigma x}(\sigma x+1))}\right]^{n-c} e^{-\frac{\log(1-R)}{\log(1-e^{-\sigma x}(\sigma x+1))}T}$ and the Bayes estimator of *R* under entropy loss function relative to the posterior (8) is obtained as

$$\hat{R}_E = \int_0^\infty \frac{N_1}{R}\; dR$$

where $\hat{R}_S$ = Reliability under SELF, $\hat{R}_E$ == Reliability under Entropy loss function.

5. BAYESIAN ESTIMATION OF θ AND *R* UNDER THE POSTULATION OF INFORMATIVE GAMMA PRIOR

5.1 Bayes Estimator of θ

Merging the prior distribution (6) with the likelihood function, the posterior density of θ is derived as follows:

$$\pi_3(\theta\,|\,\underline{x}) \propto \theta^n \sigma^{2n} \prod_{i=1}^n x_i\; e^{e^{-\sigma\sum_{i=1}^n x_i}} \prod_{i=1}^n \left[1-e^{-\sigma x_i}(\sigma x_i+1)\right]^{\theta-1} \frac{a^b}{\Gamma b} e^{-a\theta}\, \theta^{b-1}$$

$$\pi_3(\theta\,|\,\underline{x}) = S\theta^{n+b-1}\, e^{\theta\sum_{i=1}^n \ln\left[1-e^{-\sigma x_i}(\sigma x_i+1)\right]}\, e^{-a\theta}$$

$$\pi_3(\theta\,|\,\underline{x}) = S\theta^{n+b-1}\, e^{-\theta T}\, e^{-a\theta}$$

$$\pi_3(\theta\,|\,\underline{x}) = S\theta^{n+b-1}\, e^{-(a+T)\theta}$$

where

$$S^{-1} = \int_0^\infty \theta^{n+b-1}\, e^{-(a+T)\theta}\, d\theta$$

$$\Rightarrow S^{-1} = \frac{\Gamma(n+b)}{(a+T)^{n+b}}$$

Hence the posterior density of θ is given as

$$\pi_3(\theta\,|\,\underline{x}) = \frac{(a+T)^{n+b}}{\Gamma(n+b)}\theta^{n+b-1}\ e^{-(a+T)\theta} \tag{13}$$

which is density function of gamma distribution with scale parameter $(a+T)$, $T=\sum_{i=1}^{n}\ln\left(1-e^{-\sigma x_i}(\sigma x_i+1)\right)^{-1}$ and shape parameter (n+b).

5.1.1 Estimation under Squared Error Loss Function

The risk function under SELF $L(\hat{\theta},\theta)=c_1(\hat{\theta}-\theta)^2$ for some constant c_1 is given by

$$\begin{aligned}
R(\hat{\theta},\theta) &= \int_0^\infty c_1(\hat{\theta}-\theta)^2\frac{(a+T)^{n+b}}{\Gamma(n+b)}\theta^{n+b-1}\ e^{-(a+T)\theta}\quad d\theta \\
&= \frac{c_1\,(a+T)^{n+b}}{\Gamma(n+b)}\left[\hat{\theta}^2\int_0^\infty \theta^{n+b-1}e^{-(a+T)\theta}d\theta - 2\hat{\theta}\int_0^\infty \theta^{n+b}e^{-(a+T)\theta}\,d\theta + \int_0^\infty \theta^{n-c+1}e^{-(a+T)\theta}\,d\theta\right] \\
&= \frac{c_1\,(a+T)^{n+b}}{\Gamma(n+b)}\left[\hat{\theta}^2\frac{\Gamma(n+b)}{(a+T)^{n+b}} - 2\hat{\theta}\frac{\Gamma(n+b+1)}{(a+T)^{n+b+1}} + \frac{\Gamma(n+b+2)}{(a+T)^{n+b+2}}\right] \\
&= c_1\hat{\theta}^2 - 2c_1\hat{\theta}\frac{(n+b)}{(a+T)} + c_1\frac{(n+b+1)(n+b)}{(a+T)^2}
\end{aligned}$$

Now solving $\dfrac{\partial R(\hat{\theta},\theta)}{\partial\theta}=0$, we obtain the Bayes estimator as

$$\hat{\theta} = \frac{(n+b)}{(a+T)} \tag{14}$$

5.1.2 Estimation under LINEX Loss Function

The risk function under LINEX loss function $l(\theta,\hat{\theta})=\exp\left\{b_1\left(\hat{\theta}-\theta\right)\right\}-b_1\left(\hat{\theta}-\theta\right)-1$ for some constant b_1 is given by

$$\begin{aligned}
R(\hat{\theta},\theta) &= \int_0^\infty\left(\exp\left\{b_1\left(\hat{\theta}-\theta\right)\right\}-b_1\left(\hat{\theta}-\theta\right)-1\right)\frac{(a+T)^{n+b}}{\Gamma(n+b)}\theta^{n+b-1}\ e^{-\theta(a+T)}\,d\theta \\
&= \frac{(a+T)^{n+b}}{\Gamma(n+b)}\left[e^{b_1\hat{\theta}}\int_0^\infty \theta^{n+b-1}e^{-\theta(b_1+a+T)}d\theta - b_1\hat{\theta}\int_0^\infty \theta^{n+b-1}e^{-\theta(a+T)}\,d\theta + b_1\int_0^\infty \theta^{n+b}e^{-\theta(a+T)}\,d\theta - \int_0^\infty \theta^{n+b-1}e^{-\theta(aT)}\,d\theta\right]
\end{aligned}$$

$$= \frac{(a+T)^{n+b}}{\Gamma(n+b)}\left[e^{b_1\hat{\theta}} \frac{\Gamma(n+b)}{(b_1+a+T)^{n+b}} - b_1\hat{\theta}\frac{\Gamma(n+b)}{(a+T)^{n+b}} + b_1\frac{\Gamma(n+b+1)}{(a+T)^{n+b+1}} - \frac{\Gamma(n+b)}{(a+T)^{n+b}} \right]$$

$$= e^{b_1\hat{\theta}}\left(\frac{a+T}{b_1+a+T}\right)^{n+b} - b_1\hat{\theta} + b_1\frac{(n+b)}{(a+T)} - 1$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial \hat{\theta}} = 0$, we obtain the Bayes estimator as

$$\hat{\theta} = \frac{1}{b_1}\log\left(\frac{b_1+a+T}{a+T}\right)^{n+b} \tag{15}$$

5.1.3 Estimation under Entropy Loss Function

The risk function under entropy loss function $L(\delta)=b[\delta-\log\delta-1]$ for some constant b is given by

$$R(\hat{\theta},\theta) = \int_0^\infty b\left(\delta - \log(\delta) - 1\right)\pi_2(\theta\,|\,\underline{x})\,d\theta$$

$$= \int_0^\infty b\left(\frac{\hat{\theta}}{\theta} - \log\left(\frac{\hat{\theta}}{\theta}\right) - 1\right)\frac{(a+T)^{n+b}}{\Gamma(n+b)}\theta^{n+b-1}\, e^{-(a+T)\theta}\, d\theta$$

$$= b\frac{(a+T)^{n+b}}{\Gamma(n+b)}\int_0^\infty\left(\frac{\hat{\theta}}{\theta} - \log\left(\frac{\hat{\theta}}{\theta}\right) - 1\right)\theta^{n+b-1}\, e^{-(a+T)\theta}\, d\theta$$

$$= b\frac{(a+T)^{n+b}}{\Gamma(n+b)}\left[\hat{\theta}\int_0^\infty \theta^{n+b-2}\, e^{-(a+T)\theta}\,d\theta - \log(\hat{\theta})\int_0^\infty \theta^{n+b-1}\, e^{-(a+T)\theta}\,d\theta + \int_0^\infty \log(\theta)\,\theta^{n+b-1}\, e^{-(a+T)\theta}\, d\theta - \int_0^\infty \theta^{n+b-1} e^{-(a+T)\theta}\, d\theta\right]$$

$$= b\frac{(a+T)^{n+b}}{\Gamma(n+b)}\left[\hat{\theta}\frac{\Gamma(n+b-1)}{(a+T)^{n+b-1}} - \log(\hat{\theta})\frac{\Gamma(n+b)}{(a+T)^{n+b}} + \frac{\Gamma'(n+b)}{(a+T)^{n+b}} - \frac{\Gamma(n+b)}{(a+T)^{n+b}}\right]$$

$$= b\left[\frac{\hat{\theta}(a+T)}{n+b-1} - \log(\hat{\theta}) + \frac{\Gamma'(n+b)}{\Gamma(n+b)} - 1\right]$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial \hat{\theta}} = 0$, we obtain the Bayes estimator as

$$\hat{\theta} = \frac{(n+b-1)}{(a+T)} \tag{16}$$

5.2 Bayes Estimator of Reliability Function *R*(*t*) using Gamma Prior

For given t, from (3), we get,

$$\theta = \frac{\log(1-R)}{\log(1-e^{-\alpha x}(\alpha x+1))}$$

Using this transformation, the posterior pdf R given x, can be obtained from the posterior pdf (13). After simplification, it reduces to

$$\pi_4(R\,|\,\underline{x}) = \frac{(a+T)^{n+b}}{\Gamma(n+b)}\left[\frac{\log(1-R)}{\log(1-e^{-\sigma x}(\sigma x+1))}\right]^{n+b-1} e^{-\frac{\log(1-R)}{\log(1-e^{-\sigma x}(\sigma x+1))}(a+T)} \tag{17}$$

Now, the Bayes estimator of R under SELF relative to the posterior (13) is obtained as

$$\hat{R}_S = \int_0^{\infty} N_2\, R\, dR$$

where $N_2 = \frac{(a+T)^{n+b}}{\Gamma(n+b)}\left[\frac{\log(1-R)}{\log(1-e^{-\sigma x}(\sigma x+1))}\right]^{n+b-1} e^{-\frac{\log(1-R)}{\log(1-e^{-\sigma x}(\sigma x+1))}(a+T)}$ and the Bayes estimator of R under entropy loss function relative to the posterior (13) is obtained as

$$\hat{R}_E = \left[\int_0^{\infty} \frac{N_2}{R}\right]^{-1}.$$

6. SIMULATION STUDY

In simulation study we have generated a sample of sizes n=25, 50 and 100 to represent the influence of small, medium, and large samples on the estimators. The outcomes are simulated 10000 times and the results has been presented in the tables. To observe the performance of Bayes estimates are presented along with posterior risks given in parenthesis in the below tables.

Table 6.1
Bayes Estimates and Posterior Risks (in Parenthesis)
using Extended Jeffery's Prior

n	σ	$\hat{\theta}_S$	$\hat{\theta}_L$		$\hat{\theta}_E$
			b_1=0.01	b_1=0.25	
25	0.5	1.0893 (0.0359)	1.0223 (**0.0311**)	1.0359 (0.0316)	1.1255 (0.0367)
	1.0	1.7191 (0.0728)	1.6303 **(0.0648)**	1.6524 (0.0678)	1.6671 (0.0724)
	2.0	2.1586 (0.1276)	2.0404 (**0.1206)**	2.0699 (0.1224)	2.7250 (0.1290)
50	1.0	1.2079 (0.0186)	1.1663 **(0.0173)**	1.1742 (0.0175)	1.2314 (0.0185)
	1.5	1.8242 (0.0454)	1.7543 **(0.0422)**	1.7676 (0.0430)	1.8270 (0.0442)
	2.0	2.4189 (0.0790)	2.3536 **(0.0752**)	2.3639 (0.0764)	2.4170 (0.0758)
100	1.0	1.1201 (0.0093)	1.0945 **(0.0083)**	1.0963 (0.0083)	1.1292 (0.0092)
	1.5	1.6750 (0.0176)	1.6236 **(0.0122)**	1.6272 (0.0124)	1.6762 (0.0173)
	2.0	2.2201 (0.0321)	2.1802 **(0.0315)**	2.1879 (0.0320)	2.1941 (0.0335)

$\hat{\theta}_S$ = Estimate under SELF, $\hat{\theta}_E$ = Estimate under Entropy, $\hat{\theta}_L$ = Estimate under LINEX

Table 6.2
Bayes Estimates and Posterior Risks (in Parenthesis) using Gamma Prior

n	σ	$a=b$	$\hat{\theta}_S$	$\hat{\theta}_L$ b_1=0.01	$\hat{\theta}_L$ b_1=0.25	$\hat{\theta}_E$
25	0.5	0.5	0.4414 (0.0887)	0.4506 **(0.0102)**	0.4401 (0.0113)	0.4278 (0.0130)
		1.0	0.4414 (0.0887)	0.4601 **(0.0094)**	0.4497 (0.0103)	0.4377 (0.0117)
	1.0	0.5	0.8829 (0.4342)	0.8950 **(0.0418)**	0.8741 (0.0466)	0.8497 (0.0533)
		1.0	0.8829 (0.4342)	0.8985 **(0.0401)**	0.8783 (0.0446)	0.8548 (0.0509)
	1.5	0.5	1.3244 (1.1814)	1.3332 **(0.0961)**	1.3021 (0.1075)	1.2658 (0.1231)
		1.0	1.3244 (1.1814)	1.3168 **(0.0977)**	1.2872 (0.1094)	1.2527 (0.1253)
50	0.5	0.5	0.4886 (0.0162)	0.4935 **(0.0048)**	0.4877 (0.0049)	0.4809 (0.0051)
		1.0	0.4886 (0.0162)	0.4984 **(0.0047)**	0.4926 (0.0048)	0.4859 (0.0049)
	1.0	0.5	0.9773 (0.0660)	0.9833 **(0.0192)**	0.9717 (0.0197)	0.9581 (0.0207)
		1.0	0.9773 (0.0660)	0.9836 **(0.0188)**	0.9722 (0.0193)	0.9589 (0.0202)
	1.5	0.5	1.4660 (0.1519)	1.4693 **(0.0432)**	1.4519 (0.0446)	1.4316 (0.0470)
		1.0	1.4660 (0.1519)	1.4560 **(0.0426)**	1.4391 (0.0444)	1.4194 (0.0472)
100	0.5	0.5	0.5175 (0.0083)	0.5134 **(0.0028)**	0.5170 (0.0029)	0.5201 (0.0030)
		1.0	0.5175 (0.0083)	0.5159 **(0.0029)**	0.5195 (0.0030)	0.5225 (0.0031)
	1.0	0.5	1.0351 (0.0329)	1.0246 **(0.0112)**	1.0319 (0.0116)	1.0380 (0.0121)
		1.0	1.0351 (0.0329)	1.0244 **(0.0111)**	1.0315 (0.0115)	1.0377 (0.0119)
	1.5	0.5	1.5527 (0.0730)	1.5338 **(0.0250)**	1.5446 (0.0259)	1.5539 (0.0268)
		1.0	1.5527 (0.0730)	1.5257 **(0.0240)**	1.5364 (0.0247)	1.5455 (0.0254)

7. NUMERICAL EXAMPLE

In this section we provide a data analysis for a simple un- censored data set to see the performance of estimates. The data have been obtained from Nicholas and Padgett (2006), the data concerning tensile strength of 100 observations of carbon fibers and they are:

3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

By using different Loss functions i.e. Square Error loss function, LINEX loss function and Entropy loss function the Bayes estimates and Posterior Risks of the posterior distribution through non-informative prior i.e. extension of Jeffery's prior and informative gamma prior are as follow where posterior risk are in parentheses.

Table 7.1
Bayes Estimates and Posterior Risks using Jeffery's Prior

σ	C	$\hat{\theta}_S$	$\hat{\theta}_L$		$\hat{\theta}_E$
			b_1=0.01	b_1=0.25	
1.0	0.5	2.2327 (0.0498)	2.2324 (0.0223)	2.2265 (0.5581)	2.2104 (3.8069)
	1.0	2.2104 (0.0493)	2.2101 (0.0221)	2.2042 (0.5526)	2.1880 (3.8070)
	15	2.1880 (0.0488)	2.1878 (0.0218)	2.1820 (0.5470)	2.1657 (3.8070)
2.0	0.5	8.9827 (0.8068)	8.9787 (0.0898)	8.8833 (2.2456)	8.8929 (2.4149)
	1.0	8.9829 (0.7988)	8.8889 (0.0889)	8.7945 (2.2232)	8.8031 (2.4149)
	1.5	8.8031 (0.7907)	8.7991 (0.0880)	8.7057 (2.2007)	8.7132 (2.4150)
3.0	0.5	25.1416 (6.3210)	25.1101 (0.2514)	24.3831 (6.2854)	24.8902 (1.3856)
	1.0	24.8902 (6.2578)	24.8590 (0.2489)	24.1393 (6.2225)	24.6388 (1.3857)
	1.5	24.6388 (6.1946)	24.6079 (0.2463)	23.8954 (6.1597)	24.3874 (1.3857)

$\hat{\theta}_S$ = Estimate under SELF, $\hat{\theta}_E$ = Estimate under Entropy, $\hat{\theta}_L$ = Estimate under LINEX

Table 7.2
Bayes Estimates and Posterior Risks using Gamma Prior

α	a	b	$\hat{\theta}_S$	$\hat{\theta}_L$		$\hat{\theta}_E$
				b_1=0.01	b_1=0.25	
1.0	0.5	0.5	2.2191 (0.0490)	2.2188 (0.0221)	2.2130 (0.5547)	2.1970 (3.8180)
	0.5	1.0	2.2301 (0.0492)	2.2299 (0.0223)	2.2240 (0.5575)	2.2080 (3.8180)
	1.0	0.5	2.1948 (0.0479)	2.1946 (0.0219)	2.1889 (0.5487)	2.1730 (3.8290)
	1.0	1.0	2.2058 (0.0481)	2.2055 (0.0220)	2.1998 (0.5514)	2.1839 (3.8290)
2.0	0.5	0.5	8.6396 (0.7427)	8.6359 (0.0863)	8.5481 (2.1599)	8.5536 (2.4588)
	0.5	1.0	8.6826 (0.7464)	8.6788 (0.0868)	8.5906 (2.1706)	8.5966 (2.4587)
	1.0	0.5	8.2835 (0.6827)	8.2801 (0.0828)	8.1993 (2.0708)	8.2011 (2.5008)
	1.0	1.0	8.3247 (0.6861)	8.3213 (0.0832)	8.2401 (2.0811)	8.2423 (2.5008)
3.0	0.5	0.5	22.4457 (5.0130)	22.4207 (0.2244)	21.8415 (5.6114)	22.2224 (1.5040)
	0.5	1.0	22.5574 (5.0379)	22.5322 (0.2255)	21.9501 (5.6393)	22.3340 (1.5040)
	1.0	0.5	20.1910 (4.0564)	20.1707 (0.2019)	19.7003 (5.0477)	19.9901 (1.6099)
	1.0	1.0	20.2914 (4.0766)	20.2711 (0.2029)	19.7983 (5.0728)	20.0905 (1.6099)

It is seen from Tables 6.1, 6.2, 7.1 and 7.2, on comparing the Bayes posterior risk under different loss functions, it is perceived that the LINEX loss function has less Bayes posterior risk in both non informative and informative priors than other loss functions. According to the decision rule of less Bayes posterior risk we conclude that LINEX loss function is more preferable loss function. The risk becomes negative if we take $\theta < 1$ so we take $\theta \geq 1$. Also the risk of LINEX loss function becomes negative for negative values of loss parameter b_1 and for higher positive values the risk becomes higher. Hence it is advisable to take smaller positive values of b_1. Furthermore within each loss function informative prior (Gamma) provides less Bayes posterior risk than Jeffery's prior so it is more suitable for the exponentiated gamma distribution.

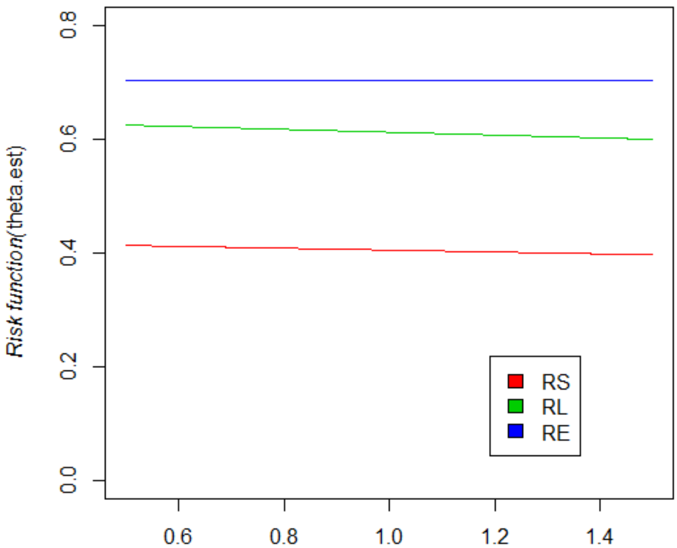

Figure 1: Posterior Risks under ext. Jeffery's Prior for Different Values of c1 with Fixed n

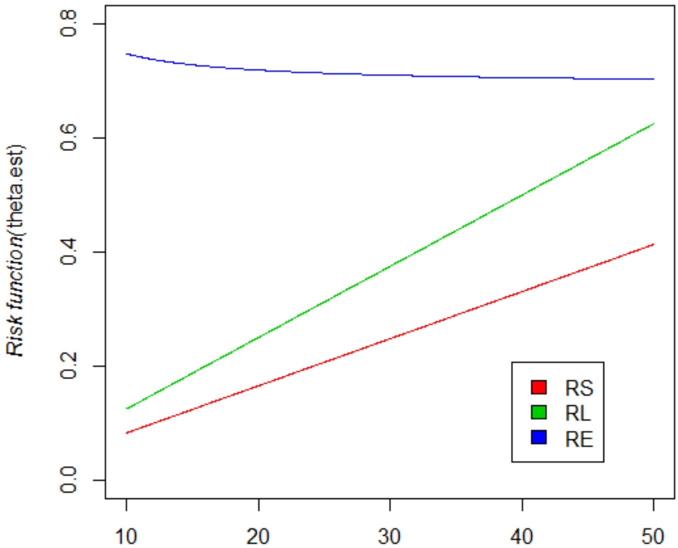

Figure 2: Posterior Risks under ext. Jeffery's Prior for Different Values of n with Fixed c1

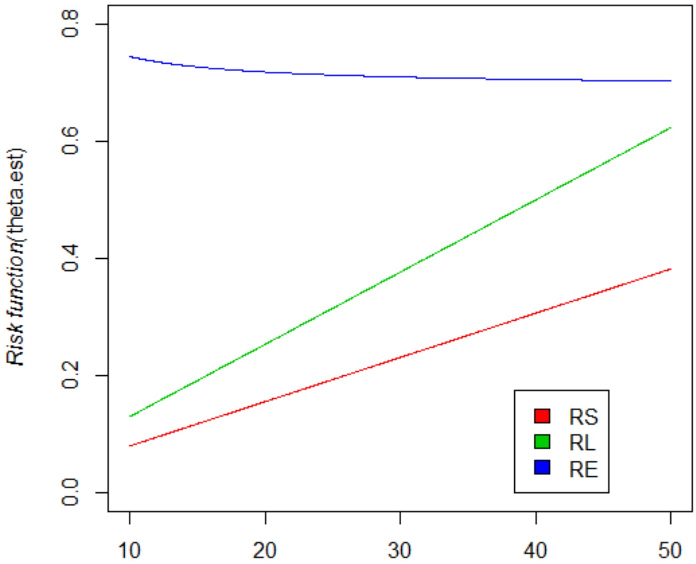

Figure 3: Posterior Risks under Gamma Prior for Different Values of n with Fixed a=b=0.5

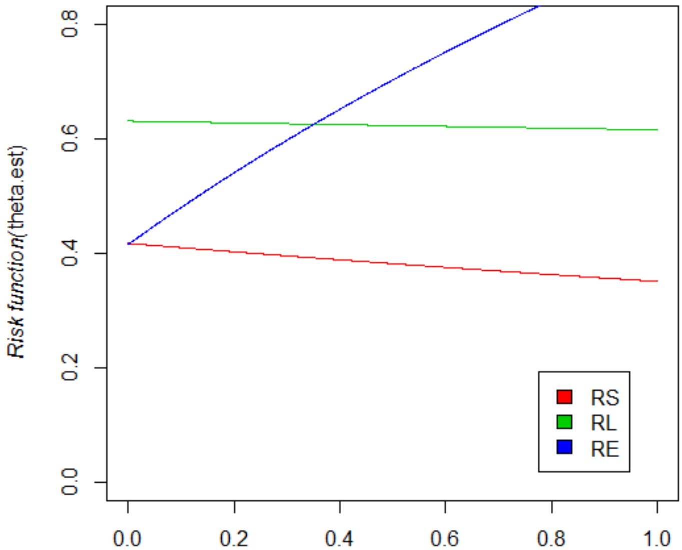

Figure 4: Posterior Risks under Gamma Prior for Different Values of a with fixed a=50, b=0.5

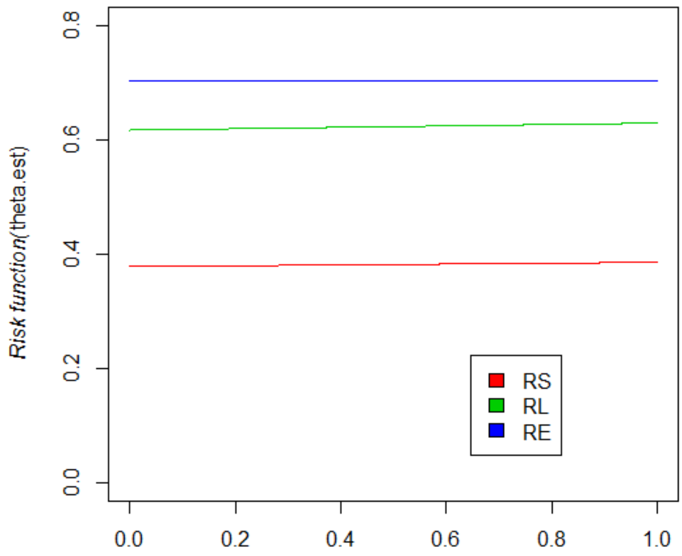

Figure 5: Posterior Risks under Gamma Prior for Different Values of b with Fixed n=50, a=0.5

8. CONCLUSION

We contemplate the Bayesian analysis of the exponentiated gamma distribution using informative Gamma prior and non-informative extension of Jeffery's prior. After exploration we conclude that the Gamma prior is well-suited for the unknown parameter of the exponentiated gamma distribution and preferable over all other competitive priors because of having less posterior risk. As far as choice of loss function is concerned, one can easily perceive based on proof of different properties as discussed above that LINEX loss function has smaller Posterior risk.

ACKNOWLEDGEMENT

The authors are vastly gratified to the editor and reviewers who gave cherished suggestions for the improvement of this paper.

REFERENCES

1. Ahmad, A., Ahmad, S.P. and Ahmed, A. (2015). Bayesian Analysis of Shape Parameter of Lomax Distribution Using Different Loss Functions. *International Journal of Statistics and Mathematics*, 2(1), 055-065.
2. Ahmed, A., Dar, A.H. and Reshi, J.A. (2017). Bayesian analysis of maxwell-boltzmann distribution under different loss functions and prior distributions. *Pakistan Journal of Statistics*, 33(6), 419-440.
3. Al-Kutubi, H.S. (2005). On Comparison Estimation Procedures for Parameter and Survival function. *Iraqi Journal of Statistical Science*, 9, 1-14.
4. Calabria, R. and Pulcini, G. (1994). An engineering approach to Bayes estimation for the Weibull distribution. *Microelectronics Reliability*, 34(5), 789-802.

5. Dey, D.K. and Liu, Pie-San L. (1992). On Comparison of Estimators in a Generalized Life Model. *Microelectronics Reliability*, 45(3), 207-221.
6. Dey, D.K., Ghosh, M. and Srinivasan, C. (1987). Simultaneous Estimation of Parameters under Entropy loss. *Journal of Statistical Planning and Interference*, 15, 347-363.
7. Gupta, R.C., Gupta, R.D. and Gupta, P.L. (1998). Modeling Failure Time Data by Lehman Alternatives. *Communication in Statistics-Theory and Methods*, 27(4), 887-904.
8. Jefferys, H. (1964). An invariant form for the Prior Probability in estimation problems. *Proceedings of the Royal Society of London, Series. A*., 186(1007), 453-461.
9. Klebanov, L.B. (1972). Universal Loss Function and Unbiased Estimation. *Dokl. Akad. Nank SSSR Soviet-Maths. Dokl. T*., 203, 1249-1251.
10. Sofi, M., Ahmad, A., Ahmad, S.P. and Ahmed, A. (2015). A Note on Bayesian estimation of Inverse Weibull Distribution under LINEX and Quadratic Loss Functions. *International Journal of Statistics and Mathematics*. 13(2), 170-177.
11. Navid, F. and Aslam, M. (2012). Bayesian Analysis of Exponentiated Gamma Distribution under Type II Censored Samples. *International Journal of Advanced Science and Technology*, 49, 37-46.
12. Nicholas, M.D. and Padgett, W.J. (2006). A bootstrap control chart for Weibull Percentiles. *Quality and Reliability Engineering International*, 22, 141-151.
13. Raqab, M.Z. and Madi, M.T. (2009). Bayesian Analysis for the Exponentiated Rayleigh Distribution. *METRON-International Journal of Statistics*, 67(3), 269-288.
14. Shawky, A.I. and Bakoban, R.A. (2008). Bayesian and Non-Bayesian Estimations on the Exponentiated Gamma Distribution. *Applied Mathematical Sciences*, 2(51), 2521-2530.
15. Varian, H.R. (1975). A Bayesian approach to real state assessment, in *Studies in Bayesian Econometrics and Statistics* in honour of L.J. Savage, Eds. Feinberg, S.E. and Zellner, A. (North Holland, Amsterdam), 195-208.