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ON DEVELOPMENT OF FOUR-PARAMETERS EXPONENTIATED GENERALIZED EXPONENTIAL DISTRIBUTION

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ABSTRACT

In this paper, a four parameter Exponentiated Generalized Exponentiated exponential distribution is derived from Exponentiated Generalized Family (EGF) of distribution. Some properties of the distribution are studied. The distribution is found to be unimodal and has a decreasing and increasing hazard rate depending on the shape parameters. The expressions for the moment, median, quantile, mean deviation, median deviation, skewness, kurtosis, Renyi entropy are obtained. Some known continuous distributions are special cases of the new derived distributions. Simulation study, maximum likelihood estimator and real life application of the model to data, shows that new distribution fits better than it's sub-models.

KEY WORDS

Moment, Hazard rate, Kurtosis, Renyi Entropy, Unimodal, skewness, quantile function.

1. INTRODUCTION

The exponential distribution (ED) also known as negative exponential distribution is a probability distribution that describes the time between event in a Poisson point process i.e. a process in which event occurs continuously and independently at a constant average rate. The ED is a very popular statistical model probably, is one of the parametric model most extensively used in several fields; Lemonte et al. (2013). The popularity of this distribution can be explained perhaps, by the simplicity of their cumulative function, which involves only one unknown parameter $\lambda > 0$ and takes a simple form $G(x) = 1 - e^{-\lambda x}$ for $x > 0$ in addition to having constant hazard rate.

Gompertz (1825) and Verhulst [(1838), (1845) and (1847)] developed several cumulative distribution functions during the first half of the nineteenth century to compare known human mortality tables and represent mortality growth. One of them is as follows

$$G(t) = (1 - \rho e^{-\lambda t})^{\alpha} \tag{1}$$

for $t > {}^{1}/_{\lambda}\, In\rho$. Where ρ, λ and α are all positive real numbers. In twentieth century, Ahuja and Nash (1967) also considered this model and made some further generalization.

The generalized exponential distribution or the exponentiated exponential distribution is defined as a particular case of the Gompertz (1825), Verhulst Verhulst [(1838), (1845) and (1847)] distribution function, when ρ = 1. Therefore, X is a two parameters generalized exponential random variable if it has the distribution function

$$G(x\colon\alpha,\lambda) = (1 - e^{-\lambda x})^{\alpha} \quad (2)$$

and the density function.

$$g(x\colon\alpha,\lambda) = \alpha\lambda(1 - e^{-\lambda x})^{\alpha-1}e^{-xt} \quad (3)$$

where α and λ play the role of the shape and scale parameters respectively. Many exponentiated families of distributions have appeared in the literature as generalizations of existing distributions. Mudholkar and Srivastava (1993) extended the Weibull distribution by introducing the 3-parameter exponentiated Weibull distribution (EWD) that has bathtub or monotone failure rate. Gupta et al. (1998) studied the general properties of the exponentiated families of distributions such as hazard function and some ordering relations. Gupta and Kundu (1999) defined a 2-parameter generalized exponential distribution, a particular case of EWD, and studied some of its properties, including hazard rate, moment generating function, distribution of sums and extreme values. They also compared the flexibility of the generalized exponential distribution to a 2-parameter gamma distribution and a 2 parameter Weibull distribution by studying the deep groove ball bearings lifetime data. They concluded that the generalized exponential distribution can be used as alternative to the 2 parameter Weibull distribution and the 2-parameter gamma distribution.

Cadeiro et al. (2013) proposed a class of distributions by adding two parameters to a continuous distribution, by extending the idea first introduced by Lehman (1953) and studied by Nadarah and Kotz (2009). This method leads to a new class of Exponentiated generalized distribution (EG) that can be interpreted as a double construction of Lehmann alternative. The distributions extend the exponentiated type distribution and obtain some of its structure properties. Given a continuous c.d.f. G(x), we define the EG class of distributions by

$$F(x) = [1 - \{1 - G(x)\}^{\alpha}]^{\beta} \quad (4)$$

and

$$f(x) = \alpha\beta\{1 - G(x)\}^{\alpha-1}[1 - \{1 - G(x)\}^{\alpha}]^{\beta-1}g(x) \quad (5)$$

where $\alpha > 0\, and\, \beta > 0$ are two additional shape parameters. The EG has tractable properties especially for simulation since its quantile function take a simple form.

$$x = Q_G\left(\left[1 - \left(1 - u^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right]\right) \quad (6)$$

where $Q_G(u)$ is the baseline quantile function.

To illustrate the flexibility of EG model, Cordero et al. (2013) applied EG to some well-known distribution such as the Frechet, normal, gamma and Gumbel distributions, with several properties for the EG class, which provide motivations to adopt this generator. The two extra parameters $\alpha\, and\, \beta$ in the density can control both tail weight, and allow generation of flexible distribution, with heavier or lighter tails, as appropriate. There is also an attractive physical interpretation of the EG model when $\alpha\, and\, \beta$ are positive integers see Cordeiro and Lemonte (2014). The EG family properties have been explored in recent works. Here, we refer to the papers: Cordeiro et al. (2014), Cordeiro and Lemonte (2014), Elbatal and Muhammed (2014), Oguntunde et al. (2014), da Silva et al. (2015), de Andrade et al. [(2016) and (2015)] Cordeiro et al. (2017), which used the EG class to extend the Burr III, Birnbaum-Saundersm, inverse Weibull, inverted exponential, generalized gamma, Gumbel, extended exponential, standardized half-logistic distributions respectively.

Numerous generalized classes of distributions have been developed and applied to explain diverse phenomena. A common feature of these generalized distributions is that they have more parameters. The interests in developing more flexible statistical distribution have remained strong in statistics profession; Alzaatreh et al. (2013). Johnson et al. (1994) stated that the use of four-parameter distributions should be sufficient for most practical purposes. According to these authors, at least three parameters are needed but they doubted any noticeable improvement arising from including a fifth or sixth parameter. We belief that additional two parameters to an existing exponentiated exponential distribution which serve as alternative distribution to weibull and gamma distribution may generate new distribution with tractable properties. This paper presents yet another four parameters statistical distribution to fit positively skewed distribution.

The rest of the paper is organized as follows. In Section 2 we define the Exponentiated Generalized Exponentiated Exponential (EGEE) distribution and outline some special cases of the distribution, the graphs of probability density function (pdf), cumulative distribution function (cdf) and hazard functions of proposed distribution and its sub-distributions are obtained. In section 3, some mathematical properties and limit behavior are derived, in section 4, estimation of the unknown parameters by method of maximum likelihood and information's criterion, in section 5, we provide some simulated result base on the mathematical properties and it real life application. We conclude in section 6 base on some significant result on the EGEE distribution.

2. EXPONENTIATED GENERALIZED EXPONENTIATED EXPONENTIAL (EGEE)

In this section we define and formulate the proposed model.

2.1 Proposed Distribution (EGEE)

We defined the Exponentiated Generalized Exponentiated Exponential (EGEE) cumulative distribution from (4) as;

$$F(x) = [1 - \{1 - (1 - e^{-\frac{x}{\theta}})^k\}^\alpha]^\beta \tag{7}$$

By inserting (2) in (4) the corresponding p.d.f (5) is

$$f(x) = \frac{\alpha\beta k}{\theta} e^{-\frac{x}{\theta}}(1 - e^{-\frac{x}{\theta}})^{k-1}\{1 - (1 - e^{-\frac{x}{\theta}})^k\}^{\alpha-1} [1 - \{1 - (1 - e^{-\frac{x}{\theta}})^k\}^\alpha]^{\beta-1} \tag{8}$$

The hazard function is;

$$h(x) = \frac{\frac{\alpha\beta k}{\theta} e^{-\frac{x}{\theta}}(1-e^{-\frac{x}{\theta}})^{k-1}\{1-(1-e^{-\frac{x}{\theta}})^{k}\}^{\alpha-1}[1-\{1-(1-e^{-x/\theta})^{k}\}^{\alpha}]^{\beta-1}}{1-[1-\{1-(1-e^{-x/\theta})^{k}\}^{\alpha}]^{\beta}} \quad (9)$$

The survival function is;

$$s(x) = 1-[1-\{1-(1-e^{-\frac{x}{\theta}})^{k}\}^{\alpha}]^{\beta} \quad (10)$$

2.2 Special Case of EGEE Distribution

- The Exponential distribution (E) with scale parameter θ is a special case of EGEE when $\alpha = \beta = k = 1$.
- For $\alpha = \beta = 1$ the EGEE gives an Exponentiated Exponential (EE) distribution
- When $k = 1$ the EGEE gives a member of Exponentiated Generalized Family which is Exponentiated Generalized Exponential (EGE) distribution.

The probability distribution plot of EGEE distribution along with its special model

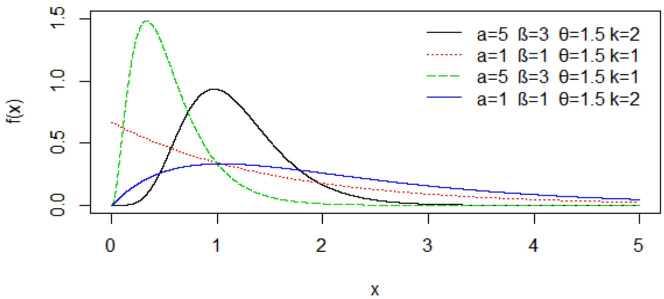

Figure 1: Probability of Density Function of EGEE and Its Sub-Models

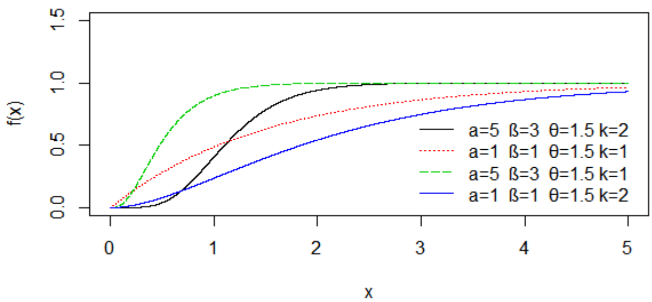

Figure 2: Distribution Function of EGEE and its Sub-Models

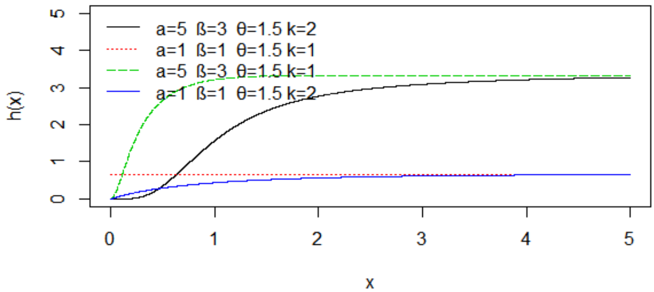

Figure 3: Hazard Function of EGEE and its Sub-Models

Table 1
Summary of EGEE and Sub-Models

Distribution	$\boldsymbol{\beta}$	$\boldsymbol{\alpha}$	$\boldsymbol{k}$	$\boldsymbol{\theta}$
EGEE	β	α	k	θ
EGE	β	α	1	θ
EE	1	1	k	θ
E	1	1	1	θ

3. MATHEMATICAL PROPERTIES OF PROPOSED DISTRIBUTION

We look at the some properties of EGEE model in this section.

3.1 Properties of Exponentiated Generalized Exponentiated Exponential (EGEE)

The properties of the proposed distribution will be derived from Exponentiated Generalized Family (EGF) distribution in equation 1 and 2.

$$F(x) = [1 - \{1 - G(x)\}^{\alpha}]^{\beta}$$

$$\boldsymbol{\alpha} > 0, \beta > 0, x\epsilon\mathbb{R}$$

$$f(x) = \alpha\beta\{1 - G(x)\}^{\alpha-1}[1 - \{1 - G(x)\}^{\alpha}]^{\beta-1}g(x)$$

Expansion for the density function

$$(1 - z)^{\beta-1} = \sum_{\lambda=0}^{\infty} \frac{(-1)^{\lambda}\Gamma(\beta)}{\Gamma(\beta - \lambda)\lambda!} z^{\lambda\, |z|<1, \beta\epsilon\mathbb{R}} \tag{11}$$

Thus using similar expansion on equation 1

$$F(x) = \sum_{\boldsymbol{\lambda}=\mathbf{0}}^{\infty} \frac{(-\mathbf{1})^{\boldsymbol{\lambda}}\boldsymbol{\Gamma}(\boldsymbol{\beta} + \mathbf{1})}{\boldsymbol{\Gamma}(\boldsymbol{\beta} - \boldsymbol{\lambda} + \mathbf{1})\boldsymbol{\lambda}!} \{1 - G(x)\}^{\alpha\lambda}$$

$$F(x) = \sum_{j=0}^{\infty} w_j \, G(x)^j \tag{12}$$

where $w_j = w_j(\alpha, \beta) = \sum_{\lambda=0}^{\infty} \frac{(-1)^{\lambda+j}\Gamma(\beta+1)\Gamma(\alpha\lambda+1)}{\Gamma(\beta-\lambda+1)\Gamma(\alpha\lambda-j+1)\, j!\lambda!}$

Differentiating equation 12 with respect to x gives the pdf

$$f(x) = \sum_{j=0}^{\infty} jw_j g(x)\,(G(x))^{j-1} \tag{13}$$

Putting equation 4 and 5 into equation 11

$$f(x) = \frac{k}{\theta}\sum_{j=0}^{\infty} j\boldsymbol{w_j}\, e^{-\frac{x}{\theta}}(1 - e^{-\frac{x}{\theta}})^{kj-1} \tag{14}$$

3.2 Quantile and Median

The quantile function for EGF distribution is expressed in equation 3 as $= Q_G([1 - (1 - u^{1/\beta})^{1/\alpha}])$. Therefore the quantile function of EGEE distribution is given by $Q(p) = F^{-1}(p)$

$$p = [1 - \{1 - (1 - e^{-x/\theta})^k\}^\alpha]^\beta$$

After simplification the quantile function takes the form

$$x = -\theta[ln(1 - \left(1 - \left\{1 - (p)^{\frac{1}{\beta}}\right\}^{\frac{1}{\alpha}})^{\frac{1}{k}}\right)\Big] \tag{15}$$

When $p = 0.25, 0.5\ and\ 0.75$ we obtain the first quantile, median and third quantile of EGEEE distribution.

In particular $Q(0.5) = F^{-1}(u)$ is the median of the probability distribution given as

$$F(x) = p_r(X \le m) = \int_0^m f(x)dx = 0.5 \tag{16}$$

The median of EGEE Distribution can be obtain by equating equation 6 to 0.5

$$[1 - \{1 - (1 - e^{-\frac{x}{\theta}})^k\}^\alpha]^\beta = 0.5$$

$$Median = -\theta[ln(1 - \left(1 - \left\{1 - (0.5)^{\frac{1}{\beta}}\right\}^{\frac{1}{\alpha}})^{\frac{1}{k}}\right)\Big] \tag{17}$$

3.3 Moment and Moment Generation Function of Proposed EGEE Distribution

The crude moment and moment generation of the EGEE distribution are expressed in 3.3.1 and 3.3.2 respectively.

3.3.1 Moment of **EGEE Distribution**

If X have the pdf, then rth moment is obtained as follow

$$\mu^r = \int_0^\infty x^r f(x)\,dx \tag{18}$$

Using the expansion of equation 12

$$\boldsymbol{\mu^r} = \int_{\mathbf{0}}^{\infty} \boldsymbol{x^r} \frac{k}{\theta} \sum_{j=0}^{\infty} j\boldsymbol{w_j}\, e^{-\frac{x}{\theta}} (1 - e^{-\frac{x}{\theta}})^{kj-1}\, \boldsymbol{dx}$$

$$\boldsymbol{\mu^r} = \frac{k}{\theta} \sum_{j=0}^{\infty} \sum_{\boldsymbol{m=0}}^{\infty} \frac{(\boldsymbol{-1})^{\boldsymbol{m}} \boldsymbol{\Gamma}(kj)}{\boldsymbol{\Gamma}(kj - \boldsymbol{m})\, \boldsymbol{m}!} j\boldsymbol{w_j} \int_{\mathbf{0}}^{\infty} \boldsymbol{x^r} e^{-\frac{x}{\theta}} e^{-\frac{xm}{\theta}} \boldsymbol{dx}$$

$$\mu^r = \frac{k\theta^r}{(m+1)^{r+1}} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(kj)}{\Gamma(kj-m)\, m!} jw_j\ \ \Gamma(\mathrm{r}+1) \tag{19}$$

The two infinite series in equation 19 is convergent for all parameters greater than zero $\beta > 0, \alpha > 0, \theta > 0, k > 0\ and\ x > 0$ (Yakubu and Doguwa, 2017).

The mean of the proposed EGEE distribution is gotten by making μ^r moment equal to one (|r=1)

$$\mu_{/}^{1} = \frac{k\theta}{(m+1)^2} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(kj)}{\Gamma(kj-m)\, m!} jw_j \tag{20}$$

When r=2

$$\boldsymbol{\mu_{/}^{2}} = \frac{k\theta^2}{(m+1)^{2+1}} \sum_{j=0}^{\infty} \sum_{\boldsymbol{m=0}}^{\infty} \frac{(\boldsymbol{-1})^{\boldsymbol{m}} \boldsymbol{\Gamma}(kj)}{\boldsymbol{\Gamma}(kj - \boldsymbol{m})\, \boldsymbol{m}!} j\boldsymbol{w_j}\ \ \boldsymbol{\Gamma}(\mathbf{2}+\mathbf{1})$$

$$\boldsymbol{\mu_{/}^{2}} = \frac{k\theta^2}{(m+1)^{3}} \sum_{j=0}^{\infty} \sum_{\boldsymbol{m=0}}^{\infty} \frac{(\boldsymbol{-1})^{\boldsymbol{m}} \boldsymbol{\Gamma}(kj)}{\boldsymbol{\Gamma}(kj - \boldsymbol{m})\, \boldsymbol{m}!} j\boldsymbol{w_j}\ \ \boldsymbol{\Gamma}(\mathbf{3})$$

$$\boldsymbol{\mu_{/}^{2}} = \frac{2k\theta^2}{(m+1)^{3}} \sum_{j=0}^{\infty} \sum_{\boldsymbol{m=0}}^{\infty} \frac{(\boldsymbol{-1})^{\boldsymbol{m}} \boldsymbol{\Gamma}(kj)}{\boldsymbol{\Gamma}(kj - \boldsymbol{m})\, \boldsymbol{m}!} j\boldsymbol{w_j} \tag{21}$$

The variance of EGEE can be obtain from equation 20 and 21.

3.3.2 Moment Generating Function of EGEE Distribution

$$M_x(t) = E(e^{tx})$$

$$E(e^{tx}) = \int_{0}^{\infty} e^{tx} f(x)\, dx \tag{22}$$

$$\int_{\mathbf{0}}^{\infty} e^{tx} \frac{k}{\theta} \sum_{j=0}^{\infty} j\boldsymbol{w_j}\, e^{-\frac{x}{\theta}} (1 - e^{-\frac{x}{\theta}})^{kj-1}\, \boldsymbol{dx}$$

$$M_x(t) = \frac{k}{(m+1-\theta t)} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \binom{kj-1}{m} j\boldsymbol{w_j} \tag{23}$$

Obtaining the first moment from the moment generating function we differentiate with respect to t.

$$\boldsymbol{\mu}_{/}^{\mathbf{1}} = M^{/}{}_x(t)$$

$$M^{/}{}_x(t) = \frac{\theta k}{(m+1-\theta t)^2} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \binom{kj-1}{m} j\boldsymbol{w_j}$$

$M^{/}{}_x(0) = \frac{\theta k}{(m+1)^2} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \binom{kj-1}{m} j\boldsymbol{w_j}$ **3.4 Order Statistics**

The density $f_{n:i}(x)$ of the ith order statistics, for $i = 1, \dots \dots n,$ from independent identical distribution random variable $Y_1 \dots \dots Y_n$ is given by

$$f_{n:i}(x) = \frac{f(x)}{B(i, n-i+1)} F(x)^{i-1} \big(1 - F(x)\big)^{n-i} \tag{24}$$

Order Statistics of Exponentiated Generalized Family Distribution

Substitute in equation 26 for pdf and cdf EGF distribution

$$f_{n:i}(x) = \frac{f(x)}{B(i, n-i+1)} F(x)^{i-1} \big(1 - F(x)\big)^{n-i}$$

$$f_{n:i}(x) = \frac{\propto \beta g(x)(1-G(x))^{\alpha-1}(1-(1-G(x))^{\alpha})^{\beta-1}}{B(i, n-i+1)}$$
$$(1-(1-G(x))^{\alpha})^{\beta(i-1)} \times \big(1-(1-(1-G(x))^{\alpha})^{\beta}\big)^{n-i}$$

$$f_{n:i}(x) = \frac{\propto \beta g(x)(1-G(x))^{\alpha-1}}{B(i, n-i+1)}$$
$$(1-(1-G(x))^{\alpha})^{\beta i-1} \times \big(1-(1-(1-G(x))^{\alpha})^{\beta}\big)^{n-i}$$

Using binomial expansion for

$$\big(1-(1-(1-G(x))^{\alpha})^{\beta}\big)^{n-i} = \sum_{q}^{n-i} (-1)^q \binom{n-i}{q} (1-(1-G(x))^{\alpha})^{\beta q}$$

$$\frac{\propto \beta g(x)(1-G(x))^{\alpha-1}}{B(i, n-i+1)}$$
$$\sum_{q}^{n-i} (-1)^q \binom{n-i}{q} (1-(1-G(x))^{\alpha})^{\beta q} \, (1-(1-G(x))^{\alpha})^{\beta i-1}$$

$$f_{n:i}(x) = \frac{\propto \beta g(x)(1-G(x))^{\alpha-1}}{B(i, n-i+1)} \sum_{q}^{n-i} (-1)^q \binom{n-i}{q} (1-(1-G(x))^{\alpha})^{\beta(q+i)-1}$$

Let

$$s_l = \sum_{q}^{n-i}\sum_{p}^{\infty}(-1)^{q+p+l}\binom{n-i}{q}\binom{\beta(q+i)-1}{p}\binom{\alpha(p+1)-1}{l}$$

$$\frac{\propto \beta}{B(i,n-i+1)}\sum_{l=o}^{\infty} s_l g(x)G(x)^l \tag{25}$$

Order Statistics of EGEE

Order statistics of EGEE distribution is obtain by replacing the cdf and pdf of exponentiated exponential in equation 25

$$\frac{\propto \beta}{B(i,n-i+1)}\sum_{l=o}^{\infty} s_l \frac{k}{\theta} e^{-\frac{x}{\theta}}\left(1-e^{-\frac{x}{\theta}}\right)^{k-1}\left(\left(1-e^{-\frac{x}{\theta}}\right)^{(k-1)}\right)^{l}$$

$$\frac{\propto \beta}{B(i,n-i+1)}\sum_{l=o}^{\infty} s_l \frac{k}{\theta} e^{-\frac{x}{\theta}}\left(1-e^{-\frac{x}{\theta}}\right)^{k-l+kl-1}$$

$$\frac{\propto \beta}{B(i,n-i+1)}\frac{k}{\theta}\sum_{l=o}^{\infty}\sum_{d=0}^{\infty}(-1)^{d}\frac{\Gamma(\mathrm{k}(l+1)-l)}{\Gamma(\mathrm{k}(l+1)-l-d)\mathrm{d}!} s_l e^{-x\left(\frac{d+1}{\theta}\right)}$$

Hence the order statistics for EGEE distribution is

$$\frac{\propto \beta}{B(i,n-i+1)}\frac{k}{\theta}$$

$$*\sum_{q=0}^{n-i}\sum_{p=0}^{\infty}\sum_{l=o}^{\infty}\sum_{d=0}^{\infty}(-1)^{p+q+l+d}\frac{\Gamma(\mathrm{n-i+1})\Gamma(\beta(\mathrm{q+i})\Gamma\alpha(\mathrm{p+1})\Gamma(\mathrm{k}(l+1)-l)}{\Gamma(\mathrm{n-i+q+1})\Gamma(\beta(\mathrm{q+i})-\mathrm{p})\,\Gamma(\alpha(\mathrm{p+1})-l)\Gamma(\mathrm{k}(l+1)-l-d)\mathrm{d!\,q!\,p!}\,l!} e^{-x\left(\frac{d+1}{\theta}\right)} \tag{26}$$

The four infinite series in equation 26 is convergent for all parameters greater than zero $\beta > 0, \alpha > 0, \theta > 0, k > 0\ and\ x > 0$ Yakubu and Doguwa (2017).

3.5 Skewness and Kurtosis of the EGEE Distribution

Two approaches are used in obtaining the skewness and kurtosis of the EGEE distribution. These approaches include the measure of kurtosis (k.u) and skewness(s.k) based on moments and quantiles. In the moments based approach,

$$S.K = \frac{\mu_3^{/} - 3\mu\mu_2^{/} + 2\mu^2}{\left(\mu_2^{/} - \mu^2\right)^{3/2}} \tag{27}$$

and

$$K.U = \frac{\mu_4^{/} - 4\mu\mu_3^{/} + 6\mu^2\mu_2^{/} - 3\mu^4}{\left(\mu_2^{/} - \mu^2\right)^{2}} \tag{28}$$

The quantile approach of evaluating skewness and kurtosis of a distribution is particularly useful when a distribution exists in closed form or in a simple analytic expression. Galton (1883) proposed a quantile measure based approach for evaluating skewness while Moore (1988) did the same for Kurtosis. Galton's skewness and Moor's kurtosis is evaluated using the relations

$$S.K = \frac{Q(6/8) - 2Q(4/8) + Q(3/8) + Q(2/8)}{Q(6/8) - Q(2/8)} \tag{29}$$

$$K.U = \frac{Q(7/8) - 2Q(5/8) + Q(3/8) + Q(1/8)}{Q(6/8) - Q(2/8)} \tag{30}$$

Since the Quantile function of the EGEE distribution exists in closed form as given in (15), then (29) and (30) can be used in evaluating the skewness and kurtosis of the EGEE Distribution.

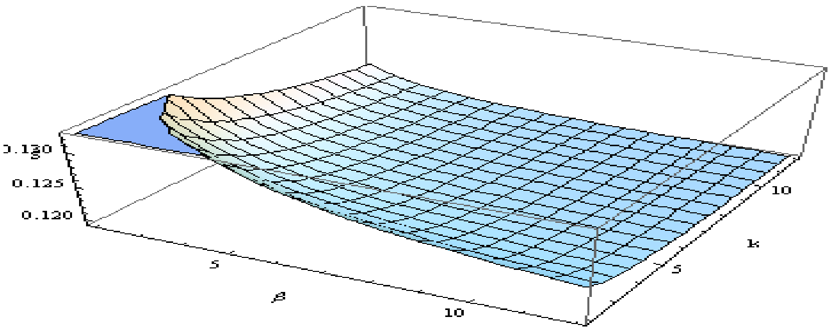

Figure 4 for EGEE Skewness

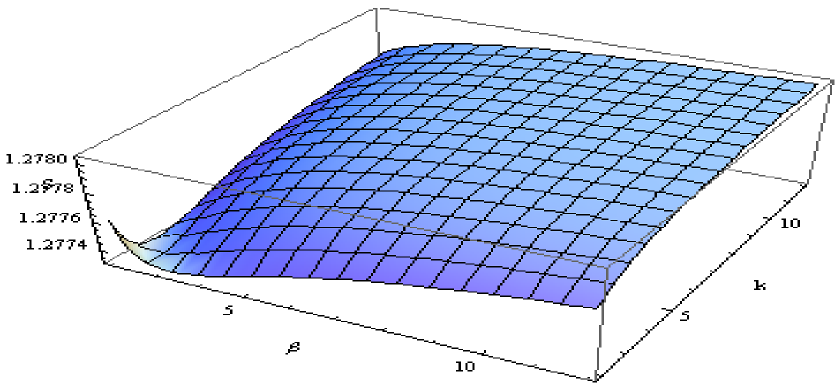

Figure 5 for EGEE Kurtosis

The 3D plot for skewness and kurtosis were plotted using the quartile function of the EGEE distribution with $\alpha = \theta = 1$ while $\beta = k$ takes values from 2 to 12.

3.6 The Mean Deviation

The deviation from the mean (in the case of the symmetric distributions) or the deviation from the median (in the case of skewed distributions) can be used as a measure of spread in the population. Let X be a EGEE random variable with mean $\mu = E(X)$ and median M. The mean deviation ($D(\mu)$) from the mean and the mean deviation ($D(M)$) from the median are defined respectively by

$$D(\mu) = E\{|X-\mu|\} = \int_{-\infty}^{\infty} |x-\mu| f_x dx \quad (31)$$

$$= \int_{-\infty}^{\mu} (\mu - x) f_x dx + \int_{\mu}^{\infty} (x-\mu) f_x dx$$

$$2\mu F_x(\mu) - 2\int_{-\infty}^{\mu} x f_x(x) dx \quad (32)$$

where $\int_{-\infty}^{\mu} x f_x(x) dx$ and $F_x(\mu)$ are incomplete moment and cumulative function respectively (Cordeiro and Lemonte, 2014).

Mean Deviation EGEE Distribution

$$D(\mu) = 2\mu F_x(\mu) - 2\int_{0}^{\mu} x f_x(x) dx \quad (33)$$

$F_x(\mu)$ is obtained from equation 31 as $F(u) = [1 - \{1 - (1 - e^{-\mu/\theta})^k\}^{\alpha}]^{\beta}$ and $\int_0^{\mu} x f_x(x) dx$ is obtain from equation 3 as

$$\frac{k\theta}{(m+1)^2} \sum_{j=0}^{\infty} \sum_{\boldsymbol{m=0}}^{\infty} \frac{(-\mathbf{1})^{\boldsymbol{m}} \boldsymbol{\Gamma}(kj)}{\boldsymbol{\Gamma}(kj-\boldsymbol{m})\, \boldsymbol{m}!} \boldsymbol{j} \boldsymbol{w_j}\ \boldsymbol{\gamma}(\mathbf{2}, \boldsymbol{\mu})$$

where $\gamma(2, \mu)$ is lower incomplete gamma function.

$$D(M) = E\{|X-\mu|\} = \int_{-\infty}^{\infty} |x-M| f_x dx \quad (34)$$

$$= \int_{-\infty}^{M} (M-x) f_x dx + \int_{M}^{\infty} (x-M) f_x dx$$

$$\mu - 2\int_{-\infty}^{M} x f_x(x) dx \quad (35)$$

$\int_0^{M} x f_x(x) dx$ is obtain from equation 31 as

$$\frac{k\theta}{(m+1)^2} \sum_{j=0}^{\infty} \sum_{\boldsymbol{m=0}}^{\infty} \frac{(-\mathbf{1})^{\boldsymbol{m}} \boldsymbol{\Gamma}(kj)}{\boldsymbol{\Gamma}(kj-\boldsymbol{m})\, \boldsymbol{m}!} \boldsymbol{j} \boldsymbol{w_j}\ \boldsymbol{\gamma}(\mathbf{2}, \mathbf{M})$$

3.7 Asymptotic Behavior

We seek to investigate the behavior of the proposed model as given in Equation 34 as $x \to 0$ and as $x \to \infty$. This involves considering $\lim_{x\to 0} f(x)$ and $\lim_{x\to\infty} f(x)$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\frac{\alpha\beta k}{\theta} e^{-\frac{x}{\theta}}(1-e^{-x/\theta})^{k-1}\{1-(1-e^{-x/})^{k}\}^{\alpha-1}[1-\{1-(1 - e^{-\frac{x}{\theta}})^{k}\}^{\alpha}]^{\beta-1}\right] = 0$$

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty}\left[\frac{\alpha\beta k}{\theta} e^{-\frac{x}{\theta}}(1-e^{-x/\theta})^{k-1}\{1-(1-e^{-x/})^{k}\}^{\alpha-1}[1-\{1-(1 - e^{-\frac{x}{\theta}})^{k}\}^{\alpha}]^{\beta-1}\right] = 0$$

These results confirm further that the proposed distribution has a mode (Oguntunde et al., 2014).

3.8 Renyi Entropy

The entropy of X is a measure of variation of the uncertainty. There are many entropy measures studied and discussed in literature but the Renyi entropy is perhaps one of the most popular (Renyi, 1961). The Renyi entropy of X with EGEE density is given by

$$I_{R(\rho)} = \frac{1}{(1-\rho)} log\left(\int_0^\infty g(x)^\rho dx\right) \tag{36}$$

where $\rho > 0$ and $\rho \neq 1$.

Inserting equation 12 into 36

$$I_{R(\rho)} = \frac{1}{(1-\rho)} log\left(\int_0^\infty \left(\frac{k}{\theta}\sum_{j=0}^{\infty} j\boldsymbol{w_j}\, e^{-\frac{x}{\theta}}(1-e^{-\frac{x}{\theta}})^{kj-1}\right)^\rho dx\right)$$

$$I_{R(\rho)} = \frac{1}{(1-\rho)} log\left(\frac{k^\rho}{\theta^\rho}\sum_{j=0}^{\infty} \left(j\boldsymbol{w_j}\right)^\rho \int_0^\infty e^{-\frac{\rho x}{\theta}}(1-e^{-\frac{x}{\theta}})^{\rho(kj-1)}dx\right)$$

$$\int_0^\infty e^{-\frac{\rho x}{\theta}}(1-e^{-\frac{x}{\theta}})^{\rho(kj-1)}dx = \binom{\rho(kj-1)}{b}(-1)^b\frac{1}{(\rho+b)}$$

$$I_{R(\rho)} = \frac{1}{(1-\rho)} log\left(\frac{k^\rho}{\theta^\rho}\sum_{j=0}^{\infty} \left(j\boldsymbol{w_j}\right)^\rho \binom{\rho(kj-1)}{b}(-1)^b\frac{1}{(\rho+b)}\right) \tag{37}$$

4. ESTIMATION OF THE UNKNOWN PARAMETERS

The method of maximum likelihood is used in estimating the unknown parameters.

4.1 Maximum Likelihood Estimation

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the EGEE distribution. For a random sample $x_1 x_2 \ldots . x_n$ of size n, the log-likelihood function of 4 parameter EGEE distribution is given by

$$L = \sum_{i=1}^{n} \ln(fx) = \sum_{i=1}^{n} ln \left(\frac{\alpha\beta k}{\theta} e^{-\frac{x}{\theta}} (1 - e^{-\frac{x}{\theta}})^{k-1} \{1 - (1 - e^{-\frac{x}{\theta}})^k\}^{\alpha-1} [1 - \{1 - (1 - e^{-\frac{x}{\theta}})^k\}^{\alpha}]^{\beta-1} \right)$$

$$nln\alpha + nln\beta + nlnk - nln\theta - \sum_{i=1}^{n} \frac{x_i}{\theta} + (k - 1) \sum_{i=1}^{n} \ln\left(1 - e^{-\frac{x_i}{\theta}}\right) + (\alpha - 1) \sum_{i=1}^{n} \ln\{1 - (1 - e^{-\frac{x_i}{\theta}})^k\} + (\beta - 1) \sum_{i=1}^{n} \ln[1 - \{1 - (1 - e^{-\frac{x_i}{\theta}})^k\}^{\alpha}]$$

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln\{1 - (1 - e^{-\frac{x_i}{\theta}})^k\} + \frac{(\beta - 1) \sum_{i=1}^{n} \quad [1 - \{1 - (1 - e^{-\frac{x_i}{\theta}})^k\}^{\alpha}] ln\{1 - (1 - e^{-\frac{x_i}{\theta}})^k\}}{[1 - \{1 - (1 - e^{-\frac{x_i}{\theta}})^k\}^{\alpha}]}$$

$$\frac{\partial L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \ln[1 - \{1 - (1 - e^{-\frac{x_i}{\theta}})^k\}^{\alpha}]$$

$$\frac{\partial L}{\partial k} = \frac{n}{k} + \sum_{i=1}^{n} \ln\left(1 - e^{-\frac{x_i}{\theta}}\right) - (\alpha - 1) \sum_{i=1}^{n} \frac{(1 - e^{-\frac{x_i}{\theta}})^k ln\left(1 - e^{-\frac{x_i}{\theta}}\right)}{\{1 - (1 - e^{-\frac{x_i}{\theta}})^k\}} + \alpha(\beta - 1) \sum_{i=1}^{n} \frac{\{1 - (1 - e^{-\frac{x_i}{\theta}})^k\}^{\alpha-1} (1 - e^{-\frac{x_i}{\theta}})^k ln\left(1 - e^{-\frac{x_i}{\theta}}\right)}{[1 - \{1 - (1 - e^{-\frac{x}{\theta}})^k\}^{\alpha}]}$$

$$\frac{\partial L}{\partial \theta} = \frac{-n}{\theta} + \sum_{i=1}^{n} \frac{x_i}{\theta^2} - (k + 1) \sum_{i=1}^{n} \frac{x_i e^{-\frac{x_i}{\theta}}}{\theta^2 \left(1 - e^{-\frac{x}{\theta}}\right)} + (\alpha - 1) \sum_{i=1}^{n} \frac{k(1 - e^{-\frac{x_i}{\theta}})^{k-1} x_i e^{-\frac{x_i}{\theta}}}{\theta^2 \left(1 - \left\{1 - e^{-\frac{x}{\theta}}\right\}^k\right)} - (\beta - 1) \sum_{i=1}^{n} \frac{\alpha k(1 - e^{-\frac{x_i}{\theta}})^{k-1} x_i e^{-\frac{x_i}{\theta}} \{1 - (1 - e^{-\frac{x_i}{\theta}})^k\}^{\alpha-1}}{\theta^2 \left(1 \left(-1 - \left\{1 - e^{-\frac{x_i}{\theta}}\right\}^k\right)^{\alpha}\right)}$$

Solving the nonlinear system of equation of $\frac{\partial L}{\partial \alpha} = 0, \frac{\partial L}{\partial \beta} = 0, \frac{\partial L}{\partial k} = 0 \; and \; \frac{\partial L}{\partial \theta} = 0$ gives the maximum likelihood estimates of $\alpha, \beta, k \; and \; \theta$ respectively. We obtain the 4×4 observed information matrix through,

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\\ \hat{k}\\ \hat{\theta}\end{pmatrix}\left[\begin{pmatrix}\alpha\\ \beta\\ k\\ \theta\end{pmatrix}\begin{pmatrix}\hat{V}_{\alpha\alpha} & \hat{V}_{\alpha\beta} & \hat{V}_{\alpha k} & \hat{V}_{\alpha\theta}\\ \hat{V}_{\beta\alpha} & \hat{V}_{\beta\beta} & \hat{V}_{\beta k} & \hat{V}_{\beta\theta}\\ \hat{V}_{k\alpha} & \hat{V}_{k\beta} & \hat{V}_{kk} & \hat{V}_{k\theta}\\ \hat{V}_{\theta\alpha} & \hat{V}_{\theta\beta} & \hat{V}_{\theta k} & \hat{V}_{\theta\theta}\end{pmatrix}\right]$$

$$V^{-1} = -E\begin{pmatrix}V_{\alpha\alpha} & V_{\alpha\beta} & V_{\alpha k} & V_{\alpha\theta}\\ V_{\beta\alpha} & V_{\beta\beta} & V_{\beta k} & V_{\beta\theta}\\ V_{k\alpha} & V_{k\beta} & V_{kk} & V_{k\theta}\\ V_{\theta\alpha} & V_{\theta\beta} & V_{\theta k} & V_{\theta\theta}\end{pmatrix}$$

where

$$V_{\alpha\alpha} = \frac{\partial^2 L}{\partial\alpha^2}, V_{\beta\beta} = \frac{\partial^2 L}{\partial\beta^2}, V_{kk} = \frac{\partial^2 L}{\partial k^2}, V_{\theta\theta} = \frac{\partial^2 L}{\partial\theta^2}$$

$$V_{\alpha\beta} = V_{\beta\alpha} = \frac{\partial^2 L}{\partial\alpha\beta}, V_{\alpha k} = V_{k\alpha} = \frac{\partial^2 L}{\partial\alpha k}, V_{\alpha\theta} = V_{\theta\alpha} = \frac{\partial^2 L}{\partial\theta\alpha},$$

$$V_{\beta k} = V_{k\beta} = \frac{\partial^2 L}{\partial\beta k}, V_{\beta\theta} = V_{\theta\beta} = \frac{\partial^2 L}{\partial\theta\beta}, V_{k\theta} = V_{\theta k} = \frac{\partial^2 L}{\partial\theta k}$$

The solution to the above inverse dispersion matrix yields the asymptotic variance and covariance of the maximum likelihood estimators $\hat{\alpha}$ $\hat{k}$ $\hat{\beta}$ and $\hat{\theta}$. The confidence interval for $\alpha, \beta, k\ and\ \theta$ is given by

$$\hat{\alpha} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{\alpha\alpha}}, \hat{\beta} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{\beta\beta}}, \hat{k} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{kk}}, \hat{\theta}, \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{\theta\theta}}$$

where $Z_{\frac{\alpha}{2}}$ is the α^{th} percentiles of the standard normal distribution.

4.2 Akaike Information Criterion (AIC)

Akaike (1973) introduced one of the most commonly used information criteria which is AIC. The idea is to select the model that minimize the negative likelihood penalized by the number of parameters as specified by the equation

$$AIC = 2k - 2lnL \tag{38}$$

where L is the likelihood function under the fitted model and k is the number of model parameters.

4.3 Bayesian information Criterion (BIC)

Schwarz (1978) studied one of the most widely used information criteria which is BIC. Unlike AIC, the BIC is an estimate of the Bayes factor for two competing models. The BIC is defined by

$$BIC = kln(n) - 2lnL \tag{39}$$

where n refers to the random sample size.

Specifically, AIC is aimed to obtain the best approximating model to the unknown true data generating process. Superficially, BIC differs from AIC only in the first term which depends on sample size n. Models that minimize the BIC are selected. From a Bayesian perspective, BIC is designed to find the most probable model given the data.

4.4 Likelihood Ratio Test (LRT)

The likelihood ratio statistics is used to compare the superiority of EGEE distribution with its existing sub-models. The values of the unrestricted and restricted log-likelihood is used in constructing the LRT statistics. The hypothesis of the test statistics is expressed as

$$H_0: \Theta = \Theta_o \text{ vs } H_1: \Theta \neq \Theta_o$$

The LR statistics use for testingH_0vs H_1 is

$$\omega = 2\left(l(\widehat{\Theta}) - l(\widehat{\Theta}_o)\right) \tag{40}$$

where $\widehat{\Theta}$ and $\widehat{\Theta}_o$ are the log-likelihood under H_1 and H_0. The statistic ω is asymptotically distributed as χ_k^2, where k is the dimension of the subset Ω of interest and degree of freedom. H_0 is rejected at a significance p-values.

5. SIMULATION STUDY

Simulation study is conducted using the quartile function of the EGEE distribution in equation 13 with the help of R-statistics package, sample sizes n=10 and 100 are used with different shape and scale parameters combination. Table 2 and 4; contains the mean, standard deviation and median of the 4-parameter EGEE distribution for different parameter values. While Table 3, 5 contains values of Skewness and Kurtosis obtained with the same combination of parameters.

Table 2
The Mean, Standard Deviation and Mean Deviation of EGEE Distribution for ϴ = 2, 4 and 6 for n=10

β	α	k	ϴ=2			ϴ=4			ϴ=6		
			Mean	SD	MD	Mean	SD	MD	Mean	SD	MD
0.5	0.5	0.5	0.9881	1.1960	0.9494	1.9763	2.3920	1.8988	2.9645	3.5880	2.8483
		0.8	1.4591	1.4541	1.5719	2.9182	2.9095	3.1438	4.3772	4.3642	4.7158
		5	4.0798	2.3439	4.7558	8.1596	4.6879	9.5117	12.2394	7.0317	14.2675
		8	4.9063	2.4738	5.6598	9.8126	4.9477	11.3196	14.7189	7.4215	16.9794
	0.8	0.5	1.7873	1.6779	1.9866	3.5747	3.3557	3.9731	5.3619	5.0336	8.9649
		0.8	2.3864	1.9472	2.7638	4.7729	3.8943	5.5276	7.1592	5.8416	8.2914
		5	5.3494	2.6859	6.1927	10.6988	5.3719	12.3854	16.0482	8.0577	18.5782
		8	6.2219	2.7717	7.1155	12.4438	5.5433	14.2309	18.6656	8.3149	21.3464
	5	0.5	6.9609	3.3771	8.1485	13.9219	6.7543	16.2971	20.8829	10.1314	24.4456
		0.8	7.8523	3.4522	9.0821	15.7045	6.9045	18.1642	23.5568	10.3567	27.2464
		5	11.4456	3.5659	12.7383	22.8912	7.1319	25.4766	34.3369	10.6978	38.2149
		8	12.3803	3.5744	13.6777	24.7607	7.1489	27.3553	37.141	10.7233	41.0329
	8	0.5	8.6479	3.5899	9.9517	17.2959	7.1798	19.9033	25.9437	10.7695	29.855
		0.8	9.5646	3.6279	10.8891	19.1292	7.2559	21.7782	28.6938	10.8839	32.6672
		5	13.1962	3.6832	14.5506	26.3925	7.3663	29.1012	39.5887	11.0495	43.6518
		8	14.1338	3.6871	15.4904	28.2676	7.3744	30.9807	12.4015	11.0616	46.4711
5	0.5	0.5	0.0219	0.0341	0.0166	0.0437	0.0683	0.0333	0.0656	0.1024	0.0499
		0.8	0.0988	0.1052	0.1025	0.1977	0.2104	0.205	0.2966	0.3156	0.3075
		5	1.6329	0.7460	1.9305	3.2659	1.4920	3.8609	4.8989	2.2382	5.7914
		8	2.3349	0.8861	2.7023	4.6698	1.7721	5.4046	7.0047	2.6582	8.1069
	0.8	0.5	0.0449	0.0540	0.0431	0.0899	0.1081	0.0861	0.1349	0.1621	0.1292
		0.8	0.1672	0.1449	0.1891	0.3344	0.2898	0.3781	0.5016	0.4346	0.5672
		5	1.9778	0.7475	2.2819	3.9555	1.4951	4.5638	5.9332	2.2426	6.8457
		8	2.7302	0.8514	3.0851	5.4603	1.7028	6.1701	8.1905	2.5542	9.2552
	5	0.5	0.2658	0.1644	0.3100	0.5316	0.3288	0.6200	0.7973	0.4932	0.9301
		0.8	0.6080	0.2884	2.1169	1.2160	0.5769	1.4113	1.8241	0.8654	2.1169
		5	3.2184	0.6224	3.4699	6.4368	1.2447	6.9398	9.6552	1.8671	10.4097
		8	9.6552	1.8671	10.4097	8.1439	1.3114	8.6781	12.2159	1.9671	13.0171
	8	0.5	0.3567	0.1928	0.4144	0.7134	0.3856	0.8289	1.0701	0.5784	1.2434
		0.8	0.7549	0.3111	0.8642	1.5098	0.6223	1.7284	2.2647	0.9334	2.5926
		5	3.5005	0.5911	3.7378	7.0001	1.1829	7.4756	10.5014	1.7734	11.2135
		8	4.3669	0.6174	4.6174	8.7339	1.2348	9.2329	13.1008	1.8522	13.8492

Table 3
The Skewness and Kurtosis of EGEE Distribution for ϴ = 2, 4 and 6 for n=10

β	α	k	ϴ=2		ϴ=4		ϴ=6	
			SK	KT	SK	KT	SK	KT
0.5	0.5	0.5	1.5700	1.6535	1.5700	1.6535	1.5700	1.6535
		0.8	1.1411	0.6917	1.1411	0.6917	1.1411	0.6917
		5	0.1097	-0.7757	0.1097	-0.7757	0.1097	-0.7757
		8	0.0036	-0.8161	0.0036	-0.8161	0.0036	-0.8161
	0.8	0.5	0.9649	0.3352	0.9649	0.3352	0.9649	0.3352
		0.8	0.6261	-0.2490	0.6261	-0.2490	0.6261	-0.2490
		5	-0.0394	-0.8538	-0.0394	-0.8538	-0.0394	-0.8538
		8	-0.1001	-0.8582	-0.1001	-0.8582	-0.1001	-0.8582
	5	0.5	-0.2396	-0.9347	-0.2396	-0.9347	-0.2396	-0.9347
		0.8	-0.2827	-0.9132	-0.2827	-0.9132	-0.2827	-0.9132
		5	-0.3461	-0.8721	-0.3461	-0.8721	-0.3461	-0.8721
		8	-0.3507	-0.8687	-0.3507	-0.8687	-0.3507	-0.8687
	8	0.5	-0.3333	-0.8971	-0.3333	-0.8971	-0.3333	-0.8971
		0.8	-0.3541	-0.8816	-0.3541	-0.8816	-0.3541	-0.8816
		5	-0.3837	-0.8574	-0.3837	-0.8574	-0.3837	-0.8574
		8	-0.3858	-0.8556	-0.3858	-0.8556	-0.3858	-0.8556
5	0.5	0.5	1.9758	2.7099	1.9758	2.7099	1.9758	2.7099
		0.8	1.3206	1.08311	1.3206	1.08311	1.3206	1.08311
		5	-0.4770	-0.9557	-0.4770	-0.9557	-0.4770	-0.9557
		8	-0.6179	-0.8201	-0.6179	-0.8201	-0.6179	-0.8201
	0.8	0.5	1.5949	1.7345	1.5949	1.7345	1.5949	1.7345
		0.8	0.8203	0.0853	0.8203	0.0853	0.8203	0.0853
		5	-0.5867	-0.8213	-0.5867	-0.8213	-0.5867	-0.8213
		8	-0.6843	-0.7056	-0.6843	-0.7056	-0.6843	-0.7056
	5	0.5	0.2204	-0.6939	0.2204	-0.6939	0.2204	-0.6939
		0.8	-0.1743	-0.6876	-0.1743	-0.6876	-0.1743	-0.6876
		5	-0.6454	-0.6876	-0.6454	-0.6876	-0.6454	-0.6876
		8	-0.6735	-0.6565	-0.6735	-0.6565	-0.6735	-0.6565
	8	0.5	0.05313	-0.3008	0.05313	-0.3008	0.05313	-0.3008
		0.8	-0.3109	-0.4021	-0.3109	-0.4021	-0.3109	-0.4021
		5	0.7379	-0.1575	0.7379	-0.1575	0.7379	-0.1575
		8	-0.7636	-0.1295	-0.7636	-0.1295	-0.7636	-0.1295

Table 4: Simulation Study of EGEE at Sample Size 100

β	α	k	θ=2			θ=4			θ=6		
			Mean	**SD**	**MD**	**Mean**	**SD**	**MD**	**Mean**	**SD**	**MD**
0.5	0.5	0.5	1.1563	2.1610	0.1596	2.3126	4.3220	0.3192	3.4689	6.4830	0.4789
		0.8	1.5416	2.4109	0.4486	3.0832	4.8219	0.8972	4.6248	7.2329	1.3458
		5	3.9956	3.1106	2.9703	7.9913	6.2212	5.9407	11.9870	9.3318	8.9111
		8	4.8154	3.1957	3.8174	9.6308	6.3915	7.6348	14.4463	9.5873	11.4523
	0.8	0.5	0.6126	1.2268	0.0684	1.2253	2.4537	0.1369	1.838	3.6806	0.2054
		0.8	0.9176	1.4645	0.2558	1.8352	2.929	0.5116	2.7528	4.3936	0.7675
		5	3.1623	2.2042	2.4921	6.3246	4.4084	4.9843	9.4869	6.6127	7.4765
		8	3.9588	2.3000	3.3108	7.9177	4.6000	6.6216	11.8767	6.9000	9.9325
	5	0.5	0.0339	0.0898	0.0020	0.0679	0.1796	0.004	0.1019	0.2694	0.0061
		0.8	0.1099	0.1881	0.0271	0.2199	0.3763	0.0543	0.3299	0.5645	0.0814
		5	1.5641	0.8074	1.3946	3.1282	1.6148	2.7893	4.6924	2.4222	4.1839
		8	2.2585	0.9178	2.1002	4.5170	1.8357	4.2005	6.7755	2.7536	6.3008
	8	0.5	0.0147	0.0410	0.0008	0.0294	0.0821	0.0016	0.0441	0.1231	0.0024
		0.8	0.0625	0.1083	0.0151	0.125	0.2166	0.0303	0.1875	0.3249	0.0454
		5	1.3501	0.6638	1.2234	2.7002	1.3277	2.4469	4.0503	1.9916	3.6704
		8	2.0177	0.7734	1.9011	4.0354	1.5469	3.8022	6.0531	2.3203	5.7034
0.8	0.5	0.5	1.8222	2.6498	0.6224	3.6445	5.2997	1.2449	5.4668	7.9496	1.8674
		0.8	2.3493	2.8795	1.1544	4.6986	5.759	2.3089	7.0480	8.6385	3.4634
		5	5.2278	3.4056	4.1820	10.4557	6.8112	8.3641	15.6835	10.2168	12.5462
		8	6.0986	3.4589	5.0735	12.1973	6.9179	10.1470	18.2960	10.3768	15.2205
	0.8	0.5	0.9758	1.5198	0.2869	1.9517	3.0396	0.5738	2.9275	4.5594	0.8608
		0.8	1.4030	1.7560	0.6688	2.8061	3.512	1.3377	4.2091	5.268	2.0066
		5	4.0622	2.3501	3.4042	8.1244	4.7002	6.8084	12.1867	7.0503	10.2126
		8	4.912	2.4137	4.2707	12.1312	7.0634	10.2736	14.7361	7.2413	12.8123
	5	0.5	0.0559	0.1143	0.0098	0.1118	0.2287	0.0197	0.1678	0.3431	0.0296
		0.8	0.1698	0.2285	0.0736	0.3397	0.457	0.1472	0.5095	0.6855	0.2208
		5	1.9175	0.7853	1.7728	3.8350	1.5706	3.5457	5.7525	2.3559	5.3186
		8	2.6653	0.8668	2.5279	5.3307	1.7336	5.0559	7.9960	2.6004	7.5838
	8	0.5	0.0243	0.0525	0.0039	0.0486	0.1050	0.0079	0.0729	0.1575	0.0118
		0.8	0.0967	0.1318	0.0413	0.1934	0.2636	0.0826	0.2901	0.3954	0.1238
		5	1.6431	0.6347	1.5382	3.2862	1.2694	3.0765	4.9294	1.9042	4.6147
		8	2.3631	0.7163	2.2644	4.7261	1.4325	4.5288	7.0892	2.1488	6.7932
5	0.5	0.5	6.7432	4.0122	5.6622	13.4863	8.0244	11.3243	20.2294	12.0366	16.9865
		0.8	7.6388	4.0533	6.5795	15.2776	8.1066	13.1591	22.9164	12.1599	19.7387
		5	11.2400	4.1140	10.2129	22.4801	8.228	20.4259	33.7201	12.342	30.6389
		8	12.1754	4.1185	11.1507	24.3508	8.237	22.3014	36.5262	12.3555	33.4521
	0.8	0.5	3.8049	2.4336	3.1149	7.6098	4.8672	6.2299	11.4147	7.3008	9.3448
		0.8	4.6351	2.5138	3.9690	9.2703	5.0277	7.9381	13.9054	7.5416	11.9072
		5	8.1386	2.6378	7.5118	16.2773	5.2756	15.0236	24.4160	7.9133	22.5354
		8	9.0667	2.6471	8.4429	18.1335	5.2943	16.8859	27.2003	7.9415	25.3289
	5	0.5	0.2615	0.2278	0.1833	0.5230	0.4556	0.3666	0.7846	0.6834	0.5500
		0.8	0.5918	0.3512	0.4925	1.1837	0.7025	0.9850	1.7756	1.0538	1.4776
		5	3.1786	0.6473	3.0637	6.3573	1.2947	6.1275	9.536	1.9420	9.1913
		8	4.0303	0.6754	3.9154	8.0607	1.3509	7.8309	12.0911	2.0263	11.7464
	8	0.5	0.1172	0.1087	0.0791	0.2345	0.2174	0.1582	0.3517	0.3261	0.2373
		0.8	0.3401	0.2054	0.2812	0.6802	0.4108	0.5625	1.0203	0.6163	0.8438
		5	2.6454	0.4920	2.5642	5.2908	0.9841	5.1283	7.9362	1.4762	7.6925
		8	3.4695	0.5219	3.3876	6.939	1.0438	6.7753	10.4085	1.5658	10.1630

Table 5: Skewness and Kurtosis at sample size 100

β	α	k	θ=2		θ=4		θ=6	
			SK	**KT**	**SK**	**KT**	**SK**	**KT**
0.5	0.5	0.5	3.455	16.6414	3.455	16.6414	3.455	16.6414
		0.8	2.9251	12.2422	2.9251	12.2422	2.9251	12.2422
		5	1.844	5.3403	1.844	5.3403	1.844	5.3403
		8	1.7359	4.8308	1.7359	4.8308	1.7359	4.8308
	0.8	0.5	3.8469	20.4196	3.8469	20.4196	3.8469	20.4196
		0.8	3.0237	13.0736	3.0237	13.0736	3.0237	13.0736
		5	1.5348	3.6594	1.5348	3.6594	1.5348	3.6594
		8	1.3982	3.1088	1.3982	3.1088	1.3982	3.1088
	5	0.5	5.7922	42.2041	5.7922	42.2041	5.7922	42.2041
		0.8	3.4497	16.9675	3.4497	16.9675	3.4497	16.9675
		5	0.7109	0.3459	0.7109	0.3459	0.7109	0.3459
		8	0.528	0.0460	0.528	0.0460	0.528	0.0460
	8	0.5	6.1811	46.9722	6.1811	46.9722	6.1811	46.9722
		0.8	3.5372	17.8128	3.5372	17.8128	3.5372	17.8128
		5	0.5932	0.0521	0.5932	0.0521	0.5932	0.0521
		8	0.4048	-0.1832	0.4048	-0.1832	0.4048	-0.1832
0.8	0.5	0.5	2.616	9.7831	2.616	9.7831	2.616	9.7831
		0.8	2.2731	7.5198	2.2731	7.5198	2.2731	7.5198
		5	1.6133	4.1072	1.6133	4.1072	1.6133	4.1072
		8	1.5528	3.862	1.5528	3.862	1.5528	3.862
	0.8	0.5	2.9067	12.0859	2.9067	12.0859	2.9067	12.0859
		0.8	2.3469	8.0295	2.3469	8.0295	2.3469	8.0295
		5	1.3670	2.9446	1.3670	2.9446	1.3670	2.9446
		8	1.2824	2.6424	1.2824	2.6424	1.2824	2.6424
	5	0.5	4.5766	28.4804	4.5766	28.4804	4.5766	28.4804
		0.8	2.6914	10.6886	2.6914	10.6886	2.6914	10.6886
		5	0.6153	0.2882	0.6153	0.2882	0.6153	0.2882
		8	0.4817	0.0893	0.4817	0.0893	0.4817	0.0893
	8	0.5	5.0456	33.6027	5.0456	33.6027	5.0456	33.6027
		0.8	2.8079	11.5959	2.8079	11.5959	2.8079	11.5959
		5	0.5075	0.1026	0.5075	0.1026	0.5075	0.1026
		8	0.3661	-0.0556	0.3661	-0.0556	0.3661	-0.0556
5	0.5	0.5	1.2706	2.5145	1.2706	2.5145	1.2706	2.5145
		0.8	1.2308	2.4016	1.2308	2.4016	1.2308	2.4016
		5	1.1722	2.2454	1.1722	2.2454	1.1722	2.2454
		8	1.1678	2.2343	1.1678	2.2343	1.1678	2.2343
	0.8	0.5	1.3568	2.8333	1.3568	2.8333	1.3568	2.8333
		0.8	1.2529	2.4782	1.2529	2.4782	1.2529	2.4782
		5	1.0994	2.0164	1.0994	2.0164	1.0994	2.0164
		8	1.0881	1.9853	1.0881	1.9853	1.0881	1.9853
	5	0.5	2.1308	7.2064	2.1308	7.2064	2.1308	7.2064
		0.8	1.4232	3.2443	1.4232	3.2443	1.4232	3.2443
		5	0.6491	0.6401	0.6491	0.6401	0.6491	0.6401
		8	0.6029	0.5456	0.6029	0.5456	0.6029	0.5456
	8	0.5	2.3714	8.9285	2.3714	8.9285	2.3714	8.9285
		0.8	1.4675	3.4686	1.4675	3.4686	1.4675	3.4686
		5	0.5613	0.4339	0.5613	0.4339	0.5613	0.4339
		8	0.5106	0.3435	0.5106	0.3435	0.5106	0.3435

In Table 2 and 4, it is observed that the mean, standard deviation and mean deviation are increasing functions of the scale parameters θ when the other parameters are held constant. Increasing the scale parameter θ increases the mean, standard deviation and median deviation for fixed $\alpha, \beta and\ k$. The mean, standard deviation and median are increasing function of the shape parameters α, β and k. An increase in one of the shape parameters when others are held constant increases the values of the mean, standard deviation and mean deviation of EGEE distribution. In Table 3 and 5, Skewness and kurtosis remain constant as the scale parameter θ increases when the shape parameters are held constant. An increase in the shape parameters reduces the values of skewness and kurtosis when the scale parameter is held constant.

5.1 Application

In this section, we fit the EGEE distribution to two (2) real data sets and compare it fitted values with that of its sub-models. The first data set is an uncensored data set from Nichols and Padgett (2006) consisting of 100 observations on breaking stress of carbon fibers (in Gba) as given in Table 6. While the second data set represent the total milk production in the first birth of 107 cows from SINDI race in Table 10. These cows are property of the Carnaúba farm which belongs to the Agropecuária Manoel Dantas Ltd (AMDA) located in Taperoá City, Paraiba (Brazil). This data is presented by Cordeiro and Brito (2012) as presented in Table 10.

Table 6
100 Observations on Breaking Stress of Carbon Fibers (in Gba)

3.70	4.42	3.75	3.15	3.31	1.41	2.17	1.59	2.48	2.03
2.74	2.41	2.43	2.35	3.31	3.68	1.17	2.00	0.85	1.80
2.73	3.19	2.95	2.55	2.85	2.97	5.08	1.22	1.61	1.57
2.50	3.22	2.97	2.59	2.56	1.36	2.48	1.12	2.79	1.08
3.60	1.69	3.39	2.38	3.56	0.98	1.18	1.71	4.70	2.03
3.11	3.28	2.96	2.81	2.81	2.76	3.19	1.84	3.51	1.61
3.27	3.09	2.53	4.20	2.77	4.91	1.57	3.65	2.17	2.12
2.87	1.87	2.67	3.33	2.17	3.68	0.81	2.05	1.69	1.89
1.47	3.15	2.93	2.55	2.83	1.84	5.56	0.39	1.25	2.88
3.11	4.90	3.22	3.39	1.92	1.59	1.73	3.68	4.38	2.82

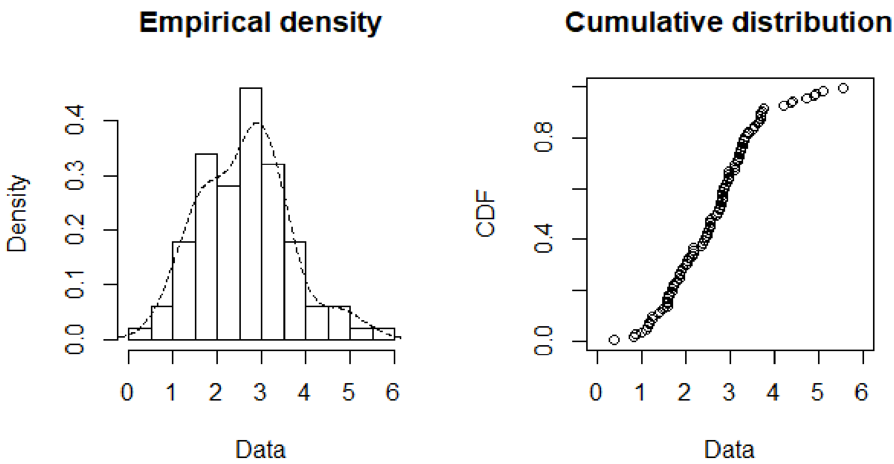

Figure 6: Histogram and CDF Plots of an Empirical Distribution for Fibers Data

Table 7
Summary Statistics Carbon Fibers Data

Min	Max	Median	Mean	S.D.	Skewness	Kurtosis
0.3900	5.5600	2.700	2.6214	1.0139	0.3738	3.1729

The statistical summary table from Table 7 shows that the fibres data is under-disperse, non-symmetry and right skewed.

Correlation matrix of EGEE:

$$\begin{pmatrix} 1.0000000 & -0.8855513 & -0.2215622 & 0.2383665 \\ -0.8855513 & 1.0000000 & -0.2214452 & 0.2382406 \\ -0.2215622 & -0.2214452 & 1.0000000 & -0.9543851 \\ 0.2383665 & 0.2382406 & -0.9543851 & 1.0000000 \end{pmatrix}$$

Correlation matrix above shows that the parameters exhibit both positive and negative correlation coefficients depending on the combination of the parameters. This allows us to see which pairs have the negative and positive correlation.

The asymptotic variance covariance matrix for the estimated parameters is

$$I_{ij}^{-1} = \begin{pmatrix} 7762.6178 & -3539.1498 & -17.207477 & 327.94615 \\ -3539.1498 & 2057.60624 & -8.854517 & 168.75244 \\ -17.207477 & -8.854517 & 0.777025 & -13.13694 \\ 327.94615 & 168.75244 & -13.13694 & 243.84074 \end{pmatrix}$$

Table 8
Maximum Likelihood Estimate of Parameters Standard Errors in Parenthesis, Loglikelihood and Information Criterion carbon Fibers

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{k}$	$\hat{\theta}$	LL	AIC	BIC
EGEE	11.3312	5.8340	3.2679	9.0031	-141.318	290.6361	301.0557
	(88.1057)	(45.3608)	(0.8815)	(15.6154)			
EGE	2.3577	7.7871		2.327	-146.182	298.3646	306.1801
	(102.0782)	(1.5042)		(100.7249)			
EG			7.79061	0.9868	-146.182	296.3646	301.575
			(1.4966)	(0.0851)			
E				0.3814	-196.371	394.7417	397.3469
				(0.0381)			

Table 6 shows the estimated values for all the shapes and scale parameters for the proposed EGEE distribution and its related sub-models (EGE, EG, and E). The estimates of log-likelihood function (LL), information criterion (Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC)) shows that EGEE distribution gives a better result, from all the values. Confidence interval at 95% for the estimates $\hat{\alpha}$ $\hat{\beta}$ $\hat{k}$ and $\hat{\theta}$ are $(-61.356, 84.0184), (-83.0733, 94.7413), (2.5477, 4.98881)$ and $(-21.6031, 39.6093)$ respectively. Likelihood ratio test (LRT) was carried out to test the significance of proposed model with its sub- models at 0.05 level of significance.

$$LR = 2\big(-141.318 - (-146.182)\big) = 9.7800$$
$$LR = 2\big(-141.318 - (-146.182)\big) = 9.7800$$
$$LR = 2\big(-141.318 - (-196.37)\big) = 110.1040$$

Table 9
Likelihood Ratio Test (LRT) Statistic for Carbon Fibers

Model	Hypothesis	LRT	$p - value$
EGE	$H_0: \beta = 1 \; vs \; H_1 : \; H_0 \; is \; false$	9.7800	0.0001
GE	$H_0: \beta = k = 1 \; vs \; H_1 : \; H_0 \; is \; false$	9.7800	0.0001
E	$H_0: \beta = k = \alpha = 1 \; vs \; H_1 : \; H_0 \; is \; false$	110.1040	0.00000

The LRT statistic shows that the EGEE is a better model when compared with the sub-models since the p-values are significant.

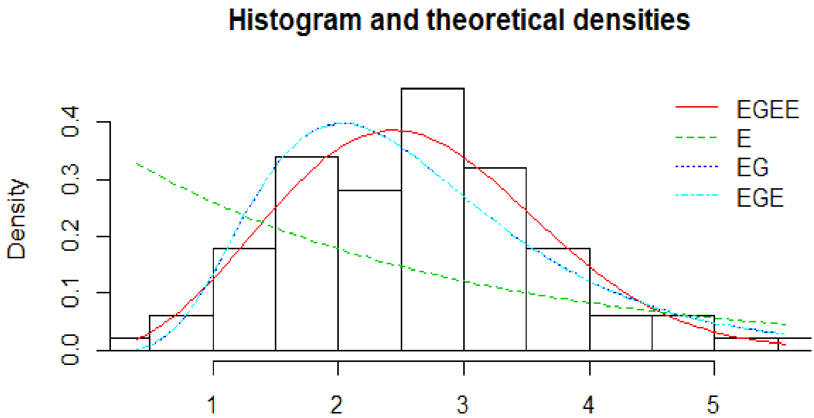

Figure 7: Fitted Plot of EGEE and Its Sub-Models on Fibers Data

Figure 7 shows the histogram of the fibers data set along with fitted distributions. The theoretical density of EGEE distribution has a better spread than its sub-models on the data set.

These cows are property of the Carnaúba farm which belongs to the Agropecuária Manoel Dantas Ltda (AMDA) located in Taperoá City, Paraíba (Brazil). This data is presented by Cordeiro and Brito (2012). These data are

Table 10
The Data Set Represent the Total Milk Production in the First Birth of 107 Cows from SINDI Race

0.43650	0.4260	0.5140	0.6907	0.7471	0.2605	0.6196	0.8781	0.4990	0.6058	0.7629
0.6891	0.5770	0.5394	0.1479	0.2356	0.6012	0.1525	0.5483	0.6927	0.7261	0.5941
0.3323	0.0671	0.2361	0.4800	0.5707	0.7131	0.5853	0.6768	0.5350	0.4151	0.6174
0.6789	0.4576	0.3259	0.2303	0.7687	0.4371	0.3383	0.6114	0.3480	0.4564	0.6860
0.7804	0.3406	0.4823	0.5912	0.5744	0.5481	0.1131	0.7290	0.0168	0.5529	0.0609
0.4530	0.3891	0.4752	0.3134	0.3175	0.1167	0.6750	0.5113	0.5447	0.4143	0.6488
0.5627	0.5150	0.0776	0.3945	0.4553	0.4470	0.5285	0.5232	0.6465	0.0650	0.2747
0.8492	0.8147	0.3627	0.3906	0.4438	0.4612	0.3188	0.2160	0.6707	0.6220	0.4741
0.5629	0.4675	0.6844	0.3413	0.4332	0.0854	0.3821	0.4694	0.3635	0.4111	0.3598
0.5349	0.3751	0.1546	0.4517	0.2681	0.4049	0.5553	0.5878			

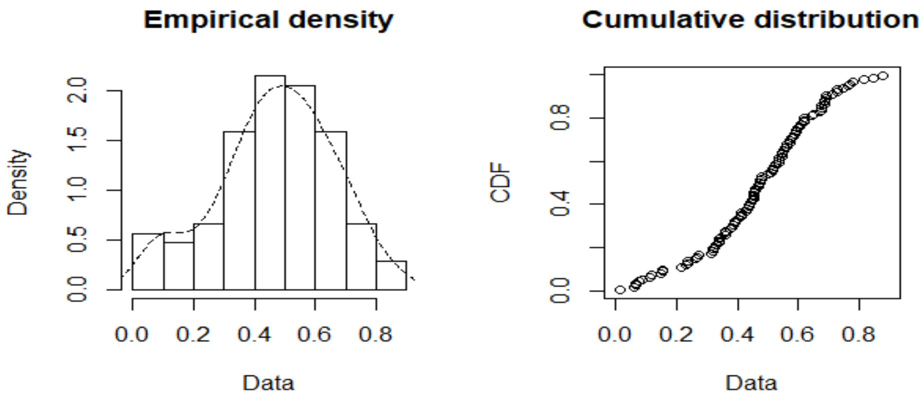

Figure 8: Histogram and CDF Plots of an Empirical Distribution for Cows Data

Table 11
Summary Statistics Cow Data

Min	Max	Median	Mean	S.D.	Skewness	Kurtosis
0.0168	0.8781	0.4741	0.4689	0.1919	-0.3306	-0.3639

The statistical summary table from table 11 shows that the cow data is over-disperse, non-symmetry and left- skewed.

Correlation matrix of EGEE:

$$\begin{pmatrix} 1.0000000 & 0.5508470 & -0.5424908 & 0.861898 \\ 0.5508470 & 1.0000000 & -0.5423818 & 0.8617364 \\ -0.5424908 & -0.5423818 & 1.0000000 & -0.7636886 \\ 0.861898 & 0.8617364 & -0.7636886 & 1.0000000 \end{pmatrix}$$

Correlation matrix above shows that the parameters exhibit both positive and negative correlation coefficients depending on the combination of the parameterss. This allows us to see which pairs has negative and positive correlation.

The asymptotic variance covariance matrix for the estimated parameters is.

$$I_{ij}^{-1} = \begin{pmatrix} 25436.86163 & 54210.79981 & -25.56504711 & 6626.11145 \\ 54210.79981 & 380755.4731 & -98.88969 & 25631.17162 \\ -25.56504711 & -98.88969 & 0.0873062 & -10.87702 \\ 6626.11145 & 25631.17162 & -10.87702 & 2323.49581 \end{pmatrix}$$

Table 12
Maximum Likelihood Estimate of Parameters Standard Errors in Parenthesis, Loglikelihood and Information Criterion Cow data

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{k}$	$\hat{\theta}$	LL	AIC	BIC
EGEE	78.11548	302.417	2.5343	27.4904	21.0898	-34.1796	-23.4883
	(159.4893)	(617.053)	(0.295)	(15.6154)			
EGE	13.7707	3.7135		3.2788	5.03874	-4.07749	3.9409
EG			3.7146	0.23801	5.038748	-6.07749	-0.73184
			(0.5657	(0.0211)			
E				2.132872	-25.9508	53.9015	56.57438
				(0.20619)			

Table 12 shows the estimated values for all the shapes and scale parameters for the proposed EGEE distribution and its related sub-models (EGE, EG and E). The estimates of log-likelihood function (LL), information criterion (Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC)) shows that EGEE distribution gives a better result, from all the values. Confidence interval at 95% for the estimates $\hat{\alpha}$ $\hat{\beta}$ $\hat{k}$ and $\hat{\theta}$ are

$$(-234.4837, 390.7147), (-907.0086, 1511.8426),$$
$$(\ 1.9552, 3.11343) \text{ and } (-66.9808, 121.973601)$$

respectively. Likelihood ratio test (LRT) was carried out to test the significance of proposed model with its sub- models at 0.05 level of significance.

$$LR = 2\big(21.0898 - (5.03875)\big) = 33.0100$$
$$LR = 2\big(21.0898 - (5.03875)\big) = 33.0100$$
$$LR = 2\big(21.0898 - (-25.9508)\big) = 94.0920$$

Table 13
Likelihood Ratio Test (LRT) Statistic for Cow Data

Model	Hypothesis	LRT	$p - value$
EGE	$H_0: \beta = 1\ vs\ H_1 :\ H_0\ is\ false$	33.0100	0.00001
GE	$H_0: \beta = k = 1\ vs\ H_1 :\ H_0\ is\ false$	33.0100	0.00001
E	$H_0: \beta = k = \alpha = 1\ vs\ H_1 :\ H_0\ is\ false$	94.0920	0.00000

The LRT statistic shows that the EGEE is a better model when compared with the sub-models since the p-values are significant.

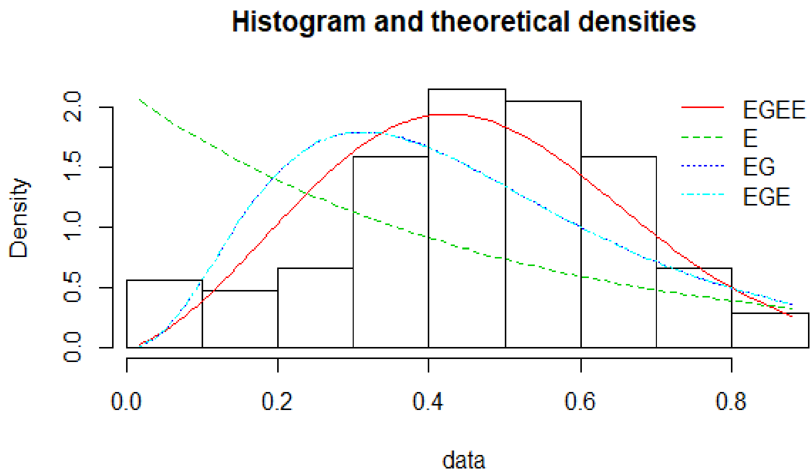

Figure 9: Fitted Plot of EGEE and Its Sub-Models on Cow Data

Figure 9 shows the histogram of the fibers data set along with fitted distributions. The theoretical density of EGEE distribution has a better spread than its sub-models on the data set.

6. CONCLUSION

We defined and derived the four parameter Exponentiated Generalized Exponentiated Exponential (EGEE) distribution using Exponentiated Generalized Family (EGF) as the generator and exponentiated exponential as base line distribution. Plot of EGEE density function with different sub-models are given in Figure 1. The graph shows that EGEE distribution can be monotonically decreasing (reversed J shape), left skewed, right skewed and unimodal depending on the shape parameters α, β and k. The cumulative distribution function (cdf) of EGEE in Figure 2 shows a satisfactory level of cdf not exceeding 1 on the y-axis. The shape parameter values have a strong influence on the shape of the graphs. The hazard function graphical displayed in Figure 3, shows an increasing and constant display for different parameters value. Since the quantile function of the proposed distribution exists in closed form, Mathematica is used in plotting the 3D plots in Figures 4 and 5 which show high level of skweness and kurtosis which could take any form (playtykurtic or leptokurtic) depending on the sample size. In Tables 2 and 4, it is observed that the mean, standard deviation and mean deviation are increasing functions of the scale parameters θ when the other parameters are held constant. Increasing the scale parameter θ increase the mean, standard deviation and median deviation for fixed $\alpha, \beta and\ k$. The mean, standard deviation and median are increasing function of the shape parameters $\alpha, \beta and\ k$. An increase in one of the shape parameters when others are held constant increases the values of the mean, standard deviation and mean deviation of EGEE distribution. In Tables 3 and 5, Skewness and kurtosis remain constant as the scale parameter θ increases when the shape parameters are held constant. An increase in the shape parameters reduces the values of skewness and kurtosis when the scale parameter is held constant. The application of the EGEE and its sub-models on fibers and cow data in Tables 6 and 10 show that the proposed model serve as a better model than its sub-models using the information criterion (AIC and BIC), loglikelihood and significant p-values using likelihood ratio test. The EGEE distribution can serve as an alternative distribution where the sub-models are applied.

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