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# CHARACTERIZATION AND BAYESIAN INFERENCE FOR EXPONENTIATED GENERALIZED STANDARD INVERSE EXPONENTIAL DISTRIBUTION

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## ABSTRACT

In this paper, we have studied several properties of Exponentiated Generalized standard Inverse Exponential (EGSIE) distribution. The model is illustrated each in the real data as well as generated data. Bayesian Analysis of unknown parameter $\beta$ of the EGSIED is examined under various priors and loss functions. Simulation study will be performed to compare the performance of the mean square error with varying sample sizes in R Software.

## KEYWORDS

EGSIED, Maximum Likelihood Estimator, Bayesian estimators, Loss functions, Priors, R Software.

## 1. INTRODUCTION

The Exponentiated Generalized Inverted Exponential (EGIE) distribution was considered by Oguntunde et al., (2014).The probability density function (p.d.f.) and cumulative distribution function (c.d.f.) of the Exponentiated Generalized Standard Inverse Exponential Distribution are respectively given by:

$$f(x;\alpha,\beta)=\frac{\alpha\beta}{x^2}e^{\frac{-1}{x}}\left\{1-e^{\frac{-1}{x}}\right\}^{\alpha-1}\left[1-\left\{1-e^{\frac{-1}{x}}\right\}^{\alpha}\right]^{\beta-1},\ x>0,\alpha,\beta>0\ , \tag{1.1}$$

$$F(x;\alpha,\beta)=\left[1-\left\{1-e^{\frac{-1}{x}}\right\}^{\alpha}\right]^{\beta},\ x>0,\alpha,\beta>0\ . \tag{1.2}$$

where $\alpha$ and $\beta$ are shape parameters and scale parameter 1.

Inverted exponential distribution as a life time model has been proposed by Lin et al. (1989). The inverted exponential distribution was generalized, by introducing a shape parameter, and considered by Abouammoh and Alshingiti (2009). The two-parameter Generalized exponential distribution, originally proposed by Gupta and Kundu (1999), (2001). Bayes estimators for generalized exponential distribution (GED) was considered by Raqab and Madi (2005) while Parviz et al. (2013) compared the Bayesian and

 

Maximum Likelihood estimation for generalized exponentiated gamma distribution. Singh et al (2014) considered the estimation of the parameters of exponentiated pareto distribution. They obtained Bayes estimator for parameters of exponentiated pareto distribution by using squared error loss function and generalized entropy loss function. They also compared the classical method with Bayesian method through Monte Carlo simulation.

Figure (1) and Figure (2) represents the probability density and cumulative distribution functions of Exponentiated Generalized Standard Inverse Exponential (EGSIE) distribution.

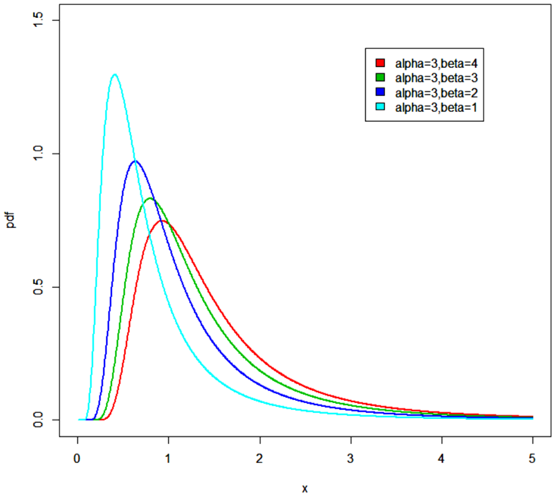


**Figure 1: The Graph of Density Function**

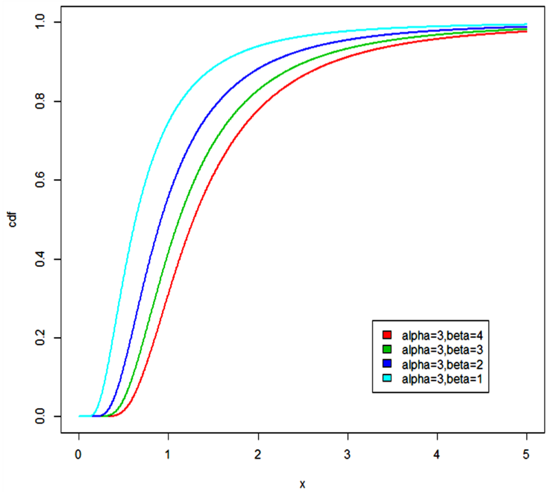


**Figure 2: The Cumulative Distribution Function**

## 2. SPECIAL CASES

**Case 1:**

When $\alpha=\beta=1$, then (EGSIE) distribution (1.1) reduces to the (SIED) with pdf as:

$$f(x) \quad = \frac{1}{x^2} e^{\frac{-1}{x}}; \quad x>0. \tag{2.1}$$

**Case 2:**

When $\beta=1$, then (EGSIE) distribution (1.1) reduces to Generalized Standard Inverse Exponential distribution (GSIED) with pdf as:

$$f(x;\alpha) = \frac{\alpha}{x^2} e^{\frac{-1}{x}} \left\{1-e^{\frac{-1}{x}}\right\}^{\alpha-1} \quad ; \; x>0, \alpha>0. \tag{2.2}$$

## 3. RELIABILITY ANALYSIS

i) R**eliability Function R(x; *α*, β)**

The EGSIED can be a useful characterization of the survival time of a given system because of it analytical structure. The reliability function is given by $R(x) = (1-F(x;\alpha,\beta))$. Thus using (1.2),

$$R(x;\alpha,\beta)=1-\left[1-\left\{1-e^{\frac{-1}{x}}\right\}^{\alpha}\right]^{\beta}, \; x>0, \alpha,\beta>0. \tag{3.1}$$

ii) H**azard Function H(x; *α*, β)**

Another characteristics of interest of a random variable is the hazard function defined by

$$H(x;\alpha,\beta)=\frac{f(x;\alpha,\beta)}{R(x;\alpha,\beta)}$$

Thus using (1.1) and (3.1), the hazard function is given by

$$H(x;\alpha,\beta)=\frac{\frac{\alpha\beta}{x^2} e^{\frac{-1}{x}} \left\{1-e^{\frac{-1}{x}}\right\}^{\alpha-1} \left[1-\left\{1-e^{\frac{-1}{x}}\right\}^{\alpha}\right]^{\beta-1}}{1-\left[1-\left\{1-e^{\frac{-1}{x}}\right\}^{\alpha}\right]^{\beta}}; x>0, \alpha,\beta>0. \tag{3.2}$$

iii) R**everse Hazard Function** $\varphi(x;\alpha,\beta)$

The reverse hazard function can be interpreted as an approximate probability of failure in $[x, x+d]$, given that the failure had occurred in $[0,x]$. The reverse hazard function $\varphi(x;\alpha,\beta)$ is defined to be

$$\varphi(x\,;\alpha,\beta)=\frac{f(x\;;\alpha,\beta)}{F(x\,;\alpha,\beta)}=\frac{\dfrac{\alpha\beta}{x^{2}}e^{\frac{-1}{x}}\left\{1-e^{\frac{-1}{x}}\right\}^{\alpha-1}}{\left[1-\left\{1-e^{\frac{-1}{x}}\right\}^{\alpha}\right]};\; x>0,\alpha,\beta>0\,. \tag{3.3}$$

Figure (3), Figure (4) and Figure (5) represents reliability, hazard and reverse hazard functions respectively.

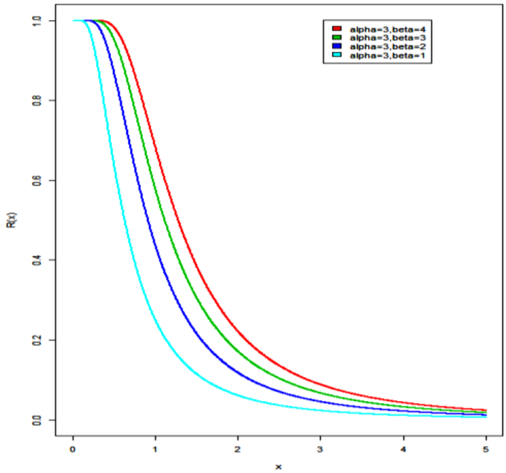


**Figure 3: The Graph of Reliability Function**

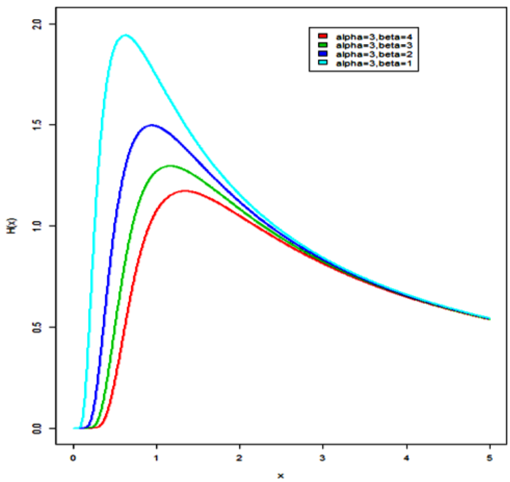


**Figure 4: The Graph of Hazard Function**

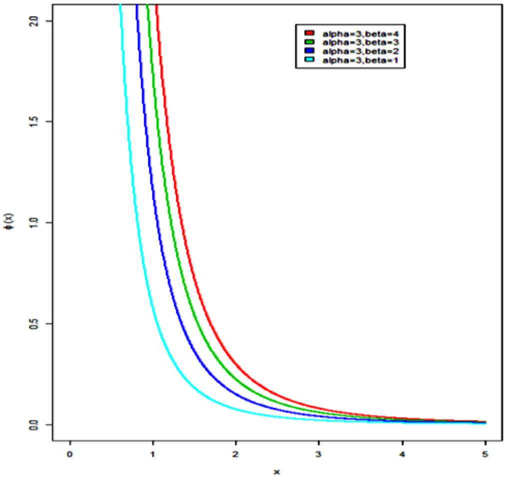


**Figure 5: The Graph of Reverse Hazard Function**

## 4. STATISTICAL PROPERTIES OF THE EGSIE DISTRIBUTION

This section provides several properties of the EGSIED.

### 4.1 Moments

The r-th moment for the EGSIED can be obtained as;

$$\mu_r = E(X^r) = \int_0^\infty x^r f(x;\alpha,\beta)dx . \tag{4.1}$$

By solving the above equation, we get:

$$\mu_r = A_{jk}\alpha\beta\Gamma(1-r)(k+1)^{(r-1)} , \tag{4.2}$$

where $A_{jk} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{j+k}\Gamma(\beta)\Gamma\alpha(j+1)}{\Gamma(\beta-j)\Gamma(\alpha(j+1)-k)j!k!}$.

We observe that Equation (4.4) only exists when $r<1$. The implication is that the first moment, second moment and other higher-order moments does not exist.

### 4.2 Harmonic Mean of EGSIE Distribution

The harmonic mean (H) is given as:

$$\frac{1}{H} = E\left(\frac{1}{X}\right) = \int_0^\infty \frac{1}{x} f(x;\alpha,\beta)dx$$

$$\frac{1}{H} = \alpha\beta\int_0^\infty \frac{1}{x^{2+1}} e^{\frac{-1}{x}} \left\{1-e^{\frac{-1}{x}}\right\}^{\alpha-1} \left[1-\left\{1-e^{\frac{-1}{x}}\right\}^{\alpha}\right]^{\beta-1} dx . \tag{4.3}$$

Using the expansions of

$$\left[1-\left\{1-e^{\frac{-1}{x}}\right\}^{\alpha}\right]^{\beta-1}=\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(\beta)}{\Gamma(\beta-j)\,j!}\left\{1-e^{\frac{-1}{x}}\right\}^{\alpha j}$$

$$\&\ \left\{1-e^{\frac{-1}{x}}\right\}^{\alpha(j+1)-1}=\sum_{k=0}^{\infty}\frac{(-1)^{k}\Gamma\alpha(j+1)}{\Gamma(\alpha(j+1)-k)k!}e^{\frac{-k}{x}}.$$

Expression (4.3) takes the following form:

$$\frac{1}{H}=A_{jk}\alpha\beta\int_{0}^{\infty}\frac{1}{x^{2+1}}\,e^{\frac{-(k+1)}{x}}dx\ ;\ \text{where}\ A_{jk}=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k}\Gamma(\beta)\Gamma\alpha(j+1)}{\Gamma(\beta-j)\Gamma(\alpha(j+1)-k)\,j!k!}.$$

On solving the above equation, we get

$$\frac{1}{H}=A_{jk}\alpha\beta\,\frac{1}{(k+1)^{2}}\quad\Rightarrow\quad H=\left(A_{jk}\alpha\beta\,\frac{1}{(k+1)^{2}}\right)^{-1}. \tag{4.4}$$

## 4.5 Moment Generating Function and Characteristic Function

The Moment generating function can be derived as:

$$M_X(t)=E(e^{tx})=\int_{0}^{\infty}e^{tx}f(x;\alpha,\beta)dx.$$

$$\Rightarrow\qquad M_X(t)=\sum_{r=0}^{\infty}\frac{t^{r}}{r!}A_{jk}\alpha\beta\,\Gamma(1-r)(k+1)^{(r-1)},$$

where $A_{jk}=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k}\Gamma(\beta)\Gamma\alpha(j+1)}{\Gamma(\beta-j)\Gamma(\alpha(j+1)-k)\,j!k!}$. (4.5)

The characteristic function can be derived as:

$$\varphi_X(t)=E(e^{itx})=\int_{0}^{\infty}e^{itx}f(x;\alpha,\beta)dx$$

$$\Rightarrow\qquad \varphi_X(t)=\sum_{r=0}^{\infty}\frac{(it)^{r}}{r!}A_{jk}\alpha\beta\,\Gamma(1-r)(k+1)^{(r-1)}. \tag{4.6}$$

## 4.6 Quantile Function and Median

Quantile function is defined as;

$$Q(u)=F^{-1}(u).$$

Hence, the quantile function for the EGSIED can be obtained as;

$$Q(u) = \left[ -\log\left[ 1 - \left(1 - u^{1/\beta}\right)^{1/\alpha} \right]\right]^{-1}. \tag{4.7}$$

Therefore, the median of the EGSIED is derived by putting $u = 0.5$ into Equation (4.7). That is,

$$\text{Median} = \left[ -\log\left[ 1 - (1 - 0.5^{1/\beta})^{1/\alpha} \right]\right]^{-1}. \tag{4.8}$$

## 5. MAXIMUM LIKELIHOOD ESTIMATION FOR THE SHAPE PARAMETER $\boldsymbol{\beta}$:

Consider a random sample of independent observations $x_1, ..., x_n$ from (1.1), and then the likelihood function is given by:

$$L(\underline{x} \mid \beta) = \alpha^n \beta^n \prod_{i=1}^{n} \frac{1}{x_i^2} e^{-\frac{1}{x_i}} \left(1 - e^{-\frac{1}{x_i}}\right)^{\alpha-1} \left[ 1 - \left\{ 1 - e^{\frac{-1}{x}} \right\}^{\alpha} \right]^{\beta-1} \tag{5.1}$$

$$\ln L(\underline{x} \mid \beta) = n \ln \alpha + n \ln \beta - 2\sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} \left(\frac{1}{x_i}\right) +$$

$$(\alpha - 1)\sum_{i=1}^{n} \ln\left(1 - e^{-\frac{1}{x_i}}\right) + (\beta - 1)\sum_{i=1}^{n} \ln\left[ 1 - \left\{ 1 - e^{-\frac{1}{x_i}} \right\}^{\alpha} \right]. \tag{5.2}$$

Now maximizing the likelihood function or log likelihood function the MLE of $\beta$ can be obtained as:

$$\frac{\partial}{\partial \beta} \ln L(\underline{x} \mid \beta) = \frac{n}{\beta} + \sum_{i=1}^{n} \ln\left[ 1 - \left\{ 1 - e^{-\frac{1}{x_i}} \right\}^{\alpha} \right] = 0.$$

$$\hat{\beta} = \frac{n}{T} \quad ; \quad \textit{where } T = \sum_{i=1}^{n} \ln\left[ 1 - \left\{ 1 - e^{-\frac{1}{x_i}} \right\}^{\alpha} \right]^{-1}. \tag{5.3}$$

## 6. THE POSTERIOR DISTRIBUTION OF UNKNOWN PARAMETER Β OF EXPONENTIATED GENERALIZED STANDARD INVERSE EXPONENTIAL (EGIE) DISTRIBUTION USING NON-INFORMATIVE PRIORS

Bayesian analysis is performed by combining the prior information $g(\beta)$ and the sample information $(x_1, x_2, \ldots, x_n)$ into what is called the posterior distribution of $\beta$ given $\underline{x} = x_1, x_2, \ldots, x_n$ from which all decisions and inferences are made. So $\pi(\beta \mid \underline{x})$ reflects the updated beliefs about $\beta$ after observing the sample $\underline{x} = x_1, x_2, \ldots, x_n$.

The posterior distributions using non-informative priors for the unknown parameter $\beta$ of an exponentiated generalized inverse exponential distribution are derived below:

### 6.1 Posterior Distribution using Uniform Prior

The uniform prior of $\beta$ is defined as:

$$g(\beta) \propto 1\,, \quad 0 < x < \infty. \tag{6.1}$$

Using the likelihood (5.1) and the prior (6.1), the posterior density of $\beta$ is given by

$$\pi(\beta \mid \underline{x}) \propto \alpha^n \beta^n \prod_{i=1}^{n} \frac{1}{x_i^2} e^{-\frac{1}{x_i}} \left(1 - e^{-\frac{1}{x_i}}\right)^{\alpha-1} \left[1 - \left\{1 - e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{\beta-1} \times 1$$

$$\pi(\beta \mid \underline{x}) = k \beta^n e^{-\beta T_1} d\beta\,, \tag{6.2}$$

where k is independent of $\beta$ and $T_1 = \sum_{i=1}^{n} \ln \left[1 - \left\{1 - e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}$.

$$k^{-1} = \int_0^{\infty} \beta^n e^{-\beta T_1} d\beta \Rightarrow k^{-1} = \frac{\Gamma(n+1)}{T_1^{n+1}}.$$

Therefore from (6.2) we have:

$$\pi_1(\beta \mid \underline{x}) = \frac{T_1^{n+1}}{\Gamma(n+1)} \beta^n e^{-\beta T_1} \quad ; \qquad \beta > 0. \tag{6.3}$$

The posterior distribution of $(\beta \mid \underline{x}) \sim G(\alpha_1, T_1)$

where $\alpha_1 = (n+1)$ & $T_1 = \sum_{i=1}^{n} \ln \left[1 - \left\{1 - e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}$.

**6.2 Posterior Distribution using the Jeffrey's Prior**

A general rule proposed by Jeffrey prescribes $g(\beta) \propto \sqrt{I(\beta)}$ where $I(\beta)$ the well-known Fisher's information matrix is.

$$I(\beta) = -nE\left[\frac{\partial^2}{\partial\beta^2}\log f(x;\alpha,\beta)\right]$$

$$g_2(\beta) \propto \sqrt{|I(\beta)|}\,.$$

The Jeffrey's prior for the shape parameter $\beta$ of the EGIE distribution is derived which is:

$$g_2(\beta) \propto \frac{1}{\beta}, \quad 0 < \beta < \infty\,. \tag{6.4}$$

Using the likelihood (5.1) and the prior (6.4), the posterior density of $\beta$ is given by:

$$\pi(\beta\,|\,\underline{x}) \propto \alpha^n\beta^n\prod_{i=1}^{n}\frac{1}{x_i^2}e^{-\frac{1}{x_i}}\left(1-e^{-\frac{1}{x_i}}\right)^{\alpha-1}\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{\beta-1}\times\frac{1}{\beta}$$

$$\pi(\beta\,|\,\underline{x}) = k\beta^{n-1}e^{-\beta T_2}\,d\beta\,. \tag{6.5}$$

where k is independent of $\beta$ and $T_2 = \sum_{i=1}^{n}\ln\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}$.

$$k^{-1} = \int_0^{\infty}\beta^{n-1}e^{-\beta T_2}\,d\beta \Rightarrow k^{-1} = \frac{\Gamma(n)}{T_2^{\,n}}\,.$$

Therefore from (6.5) we have:

$$\pi_2(\beta\,|\,\underline{x}) = \frac{T_2^{\,n}}{\Gamma(n)}\beta^{n-1}e^{-\beta T_2} \qquad ;\beta > 0\,. \tag{6.6}$$

The posterior distribution of $(\beta\,|\,\underline{x}) \sim G(\alpha_2, T_2)$

where $\alpha_2 = (n)$ & $T_2 = \sum_{i=1}^{n}\ln\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}$.

## 7. THE POSTERIOR DISTRIBUTION OF UNKNOWN PARAMETER Β OF EXPONENTIATED GENERALIZED STANDARD INVERTED EXPONENTIAL (EGIE) DISTRIBUTION USING INFORMATIVE PRIORS

Here, we use gamma and Chi-square distribution as informative priors.

### 7.1 Posterior Distribution using Gamma Prior

It is assumed that the prior distribution of $\beta$ is the gamma distribution with hyper parameters $a$ and $b$, which is given as:

$$g_3(\beta) \propto \frac{b^a}{\Gamma(a)}\beta^{a-1}e^{-b\beta}, \quad 0<\beta<\infty\,, a,b>0 \quad . \tag{7.1}$$

Using the likelihood (5.1) and the prior (7.1), the posterior density of $\beta$ is given by:

$$\pi(\beta\,|\,\underline{x}) \propto \alpha^n\beta^n\prod_{i=1}^{n}\frac{1}{x_i^2}e^{-\frac{1}{x_i}}\left(1-e^{-\frac{1}{x_i}}\right)^{\alpha-1}\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{\beta-1}\times\frac{b^a}{\Gamma(a)}\beta^{a-1}e^{-b\beta}$$

$$\pi(\beta\,|\,\underline{x}) = k\beta^{n+a-1}e^{-\beta T_3}d\beta\,, \tag{7.2}$$

where k is independent of $\beta$ and $T_3 = \left\{b+\sum_{i=1}^{n}\ln\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}\right\}$.

$$k^{-1} = \int_0^\infty \beta^{n+a-1}e^{-\beta T_3}d\beta \Rightarrow \qquad k^{-1} = \frac{\Gamma(n+a)}{T_3^{(n+a)}}.$$

Therefore from (7.2) we have:

$$\pi_3(\beta\,|\,\underline{x}) = \frac{T_3^{\,n+a}}{\Gamma(n+a)}\beta^{n+a-1}e^{-\beta T_3} \quad ; \quad \beta>0. \tag{7.3}$$

The posterior distribution of $(\beta\,|\,\underline{x}) \sim G(\alpha_3, T_3)$

where $\alpha_3 = (n+a)$ and $T_3 = \left\{b+\sum_{i=1}^{n}\ln\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}\right\}$.

**Remark 1**:

1. For $a=b=0$ in (7.3), the posterior distribution under the gamma prior reduces to posterior distribution under the Jeffrey's prior.
2. For $a=1, b=0$ in (7.3), the posterior distribution under the gamma prior reduces to posterior distribution under the Uniform prior.

**7.2 Posterior Distribution using Chi-square Prior:**

Another informative prior is assumed to be the Chi-square prior with hyper parameter $a_2$, which is given as:

$$g_4(\beta) \propto \frac{1}{\Gamma(a_2/2)2^{a_2/2}} \beta^{\frac{a_2}{2}-1} e^{\frac{-\beta}{2}}, \quad 0<\beta<\infty, a_2,>0 \ . \tag{7.4}$$

Using the likelihood (5.1) and the prior (7.4), the posterior density of $\beta$ is given by:

$$\pi(\beta\,|\,\underline{x}) \propto \alpha^n \beta^n \prod_{i=1}^{n} \frac{1}{x_i^2} e^{-\frac{1}{x_i}} \left(1-e^{-\frac{1}{x_i}}\right)^{\alpha-1} \left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{\beta-1} \times \frac{1}{\Gamma(a_2/2)2^{a_2/2}} \beta^{\frac{a_2}{2}-1} e^{\frac{-\beta}{2}}$$

$$\pi(\beta\,|\,\underline{x}) = k\beta^{n+\frac{a_2}{2}-1} e^{-\beta T_4} d\beta, \tag{7.5}$$

where k is independent of $\beta$ and $T_4 = \left\{\frac{1}{2} + \sum_{i=1}^{n} \ln\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}\right\}$.

$$k^{-1} = \int_0^\infty \beta^{n+\frac{a_2}{2}-1} e^{-\beta T_4} d\beta \Rightarrow k^{-1} = \frac{\Gamma\left(n+\frac{a_2}{2}\right)}{T_4^{\left(n+\frac{a_2}{2}\right)}} .$$

Therefore from (7.5) we have:

$$\pi_4(\beta\,|\,\underline{x}) = \frac{T_4^{\,n+\frac{a_2}{2}}}{\Gamma\left(n+\frac{a_2}{2}\right)} \beta^{n+\frac{a_2}{2}-1} e^{-\beta T_4} \quad ; \beta>0 \, . \tag{7.6}$$

The posterior distribution of $(\beta\,|\,\underline{x}) \sim G(\alpha_3, T_4)$

where $\alpha_4 = \left(n+\frac{a_2}{2}\right)$ and $T_4 = \left\{\frac{1}{2} + \sum_{i=1}^{n} \ln\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}\right\}$.

## 8. ASSUMPTION OF PRIORS RELATING TO POSTERIOR VARIANCES

The variances of the posterior distribution among the priors can be obtained as:

$$V(\beta \mid X) = \frac{\alpha_i}{T_i^2}; i = 1, 2, 3, 4. \tag{8.1}$$

## 9. BAYESIAN ESTIMATION BY USING UNIFORM PRIOR UNDER DIFFERENT LOSS FUNCTIONS

**Theorem 9.1**

Assuming the loss function $L_q(\hat{\beta}, \beta)$, the Bayesian estimator of $\beta$, takes the following form:

$$\hat{\beta}_{1q} = \frac{(n-1)}{T_1}, \ \mathrm{T}_1 = \sum_{i=1}^{n} \ln\left[1 - \left\{1 - e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}.$$

**Proof:**

The risk function of the estimator $\beta$ with quadratic loss function $L_q(\hat{\beta}, \beta)$ is using the formula:

$$R(\hat{\beta}) = \int_0^{\infty} \left(\frac{\beta - \hat{\beta}}{\beta}\right)^2 \pi_1(\beta \mid \underline{x})\, d\beta \tag{9.1}$$

On substituting (6.3) in (9.1), we have:

$$R(\hat{\beta}) = \frac{T_1^{n+1}}{\Gamma(n+1)}\left[\int_0^{\infty} \beta^{n+1-1} e^{-\beta T_1} d\beta + \hat{\beta}^2 \int_0^{\infty} \beta^{n-1-1} e^{-\beta T_1} d\beta - 2\hat{\beta} \int_0^{\infty} \beta^{n-1} e^{-\beta T_1} d\beta\right]. \tag{9.2}$$

On solving (9.2), we get:

$$R(\hat{\beta}) = 1 + \frac{\hat{\beta}^2 T_1^2}{n(n-1)} - \frac{2\hat{\beta} T_1}{n}. \tag{9.3}$$

Minimization of the risk with respect to $\hat{\beta}$ gives us the optimal estimator

$$\hat{\beta}_{1q} = \frac{(n-1)}{T_1}, \quad \mathrm{T}_1 = \sum_{i=1}^{n} \ln\left[1 - \left\{1 - e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}. \tag{9.4}$$

**Theorem 9.2**

Assuming the loss function $L_l(\hat{\beta},\beta)$, the Bayesian estimator of $\beta$, takes the following form:

$$\hat{\beta}_{1l} = \frac{1}{c_1}\ln\left(\frac{c_1}{T_1}+1\right)^{n+1}, \quad T_1 = \sum_{i=1}^{n}\ln\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}$$

**Proof:**

The risk function of the $\beta$ with linex loss function $L_l(\hat{\beta},\beta)$ is using the formula:

$$R(\hat{\beta}) = \int_0^{\infty}\left[\exp\left\{c_1\left(\hat{\beta}-\beta\right)\right\} - c_1\left(\hat{\beta}-\beta\right)-1\right]\pi_1(\beta \mid \underline{x})\,d\beta \tag{9.5}$$

On substituting (6.3) in (9.5), we have

$$R(\hat{\beta}) = \frac{T_1^{n+1}}{\Gamma(n+1)}\left[\begin{array}{l} e^{c_1\hat{\beta}}\int_0^{\infty}\beta^{n+1-1}e^{-\beta(c_1+T_1)}d\beta - c_1\hat{\beta}\int_0^{\infty}\beta^{n+1-1}e^{-\beta T_1}d\beta \\ \qquad + c_1\int_0^{\infty}\beta^{n+2-1}e^{-\beta T_1}d\beta - \int_0^{\infty}\beta^{n+1-1}e^{-\beta T_1}d\beta \end{array}\right] \tag{9.6}$$

On solving (9.6), we get:

$$R(\hat{\beta}) = e^{c_1\hat{\beta}}\left(\frac{T_1}{c_1+T_1}\right)^{n+1} - c_1\hat{\beta} + \frac{c_1(n+1)}{T_1} - 1\,. \tag{9.7}$$

Minimization of the risk with respect to $\hat{\beta}$ gives us the optimal estimator

$$\hat{\beta}_{1l} = \frac{1}{c_1}\ln\left(\frac{c_1}{T_1}+1\right)^{n+1}, \; T_1 = \sum_{i=1}^{n}\ln\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}. \tag{9.8}$$

**Theorem 9.3**

Assuming the loss function $L_p(\hat{\beta},\beta)$, the Bayesian estimator of $\beta$, takes the following form:

$$\hat{\beta}_{1p} = \frac{\left[(n+2)(n+1)\right]^{1/2}}{T_1}, \; T_1 = \sum_{i=1}^{n}\ln\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}.$$

**Proof:**

The risk function of the estimator $\beta$ under the precautionary loss function $L_p(\hat{\beta},\beta)$ is using the formula:

$$R(\hat{\beta}) = \int_0^\infty \frac{(\hat{\beta}-\beta)^2}{\hat{\beta}} \pi_1(\beta \mid \underline{x})\, d\beta \tag{9.9}$$

By substituting (6.3) in (9.9), we have:

$$R(\hat{\beta}) = \frac{T_1^{n+1}}{\Gamma(n+1)} \left[ \hat{\beta}\int_0^\infty \beta^{n+1-1} e^{-\beta T_1} d\beta + \frac{1}{\hat{\beta}} \int_0^\infty \beta^{n+3-1} e^{-\beta T_1} d\beta - 2\hat{\beta}\int_0^\infty \beta^{n+2-1} e^{-\beta T_1} d\beta \right]. \tag{9.10}$$

By solving (9.10), we get:

$$R(\hat{\beta}) = \hat{\beta} + \frac{(n+2)(n+1)}{\hat{\beta} T_1^2} - \frac{2(n+1)}{T_1}. \tag{9.11}$$

Minimization of the risk with respect to $\hat{\beta}$ gives us the optimal estimator

$$\hat{\beta}_{1p} = \frac{\left[(n+2)(n+1)\right]^{1/2}}{T_1} \quad , \mathrm{T}_1 = \sum_{i=1}^{n} \ln \left[ 1 - \left\{ 1 - e^{\frac{-1}{x_i}} \right\}^{\alpha} \right]^{-1}. \tag{9.12}$$

## 10. BAYESIAN ESTIMATION BY USING JEFFREY'S PRIOR UNDER DIFFERENT LOSS FUNCTIONS

**Theorem 10.1**

Assuming the loss function $L_q(\hat{\beta}, \beta)$, the Bayesian estimator of $\beta$, takes the following form:

$$\hat{\beta}_{2q} = \frac{(n-2)}{T_2}, \quad \mathrm{T}_2 = \sum_{i=1}^{n} \ln \left[ 1 - \left\{ 1 - e^{\frac{-1}{x_i}} \right\}^{\alpha} \right]^{-1}.$$

**Proof:**

The risk function of the estimator $\beta$ with quadratic loss function $L_q(\hat{\beta}, \beta)$ is using the formula:

$$R(\hat{\beta}) = \int_0^\infty \left( \frac{\beta - \hat{\beta}}{\beta} \right)^2 \pi_2(\beta \mid \underline{x})\, d\beta \tag{10.1}$$

Using (6.6) in (10.1), we have:

$$R(\hat{\beta}) = \frac{T_2^{\,n}}{\Gamma(n)} \left[ \int_0^\infty \beta^{n-1} e^{-\beta T_2} d\beta \ + \hat{\beta}^2 \int_0^\infty \beta^{n-2-1} e^{-\beta T_2} d\beta - 2\hat{\beta} \int_0^\infty \beta^{n-1-1} e^{-\beta T_2} d\beta \right]. \tag{10.2}$$

By solving (10.2), we get:

$$R(\hat{\beta}) = 1 + \frac{\hat{\beta}^2 {T_2}^2}{(n-1)(n-2)} - \frac{2\hat{\beta} T_2}{(n-1)} . \tag{10.3}$$

Minimization of the risk with respect to $\hat{\beta}$ gives us the optimal estimator

$$\hat{\beta}_{2q} = \frac{(n-2)}{T_2} \ , \ \mathrm{T}_2 = \sum_{i=1}^{n} \ln\left[1 - \left\{1 - e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1} . \tag{10.4}$$

**Theorem 10.2**

Assuming the loss function $L_l(\hat{\beta}, \beta)$, the Bayesian estimator of $\beta$, takes the following form:

$$\hat{\beta}_{2l} = \frac{1}{c_1} \ln\left(\frac{c_1}{T_2} + 1\right)^{n}, \mathrm{T}_2 = \sum_{i=1}^{n} \ln\left[1 - \left\{1 - e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1} .$$

**Proof:**

The risk function of $\beta$ with linex loss function $L_l(\hat{\beta}, \beta)$ using the formula

$$R(\hat{\beta}) = \int_0^\infty \left[\exp\left\{c_1\left(\hat{\beta} - \beta\right)\right\} - c_1\left(\hat{\beta} - \beta\right) - 1\right] \pi_2(\beta \mid \underline{x})\, d\beta . \tag{10.5}$$

Using (6.6) in (10.5), we have:

$$R(\hat{\beta}) = \frac{{T_2}^n}{\Gamma(n)} \left[\begin{array}{l} e^{c_1\hat{\beta}} \int_0^\infty \beta^{n-1} e^{-\beta(c_1+T_2)} d\beta - c_1\hat{\beta} \int_0^\infty \beta^{n-1} e^{-\beta T_2} d\beta \\ \qquad + c_1 \int_0^\infty \beta^{n+1-1} e^{-\beta T_2} d\beta - \int_0^\infty \beta^{n-1} e^{-\beta T_2} d\beta \end{array}\right] \tag{10.6}$$

By solving (10.6), we get:

$$R(\hat{\beta}) = e^{c_1\hat{\beta}} \left(\frac{T_2}{c_1 + T_2}\right)^{n} - c_1\hat{\beta} + \frac{c_1(n)}{T_2} - 1 . \tag{10.7}$$

Minimization of the risk with respect to $\hat{\beta}$ gives us the optimal estimator

$$\hat{\beta}_{2l} = \frac{1}{c_1} \ln\left(\frac{c_1}{T_2} + 1\right)^{n} \ , \ \mathrm{T}_2 = \sum_{i=1}^{n} \ln\left[1 - \left\{1 - e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1} . \tag{10.8}$$

**Theorem 10.3**

Assuming the loss function $L_p(\hat{\beta},\beta)$, the Bayesian estimator of $\beta$, takes the following form:

$$\hat{\beta}_{2p} = \frac{\sqrt{(n+1)(n)}}{T_2} \quad, \mathrm{T}_2 = \sum_{i=1}^{n} \ln\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}.$$

**Proof:**

The risk function of $\beta$ under the precautionary loss function $L_p(\hat{\beta},\beta)$ is using the formula:

$$R(\hat{\beta}) = \int_0^{\infty} \frac{(\hat{\beta}-\beta)^2}{\hat{\beta}} \pi_2(\beta \mid \underline{x})\, d\beta . \tag{10.9}$$

Using (6.6) in (10.9), we have:

$$R(\hat{\beta}) = \frac{T_2{}^n}{\Gamma(n)}\left[\hat{\beta}\int_0^{\infty} \beta^{n-1} e^{-\beta T_2} d\beta + \frac{1}{\hat{\beta}}\int_0^{\infty} \beta^{n+2-1} e^{-\beta T_2} d\beta - 2\hat{\beta}\int_0^{\infty} \beta^{n+1-1} e^{-\beta T_2} d\beta\right] \tag{10.10}$$

By solving (10.10), we get

$$R(\hat{\beta}) = \hat{\beta} + \frac{(n+1)(n)}{\hat{\beta} T_2{}^2} - \frac{2(n)}{T_2} . \tag{10.11}$$

Minimization of the risk with respect to $\hat{\beta}$ gives us the optimal estimator

$$\hat{\beta}_{2p} = \frac{\sqrt{(n+1)(n)}}{T_2} \quad, \quad \mathrm{T}_2 = \sum_{i=1}^{n} \ln\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}. \tag{10.12}$$

## 11. BAYESIAN ESTIMATION BY USING GAMMA PRIOR UNDER DIFFERENT LOSS FUNCTIONS

**Theorem 11.1**

Assuming the loss function $L_q(\hat{\beta},\beta)$, the Bayesian estimator of $\beta$, takes the following form:

$$\hat{\beta}_{3q} = \frac{(n+a-2)}{T_3}, \; T_3 = \left\{ b + \sum_{i=1}^{n} \ln\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1} \right\}.$$

**Proof:**

The risk function of the estimator $\beta$ with quadratic loss function $L_q(\hat{\beta},\beta)$ is using the formula:

$$R(\hat{\beta})=\int_0^{\infty}\left(\frac{\beta-\hat{\beta}}{\beta}\right)^2\pi_3(\beta\,|\,\underline{x})\,d\beta \tag{11.1}$$

Using (7.3) in (11.1), we have:

$$R(\hat{\beta})=\frac{T_3^{\,n+a}}{\Gamma(n+a)}\left[\int_0^{\infty}\beta^{n+a-1}e^{-\beta T_3}d\beta\ +\hat{\beta}^2\int_0^{\infty}\beta^{n+a-2-1}e^{-\beta T_3}d\beta-2\hat{\beta}\int_0^{\infty}\beta^{n+a-1-1}e^{-\beta T_3}d\beta\right]. \tag{11.2}$$

By solving (11.2), we get

$$R(\hat{\beta})=1+\frac{\hat{\beta}^2T_3^{\,2}}{(n+a-1)(n+a-2)}-\frac{2\hat{\beta}T_3}{(n+a-1)}. \tag{11.3}$$

Minimization of the risk with respect to $\hat{\beta}$ gives us the optimal estimator

$$\hat{\beta}_{3q}=\frac{(n+a-2)}{T_3}\quad ,T_3=\left\{b+\sum_{i=1}^{n}\ln\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}\right\}. \tag{11.4}$$

**Theorem 11.2**

Assuming the loss function $L_l(\hat{\beta},\beta)$, the Bayesian estimator of $\beta$, takes the following form:

$$\hat{\beta}_{3l}=\frac{1}{c_1}\ln\left(\frac{c_1}{T_3}+1\right)^{n+a},\ T_3=\left\{b+\sum_{i=1}^{n}\ln\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}\right\}.$$

**Proof:**

The risk function of $\beta$ with linex loss function $L_l(\hat{\beta},\beta)$ is using the formula:

$$R(\hat{\beta})=\int_0^{\infty}\left[\exp\left\{c_1\left(\hat{\beta}-\beta\right)\right\}-c_1\left(\hat{\beta}-\beta\right)-1\right]\pi_3(\beta\,|\,\underline{x})\,d\beta \tag{11.5}$$

Using (7.3) in (11.5), we have:

$$R(\hat{\beta})=\frac{T_3^{\,n+a}}{\Gamma(n+a)}\left[\begin{array}{l} e^{c_1\hat{\beta}}\int_0^\infty \beta^{n+a-1}e^{-\beta(c_1+T_3)}d\beta-c_1\hat{\beta}\int_0^\infty \beta^{n+a-1}e^{-\beta T_3}d\beta+ \\ \qquad c_1\int_0^\infty \beta^{n+a+1-1}e^{-\beta T_3}d\beta-\int_0^\infty \beta^{n+a-1}e^{-\beta T_3}d\beta \end{array}\right]. \tag{11.6}$$

By solving (11.6), we get:

$$R(\hat{\beta})=e^{c_1\hat{\beta}}\left(\frac{T_3}{c_1+T_3}\right)^{n+a}-c_1\hat{\beta}+\frac{c_1(n+a)}{T_3}-1. \tag{11.7}$$

Minimization of the risk with respect to $\hat{\beta}$ gives us the optimal estimator

$$\hat{\beta}_{3l}=\frac{1}{c_1}\ln\left(\frac{c_1}{T_3}+1\right)^{n+a},T_3=\left\{b+\sum_{i=1}^{n}\ln\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}\right\}. \tag{11.8}$$

**Theorem 11.3**

Assuming the loss function $L_p(\hat{\beta},\beta)$, the Bayesian estimator of $\beta$, takes the following form:

$$\hat{\beta}_{3p}=\frac{\sqrt{(n+a)(n+a+1)}}{T_3},T_3=\left\{b+\sum_{i=1}^{n}\ln\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}\right\}.$$

**Proof:**

The risk function of $\beta$ with precautionary loss function $L_p(\hat{\beta},\beta)$ is using the formula:

$$R(\hat{\beta})=\int_0^\infty \frac{(\hat{\beta}-\beta)^2}{\hat{\beta}}\pi_3(\beta\,|\,\underline{x})\,d\beta \tag{11.9}$$

Using (7.3) in (11.9), we have:

$$R(\hat{\beta})=\frac{T_3^{\,n+a}}{\Gamma(n+a)}\left[\hat{\beta}\int_0^\infty \beta^{n+a-1}e^{-\beta T_3}d\beta+\frac{1}{\hat{\beta}}\int_0^\infty \beta^{n+a+2-1}e^{-\beta T_3}d\beta-2\hat{\beta}\int_0^\infty \beta^{n+a+1-1}e^{-\beta T_3}d\beta\right] \tag{11.10}$$

By solving (11.10), we get:

$$R(\hat{\beta})=\hat{\beta}+\frac{(n+a+1)(n+a)}{\hat{\beta}T_3^{\,2}}-\frac{2(n+a)}{T_3} \tag{11.11}$$

Minimization of the risk with respect to $\hat{\beta}$ gives us the optimal estimator

$$\hat{\beta}_{3p}=\frac{\sqrt{(n+a+1)(n+a)}}{T_3}\quad ,T_3=\left\{b+\sum_{3\,i=1}^{n}\ln\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}\right\} \tag{11.12}$$

## 12. BAYESIAN ESTIMATION BY USING CHI-SQUARE PRIOR UNDER DIFFERENT LOSS FUNCTIONS

**Theorem 12.1**

Assuming the loss function $L_q(\hat{\beta},\beta)$, the Bayesian estimator of $\beta$, takes the following form:

$$\hat{\beta}_{4q}=\frac{\left(n+\frac{a_2}{2}-2\right)}{T_4},T_4=\left\{\frac{1}{2}+\sum_{i=1}^{n}\ln\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}\right\}.$$

**Proof:**

The risk function of the estimator $\beta$ under the quadratic loss function $L_q(\hat{\beta},\beta)$ is using the formula:

$$R(\hat{\beta})=\int_0^{\infty}\left(\frac{\beta-\hat{\beta}}{\beta}\right)^2\pi_4(\beta\,|\,\underline{x})\,d\beta\,. \tag{12.1}$$

Using (7.6) in (12.1), we have:

$$R(\hat{\beta})=\frac{T_4^{\,n+\frac{a_2}{2}}}{\Gamma\left(n+\frac{a_2}{2}\right)}\left[\begin{array}{l}\int_0^{\infty}\beta^{n+\frac{a_2}{2}-1}e^{-\beta T_4}\,d\beta\;+\hat{\beta}^2\int_0^{\infty}\beta^{n+\frac{a_2}{2}-2-1}e^{-\beta T_4}\,d\beta\\ \qquad\qquad\qquad -2\hat{\beta}\int_0^{\infty}\beta^{n+\frac{a_2}{2}-1-1}e^{-\beta T_4}\,d\beta\end{array}\right]. \tag{12.2}$$

By solving (12.2), we get:

$$R(\hat{\beta})=1+\frac{\hat{\beta}^2T_4^{\,2}}{\left(n+\frac{a_2}{2}-1\right)\left(n+\frac{a_2}{2}-2\right)}-\frac{2\hat{\beta}T_4}{\left(n+\frac{a_2}{2}-1\right)}\,. \tag{12.3}$$

Minimization of the risk with respect to $\hat{\beta}$ gives us the optimal estimator

$$\hat{\beta}_{4q} = \frac{(n+\frac{a_2}{2}-2)}{T_4} \quad T_4 = \left\{ \frac{1}{2} + \sum_{i=1}^{n} \ln\left[ 1 - \left\{ 1 - e^{\frac{-1}{x_i}} \right\}^{\alpha} \right]^{-1} \right\}. \tag{12.4}$$

**Theorem 12.2**

Assuming the loss function $L_l(\hat{\beta}, \beta)$, the Bayesian estimator of $\beta$, takes the following form:

$$\hat{\beta}_{4l} = \frac{1}{c_1} \ln\left( \frac{c_1}{T_4} + 1 \right)^{n+\frac{a_2}{2}}, T_4 = \left\{ \frac{1}{2} + \sum_{i=1}^{n} \ln\left[ 1 - \left\{ 1 - e^{\frac{-1}{x_i}} \right\}^{\alpha} \right]^{-1} \right\}.$$

**Proof:**

The risk function of the estimator $\beta$ under the linex loss function $L_l(\hat{\beta}, \beta)$ is using the formula:

$$R(\hat{\beta}) = \int_0^{\infty} \left[ \exp\left\{ c_1 \left( \hat{\beta} - \beta \right) \right\} - c_1 \left( \hat{\beta} - \beta \right) - 1 \right] \pi_4(\beta \mid \underline{x})\, d\beta . \tag{12.5}$$

Using (7.6) in (12.5), we have:

$$R(\hat{\beta}) = \frac{T_4^{\,n+\frac{a_2}{2}}}{\Gamma\left( n + \frac{a_2}{2} \right)} \left[ \begin{array}{l} e^{c_1\hat{\beta}} \int_0^{\infty} \beta^{n+\frac{a_2}{2}-1} e^{-\beta(c_1+T_4)} d\beta - c_1\hat{\beta} \int_0^{\infty} \beta^{n+\frac{a_2}{2}-1} e^{-\beta T_4} d\beta \\ \qquad + c_1 \int_0^{\infty} \beta^{n+\frac{a_2}{2}+1-1} e^{-\beta T_4} d\beta - \int_0^{\infty} \beta^{n+\frac{a_2}{2}-1} e^{-\beta T_4} d\beta \end{array} \right]. \tag{12.6}$$

By solving (12.6), we get:

$$R(\hat{\beta}) = e^{c_1\hat{\beta}} \left( \frac{T_4}{c_1 + T_4} \right)^{n+\frac{a_2}{2}} - c_1\hat{\beta} + \frac{c_1\left( n + \frac{a_2}{2} \right)}{T_4} - 1 . \tag{12.7}$$

Minimization of the risk with respect to $\hat{\beta}$ gives us the optimal estimator

$$\hat{\beta}_{4l} = \frac{1}{c_1} \ln\left( \frac{c_1}{T_4} + 1 \right)^{n+\frac{a_2}{2}}, \ T_4 = \left\{ \frac{1}{2} + \sum_{i=1}^{n} \ln\left[ 1 - \left\{ 1 - e^{\frac{-1}{x_i}} \right\}^{\alpha} \right]^{-1} \right\} . \tag{12.8}$$

**Theorem 12.3**

Assuming the loss function $L_p(\hat{\beta}, \beta)$, the Bayesian estimator of $\beta$, takes the following form:

$$\hat{\beta}_{4p} = \frac{\sqrt{\left(n+\frac{a_2}{2}\right)\left(n+\frac{a_2}{2}+1\right)}}{T_4}, T_4 = \left\{\frac{1}{2}+\sum_{i=1}^{n}\ln\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}\right\}.$$

**Proof:**

The risk function of $\beta$ with precautionary loss function $L_p(\hat{\beta},\beta)$ is using the formula:

$$R(\hat{\beta}) = \int_0^{\infty} \frac{(\hat{\beta}-\beta)^2}{\hat{\beta}} \pi_4(\beta \mid \underline{x})\, d\beta. \tag{12.9}$$

Using (7.6) in (12.9), we have:

$$R(\hat{\beta}) = \frac{T_4^{\,n+\frac{a_2}{2}}}{\Gamma\left(n+\frac{a_2}{2}\right)} \left[\hat{\beta}\int_0^{\infty}\beta^{n+\frac{a_2}{2}-1} e^{-\beta T_4} d\beta + \frac{1}{\hat{\beta}}\int_0^{\infty}\beta^{n+\frac{a_2}{2}+2-1} e^{-\beta T_4} d\beta - 2\hat{\beta}\int_0^{\infty}\beta^{n+\frac{a_2}{2}+1-1} e^{-\beta T_4} d\beta\right]. \tag{12.10}$$

By solving (12.10), we get

$$R(\hat{\beta}) = \hat{\beta} + \frac{\left(n+\frac{a_2}{2}+1\right)\left(n+\frac{a_2}{2}\right)}{\hat{\beta} T_4^{2}} - \frac{2\left(n+\frac{a_2}{2}\right)}{T_4}. \tag{12.11}$$

Minimization of the risk with respect to $\hat{\beta}$ gives us the optimal estimator

$$\hat{\beta}_{4p} = \frac{\sqrt{\left(n+\frac{a_2}{2}+1\right)\left(n+\frac{a_2}{2}\right)}}{T_4}, \quad T_4 = \left\{\frac{1}{2}+\sum_{i=1}^{n}\ln\left[1-\left\{1-e^{\frac{-1}{x_i}}\right\}^{\alpha}\right]^{-1}\right\}. \tag{12.12}$$

## 13. REAL DATA AND GENERATED DATA

In this section, we consider a real data set as well as simulated data to illustrate that the EGSIED can be a better model than the GSIED. The real data set is the strength data of glass of the aircraft window reported by Fuller *et al* (1994):

18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.50, 25.52, 25.80, 26.69, 26.77, 26.78, 27.05, 27.67, 29.90, 31.11, 33.20, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09,39.58, 44.045, 45.29, 45.381.

These data are used here only for illustrative purposes. The required numerical evaluations are carried out using the Package of R software.

We have fitted generalized standard inverted exponential and Exponentiated generalized standard inverted exponential models to this data. These two models are fitted to the subject data using MLE. The maximum likelihood estimates of the parameters with standard errors in parenthesis and the resulting log-likelihood values, AIC and BIC are present in Tables 1 and 2.

**Table 1**
**MLEs of the Model Parameters using Generated Data, the Resulting SEs in Parenthesis and Criteria for Comparison**

| N | Distribution | $\alpha$ | $\beta$ | Log-likelihood | AIC | BIC |
|---|---|---|---|---|---|---|
| 10 | EGIED | 4.736147 (1.518092) | 3.325465 (1.748487) | -4.193659 | 12.38732 | 12.99249 |
| | GIED | 2.4713419 (0.7815068) | 1 | -6.557206 | 15.11441 | 15.417 |
| 15 | EGIED | 5.575297 (1.442356) | 3.938586 (1.782244) | -3.959221 | 11.91844 | 13.33454 |
| | GIED | 2.6911243 (0.6948452) | 1 | -8.179809 | 18.35962 | 19.06767 |
| 20 | EGIED | 4.946544 (1.124837) | 3.797352 (1.494153) | -8.276666 | 20.55333 | 22.5448 |
| | GIED | 2.4143359 (0.5398618) | 1 | -13.59844 | 29.19687 | 30.1926 |

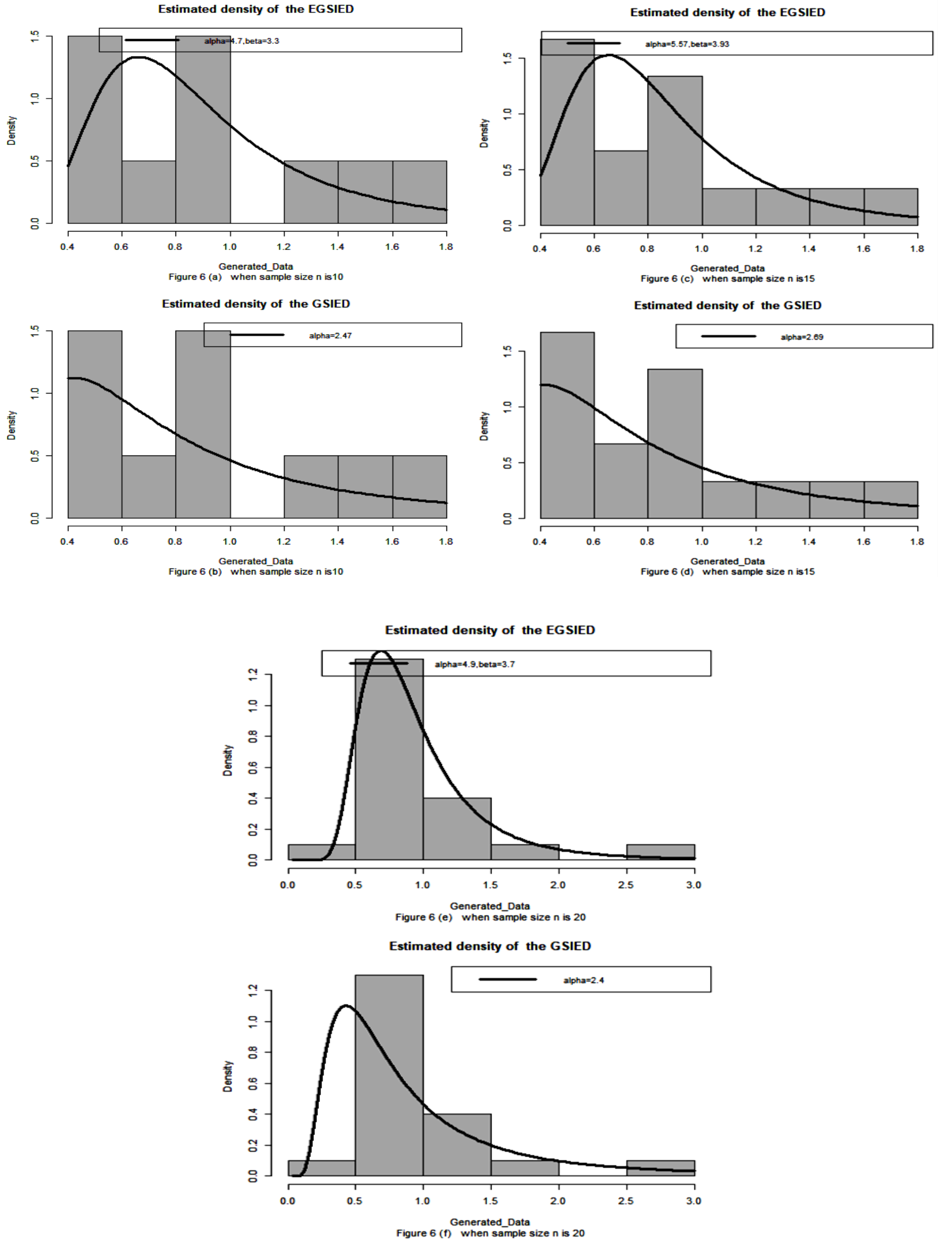


**Figure 6: Plots of the Fitted Densities of EGSIE and GSIE Distributions for the Generated Data Sets**

**Table 2**
**MLEs of the Model Parameters using Real Life Data Set, the Resulting SEs in Parenthesis and Criteria for Comparison**

| | Distribution | $\alpha$ | $\beta$ | Log-likelihood | AIC | BIC |
|---|---|---|---|---|---|---|
| Data set | EGIED | 5.0122374 (0.9307950) | 3.1979104 (0.9624979) | -11.0494 | 26.09881 | 28.96678 |
| | GIED | 2.6465858 (0.4753408) | 1 | -17.86831 | 37.73661 | 39.1706 |

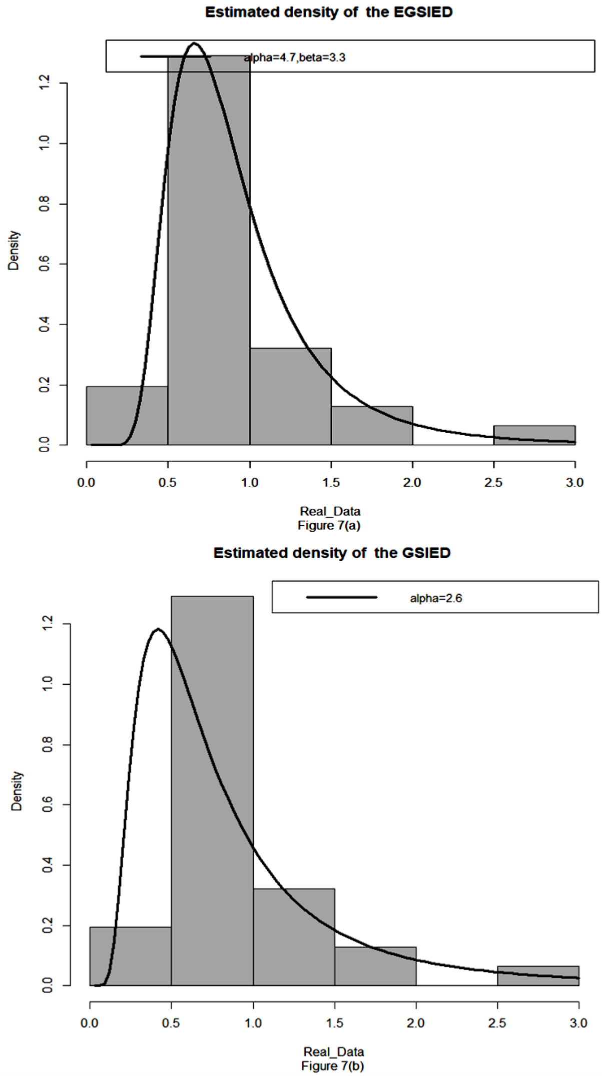


**Figure 7: Plots of the Fitted Densities of EGSIE and GSIE Distributions for the Real Data Set**

In order to evaluate the two models, we observe the criteria like AIC (Akaike information criterion) and BIC (Bayesian information criterion) for real life data set as well as generated data have been computed. The superior distribution corresponds to lesser AIC and BIC values.

$$\text{AIC} = 2k - 2\log L \text{ and } \text{BIC} = k\log n - 2\log L$$

From Tables 1 and 2, it has been examined that the EGSIED have the lesser AIC and BIC values as compared to GSIED. Hence, we can conclude that the EGSIED leads to a better fit than the GIED.

**Table 3**
**The Baye's Estimators of the Shape Parameter β for EGSIED**

| Prior | Quadratic Loss | LINEX Loss | Precautionary Loss |
|---|---|---|---|
| Uniform | $\hat{\beta}_{1q} = \frac{(n-1)}{T_1}$ | $\hat{\beta}_{1l} = \frac{1}{c_1}\ln\left(\frac{c_1+T_1}{T_1}\right)^{n+1}$ | $\hat{\beta}_{1p} = \frac{\left[(n+2)(n+1)\right]^{1/2}}{T_1}$ |
| Jeffery | $\hat{\beta}_{2q} = \frac{(n-2)}{T_2}$ | $\hat{\beta}_{2l} = \frac{1}{c_1}\ln\left(\frac{c_1}{T_2}+1\right)^{n}$ | $\hat{\beta}_{2p} = \frac{\sqrt{(n+1)(n)}}{T_2}$ |
| Gamma | $\hat{\beta}_{3q} = \frac{(n+a-2)}{T_3}$ | $\hat{\beta}_{3l} = \frac{1}{c_1}\ln\left(\frac{c_1}{T_3}+1\right)^{n+a}$ | $\hat{\beta}_{3p} = \frac{\sqrt{(n+a+1)(n+a)}}{T_3}$ |
| Chi-square | $\hat{\beta}_{4q} = \frac{(n+\frac{a_2}{2}-2)}{T_4}$ | $\hat{\beta}_{4l} = \frac{1}{c_1}\ln\left(\frac{c_1}{T_4}+1\right)^{n+\frac{a_2}{2}}$ | $\hat{\beta}_{4p} = \frac{\sqrt{(n+\frac{a_2}{2}+1)(n+\frac{a_2}{2})}}{T_4}$ |

## 14. SIMULATION STUDY

In this study, samples of size 25, 50 and 100 have been generated from the EGSIED using inverse cdf method. The shape parameter $\beta$ is estimated for EGSIED with Classical and Bayesian method of estimation using uniform, Jeffrey's, gamma and Chi Square priors under different loss functions. Mean square error has been computed to compare the performance of the estimates under different situations. The results are presented in the following tables.

**Table 4**

**MSE for $\hat{\beta}$ under Uniform Prior using Different Loss Functions**

| **N** | $\alpha$ | $\beta$ | $\hat{\beta}_{ML}$ | $\hat{\beta}_{QL}$ | $\hat{\beta}_{LL}$ | | $\hat{\beta}_{PL}$ |
|---|---|---|---|---|---|---|---|
| | | | | | **$C_1$=1.2** | **$C_1$= -1.2** | |
| 25 | 0.5 | 1.5 | 0.051551 | 0.068799 | 0.049073 | 0.035343 | 0.036965 |
| | | 2.0 | 0.453783 | 0.524345 | 0.442684 | 0.336223 | 0.359911 |
| | | 2.5 | 0.556661 | 0.535189 | 0.542519 | 0.737851 | 0.630823 |
| 50 | 0.5 | 1.5 | 0.087472 | 0.073461 | 0.087093 | 0.124263 | 0.112885 |
| | | 2.0 | 0.213770 | 0.182365 | 0.200989 | 0.311681 | 0.269110 |
| | | 2.5 | 0.208757 | 0.238796 | 0.217005 | 0.152500 | 0.170859 |
| 100 | 0.5 | 1.5 | 0.020444 | 0.021183 | 0.020388 | 0.020254 | 0.020159 |
| | | 2.0 | 0.0360291 | 0.037629 | 0.036253 | 0.0352442 | 0.035082 |
| | | 2.5 | 0.052821 | 0.056001 | 0.054118 | 0.050119 | 0.050299 |

**Table 5**

**MSE for $\hat{\beta}$ under Jeffery Prior using Different Loss Functions**

| N | $\alpha$ | $\beta$ | $\hat{\beta}_{ML}$ | $\hat{\beta}_{QL}$ | $\hat{\beta}_{LL}$ | | $\hat{\beta}_{PL}$ |
|---|---|---|---|---|---|---|---|
| | | | | | $C_1$=1.2 | $C_1$= -1.2 | |
| 25 | 0.5 | 1.5 | 0.050218 | 0.090721 | 0.063301 | 0.040189 | 0.043901 |
| | | 2.0 | 0.450352 | 0.597714 | 0.506453 | 0.393597 | 0.417726 |
| | | 2.5 | 0.536165 | 0.515765 | 0.512389 | 0.627421 | 0.555139 |
| 50 | 0.5 | 1.5 | 0.086712 | 0.061060 | 0.072639 | 0.104182 | 0.094566 |
| | | 2.0 | 0.212073 | 0.153707 | 0.170371 | 0.266293 | 0.229353 |
| | | 2.5 | 0.206712 | 0.270581 | 0.245258 | 0.172093 | 0.193179 |
| 100 | 0.5 | 1.5 | 0.020244 | 0.022162 | 0.020871 | 0.019955 | 0.020040 |
| | | 2.0 | 0.035681 | 0.039658 | 0.037617 | 0.034784 | 0.035173 |
| | | 2.5 | 0.052326 | 0.059883 | 0.057285 | 0.049805 | 0.051188 |

**Table 6**

**MSE for $\hat{\beta}$ under Gamma Prior using Different Loss Functions**

| N | $\alpha$ | $\beta$ | $a=b$ | $\hat{\beta}_{ML}$ | $\hat{\beta}_{QL}$ | $\hat{\beta}_{LL}$ | | $\hat{\beta}_{PL}$ |
|---|---|---|---|---|---|---|---|---|
| | | | | | | $C_1$=1.2 | $C_1$= -1.2 | |
| 25 | 0.5 | 1.5 | 0.4 | 0.049783 | 0.092994 | 0.065269 | 0.041374 | 0.045290 |
| | | 2.0 | 0.4 | 0.447780 | 0.604076 | 0.513597 | 0.402304 | 0.425907 |
| | | 2.5 | 0.4 | 0.489497 | 0.479377 | 0.469252 | 0.532838 | 0.484555 |
| 50 | 0.5 | 1.5 | 0.4 | 0.086186 | 0.057921 | 0.068794 | 0.098361 | 0.089373 |
| | | 2.0 | 0.4 | 0.210002 | 0.139920 | 0.155221 | 0.241819 | 0.208517 |
| | | 2.5 | 0.4 | 0.203901 | 0.283424 | 0.256965 | 0.181236 | 0.203159 |
| 100 | 0.5 | 1.5 | 0.4 | 0.020100 | 0.022276 | 0.020901 | 0.019844 | 0.019962 |
| | | 2.0 | 0.4 | 0.035306 | 0.040355 | 0.038080 | 0.034583 | 0.035177 |
| | | 2.5 | 0.4 | 0.051650 | 0.062147 | 0.059165 | 0.049811 | 0.051859 |

**Table 7**

**MSE for $\hat{\beta}$ under Chi-Square Prior using Different Loss Functions**

| N | $\alpha$ | $\beta$ | $a_2$ | $\hat{\beta}_{ML}$ | $\hat{\beta}_{QL}$ | $\hat{\beta}_{LL}$ | | $\hat{\beta}_{PL}$ |
|---|---|---|---|---|---|---|---|---|
| | | | | | | $C_1$=1.2 | $C_1$= -1.2 | |
| 25 | 0.5 | 1.5 | 0.3 | 0.049226 | 0.102658 | 0.072211 | 0.045250 | 0.049784 |
| | | 2.0 | 0.3 | 0.446027 | 0.632375 | 0.538801 | 0.426021 | 0.449510 |
| | | 2.5 | 0.3 | 0.472674 | 0.476484 | 0.460236 | 0.489910 | 0.456197 |
| 50 | 0.5 | 1.5 | 0.3 | 0.085798 | 0.053719 | 0.063692 | 0.090948 | 0.082670 |
| | | 2.0 | 0.3 | 0.208928 | 0.128887 | 0.143094 | 0.222772 | 0.192087 |
| | | 2.5 | 0.3 | 0.202535 | 0.299146 | 0.271152 | 0.192091 | 0.215030 |
| 100 | 0.5 | 1.5 | 0.3 | 0.019995 | 0.022785 | 0.021215 | 0.019859 | 0.020045 |
| | | 2.0 | 0.3 | 0.035094 | 0.041553 | 0.038985 | 0.034715 | 0.035540 |
| | | 2.5 | 0.3 | 0.051314 | 0.064672 | 0.061335 | 0.050371 | 0.052973 |

From these Tables, we conclude that Bayes estimator with LINEX Loss function under all the assumed priors provide the minimum values in most cases particularly as loss parameter $C_1$ is (-1.2) and among the priors Gamma prior provides the Bayes estimators with least MSE.

## 15. CONCLUSION

This article defined a two-parameter EGSIED and studied various properties of the distribution. The application of the distribution has also been demonstrated with each in the real life data as well as simulated data sets. The results are compared with one-parameter generalized standard inverted exponential distribution, showing that the EGIED provides a better fit than the GSIED.

On the basis of simulation study, we study that Bayesian method of estimation is better than classical method of estimation. Also, Bayesian estimator under LINEX loss function gives the minimum MSE values under all the assumed priors as compare to other loss functions and the classical estimator as the loss parameter $C_1$ is (-1.2). Hence we can say that LINEX loss is better than other loss functions. It is also studied that among the priors, Gamma prior provides the Bayes estimators with least MSE.

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