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THE MARSHALL-OLKIN GENERALIZED G POISSON FAMILY OF DISTRIBUTIONS

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ABSTRACT

In this paper, we propose a new class of lifetime distributions called the Marshall-Olkin Generalized G Poisson family. The proposed family of distributions is constructed by compounding the Marshall-Olkin Generalized distribution with the truncated Poisson distribution. It can provide better fits than some of the known lifetime distributions and this fact represents a good characterization of this new family. Some useful characterizations for the new family are presented. The maximum likelihood method is used for estimating the model parameters. The importance and flexibility of the new family are illustrated by means of an application to a real data set.

KEYWORDS

Marshall-Olkin Generalized-G Family; Truncated Poisson Distribution; Maximum Likelihood Estimation; Quantile Function; Generating Function.

1. INTRODUCTION AND PHYSICAL MOTIVATION

The statistical literature contains many new classes of distributions which have been constructed by extending known families of continuous distributions providing more flexibility via adding one or more parameters to the baseline model. These new families have been used for modeling data in many applied areas such as engineering, economics, biological studies and environmental sciences. Gupta et al. (1998) defined the exponentiated-G (Exp-G) class which consists of raising the cumulative distribution function (CDF) to a positive power parameter and proposed the Exp exponential (Exp-E) distribution with the CDF (for $x > 0$)

$$F(x) = \left[1 - \exp\left(-\lambda x\right)\right]^{\alpha},$$

where $\lambda, \alpha > 0$. This CDF is simply the α^{th} power of the standard exponential cumulative distribution function. Further details were explored by Gupta and Kundu (2001).

The generalized distributions were pioneered by Marshall and Olkin (1997), Eugene et al. (2002), Cordeiro et al. (2013), Alzaatreh et al. (2013), Yousof et al. (2015), Merovc et al. (2016), Yousof et al. (2016), Alizadeh et al. (2016a,b), Afify et al. (2016a,b,c), Aryal and Yousof (2017), Yousof et al. (2017a,b), Brito et al. (2017), Alizadeh et al. (2017), Hamedani et al. (2017), Cordeiro et al. (2017), Nofal et al. (2017), Merovci et al. (2017), Korkmaz and Genc (2017), Afify et al. (2017), Korkmaz et al. (2018), Alizadeh et al. (2018a,b), Yousof et al. (2018), Cordeiro et al. (2018), and Hamedani et al. (2018), among others. Consider a baseline reliability function (RF).

$$\overline{G}(x;\boldsymbol{\psi}) = 1 - G(x;\boldsymbol{\psi}),$$

and probability density function (PDF) $g(x;\boldsymbol{\psi})$ with a parameter vector $\boldsymbol{\psi}$, then the generalized-G (GG) family with CDF can be given by

$$H_{GG}(x;a,\boldsymbol{\psi}) = 1 - \overline{G}(x;\boldsymbol{\psi})^{a},\ x \in \mathrm{R},$$

with the corresponding PDF

$$h_{GG}(x;a,\boldsymbol{\psi}) = a g(x;\boldsymbol{\psi})\overline{G}(x;\boldsymbol{\psi})^{a-1},\ x \in \mathrm{R}.$$

We note that GG family is known as exponentiated Lehmann type **II** family of distribution in the literature. Marshall and Olkin (1997) introduced a new method of adding a parameter to a family of distributions called the Marshall-Olkin-G (MOG) family. According to them if $\overline{H}(x)$ and $h(x)$ denote the RF and hazard rate function (HRF) of a continuous random variable X, then the MO-G family has CDF

$$F_{MOG}(x;\delta,\boldsymbol{\psi}) = 1 - \delta\overline{H}(x;\boldsymbol{\psi})\left[1-(1-\delta)\overline{H}(x;\boldsymbol{\psi})\right]^{-1},\ x \in \mathrm{R},\ \delta > 0.$$

Clearly, when $\delta = 1$, we have the baseline distribution. The corresponding PDF is given by

$$f_{MOG}(x;\delta,\boldsymbol{\psi}) = \delta h(x;\boldsymbol{\psi})\left[1-(1-\delta)\overline{H}(x;\boldsymbol{\psi})\right]^{-2},\ x \in \mathrm{R},\ \delta > 0.$$

Suppose that a system has N subsystems functioning independently at a given time where N has zero truncated Poisson (ZTP) distribution with parameter λ. The ZTP distribution is the conditional probability distribution of a Poisson distributed random variable, given that the value of the random variable is not zero. The probability mass function (pmf) of N is given by

$$P(N=n) = \frac{\lambda^{n}}{n!\left(1-e^{-\lambda}\right)} e^{-\lambda},\ \text{for } n = 1,2,.... \tag{1}$$

Note that for ZTP variable the expected value and variance are respectively given by

$$E(N)=\lambda\left(1-e^{-\lambda}\right)^{-1}$$

and

$$Var(N)=\left(\lambda+\lambda^{2}\right)\left(1-e^{-\lambda}\right)^{-1}-\lambda^{2}\left(1-e^{-\lambda}\right)^{-2}.$$

Suppose that the failure time of each subsystem has the Marshall- Olkin Generalized-G (MOGG (δ, a) for short) distribution with CDF

$$H_{MOGG}\left(x;\delta,a,\psi\right)=\left[1-\overline{G}(x;\psi)^{a}\right]\left[1-\left(1-\delta\right)\overline{G}(x;\psi)^{a}\right]^{-1},\ x\in\mathrm{R},\ \delta>0, \tag{2}$$

where δ and a are positive shape parameters representing the different patterns of the MOGG family. The corresponding PDF of the MOGG family is given by

$$h_{MOGG}\left(x;\delta,a,\psi\right)=\delta a g\left(x;\psi\right)\overline{G}(x;\psi)^{a-1}\left[1-\left(1-\delta\right)\overline{G}(x;\psi)^{a}\right]^{-2},\ x\in\mathrm{R}. \tag{3}$$

Let Y_i denote the failure time of the i^{th} subsystem and X denote the time to failure of the first out of the N functioning subsystems. We can write $X = \min\{Y_1,...,Y_N\}$. The conditional CDF of X given N is

$$\begin{aligned}F\left(x\mid N\right)&=1-P_r\left(X>x\mid N\right)\\&=1-P_r\left(Y_1>x\right)^{N}\\&=1-P_r\left(1-H_{MOGG}\left(x;\delta,a,\psi\right)\right)^{N},\end{aligned}$$

and the marginal CDF of X is

$$F_{MOGGP}\left(x\right)=\left(1-e^{-\lambda}\right)^{-1}\left(1-\exp\left\{-\lambda\left[1-\frac{\delta\overline{G}(x)^{a}}{1-\overline{\delta}\overline{G}(x)^{a}}\right]\right\}\right),\ x\in\mathrm{R}, \tag{4}$$

which also can be simplified as

$$F_{MOGGP}\left(x\right)=\left(1-e^{-\lambda}\right)^{-1}\left(1-\exp\left\{-\lambda\left[\frac{1-\overline{G}\left(x\right)^{\alpha}}{1-\overline{\delta}\overline{G}\left(x\right)^{\alpha}}\right]\right\}\right),\ x\in\mathrm{R}, \tag{5}$$

where $\overline{\delta}=1-\delta.$ The CDF in (5) is called the Marshall-Olkin Generalized-G Poisson (MOGGP) distribution. The corresponding PDF is

$$f_{MOGGP}\left(x\right)=\delta a\lambda\frac{g\left(x\right)\overline{G}(x)^{a-1}}{\left(1-e^{-\lambda}\right)\left\{1-\overline{\delta}\overline{G}(x)^{a}\right\}^{2}}\exp\left\{-\lambda\left[1-\frac{\delta\overline{G}(x)^{a}}{1-\overline{\delta}\overline{G}(x)^{a}}\right]\right\},\ x\in\mathrm{R}.$$

The RF, HRF, reversed hazard rate and cumulative hazard rate functions of X are, respectively, given by

$$R(x)=1-\left[\left(1-e^{-\lambda}\right)^{-1}\left(1-\exp\left\{-\lambda\left[1-\frac{\delta\overline{G}(x)^{a}}{1-\bar{\delta}\overline{G}(x)^{a}}\right]\right\}\right)\right],$$

$$h(x)=\frac{\delta a\lambda g\left(x\right)\overline{G}(x)^{a-1}\exp\left\{-\lambda\left[1-\frac{\delta\overline{G}(x)^{a}}{1-\bar{\delta}\overline{G}(x)^{a}}\right]\right\}}{\left(1-e^{-\lambda}\right)\left\{1-\bar{\delta}\overline{G}(x)^{a}\right\}^{2}\left\{1-\frac{1-\exp\left\{-\lambda\left[1-\frac{\delta\overline{G}(x)^{a}}{1-\bar{\delta}\overline{G}(x)^{a}}\right]\right\}}{\left(1-e^{-\lambda}\right)}\right\}},$$

$$r\left(x\right)=\frac{\delta a\lambda g\left(x\right)\overline{G}(x)^{a-1}\exp\left\{-\lambda\left[1-\frac{\delta\overline{G}(x)^{a}}{1-\bar{\delta}\overline{G}(x)^{a}}\right]\right\}}{\left[1-\bar{\delta}\overline{G}(x)^{a}\right]^{2}\left(1-\exp\left\{-\lambda\left[1-\frac{\delta\overline{G}(x)^{a}}{1-\bar{\delta}\overline{G}(x)^{a}}\right]\right\}\right)}$$

and

$$H(x)=-\left[\log\left(1-\frac{1-\exp\left\{-\lambda\left[1-\frac{\delta\overline{G}(x)^{a}}{1-\bar{\delta}\overline{G}(x)^{a}}\right]\right\}}{\left(1-e^{-\lambda}\right)}\right)\right].$$

For simulating data from this family, if $U\sim u(0,1)$, then

$$x_{U}=G^{-1}\left[1-\left(\frac{\left\{1+\frac{1}{\lambda}\ln\left[1-U\left(1-e^{-\lambda}\right)\right]\right\}}{\delta+(1-\delta)\left\{1+\frac{1}{\lambda}\ln\left[1-U\left(1-e^{-\lambda}\right)\right]\right\}}\right)^{\frac{1}{a}}\right]$$

has CDF (5).

The justification for the practicality of MOGGP family is based on the wide use of the MO-G family. We are motivated to introduce the MOGGP family of distributions because it contains a number of known lifetime models as illustrated in Subsection 3.1; The MOGGP family of distributions exhibits increasing, decreasing, upside-down, constant as well as bathtub hazard rates as illustrated in Figures 1 and 2. It is shown in Subsection 3.4 that the MOGGP family of distributions can be viewed as a mixture of the exponentiated-G distributions. The new family can be viewed as a suitable model for fitting the left-skewed, right-skewed, symmetric and bimodal data, The MOGGP family of distributions outperforms several of the well-known lifetime distributions with respect to a real data application as illustrated in Section 5.

This paper is organized as follows. Some useful characterizations are presented in Section 2. In Section 3, we derive some of the mathematical properties of the new family. Maximum likelihood estimation for the model parameters under uncensored data is addressed in Section 4. In Section 5, potentiality of the proposed class is illustrated by means of a real data set. A simulation study is performed in Section 6 to assess the performance of the estimators. Finally, Section 7 provides some concluding remarks.

2. CHARACTERIZATIONS

In this section we present certain characterizations of MOGGP distribution. These characterizations are in terms of the ratio of two truncated moments. One of the advantages of these characterizations is that the CDF is not required to have a closed form. Due to the nature of our CDF, we believe that our characterizations may be the only possible ones. Our first characterization result employs a theorem due to Glänzel (1987), see Theorem 1 of **Appendix A**. Note that the result holds also when the interval H is not closed. As shown in Glänzel (1990), this characterization is stable in the sense of weak convergence.

Proposition 2.1

Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $q_1(x) = \exp\left\{\lambda\left[\frac{1-\bar{G}(x)^\alpha}{1-\bar{\delta}\bar{G}(x)^\alpha}\right]\right\}$ and $q_2(x) = q_1(x)\left[1-\bar{\delta}\bar{G}(x)^\alpha\right]^{-1}$ for $x \in \mathbb{R}$. The random variable X has PDF (5) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left\{\left[1-\bar{\delta}\bar{G}(x)^\alpha\right]^{-1} - 1\right\}, \quad x \in \mathbb{R}.$$

Proof:

Let X be a random variable with PDF (5), then

$$(1-F(x))E\left[q_1(X) \mid X \geq x\right] = \frac{\delta\lambda}{\bar{\delta}\left(1-e^{-\lambda}\right)}\left\{\left[1-\bar{\delta}\bar{G}(x)^\alpha\right]^{-1} - 1\right\}, \quad x \in \mathbb{R},$$

and

$$(1-F(x))E\left[q_2(X) \mid X \geq x\right] = \frac{\delta\lambda}{2\bar{\delta}\left(1-e^{-\lambda}\right)}\left\{\left[1-\bar{\delta}\bar{G}(x)^\alpha\right]^{-2} - 1\right\}, \quad x \in \mathbb{R}, \text{ and}$$

finally

$$\eta(x)q_1(x) - q_2(x) = -\frac{q_1(x)}{2}\left\{\left[1-\bar{\delta}\bar{G}(x)^\alpha\right]^{-1} + 1\right\} < 0 \quad \textit{for} \quad x \in \mathbb{R}.$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{\alpha\bar{\delta}g(x)\bar{G}(x)^{\alpha-1}\left[1-\bar{\delta}\bar{G}(x)^\alpha\right]^{-2}}{\left[1-\bar{\delta}\bar{G}(x)^\alpha\right]^{-1} + 1} \quad x \in \mathbb{R},$$

and hence

$$s(x) = -\ln\left\{\left[1-\bar{\delta}\bar{G}(x)^{\alpha}\right]^{-1}+1\right\}, \quad x \in \mathrm{R}.$$

Now, in view of Theorem 1, X has density (5).

Corollary 2.1:

Let $X : \Omega \to \mathrm{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2.1. The PDF of X is (5) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)} = \frac{\alpha\bar{\delta}g(x)\bar{G}(x)^{\alpha-1}\left[1-\bar{\delta}\bar{G}(x)^{\alpha}\right]^{-2}}{\left[1-\bar{\delta}\bar{G}(x)^{\alpha}\right]^{-1}+1} \quad x \in \mathrm{R}.$$

The general solution of the differential equation in Corollary 2.1 is

$$\eta(x) = \left\{\left[1-\bar{\delta}\bar{G}(x)^{\alpha}\right]^{-1}+1\right\}^{-1}$$
$$\left[-\int \alpha\bar{\delta}g(x)\bar{G}(x)^{\alpha-1}\left[1-\bar{\delta}\bar{G}(x)^{\alpha}\right]^{-2}\left(q_1(x)\right)^{-1}q_2(x)+D\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 2.1 with $D=\frac{1}{2}$. However, it should be also noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 1.

3. SOME MATHEMATICAL PROPERTIES

3.1 Special MOGGP Models

The MOGGP family includes flexible new sub-families depending on the parameters values. These sub-families are the followings:

- for $\delta = 1$, Generalized Poisson-G (GPG) family,
- for $a = 1$, Marshall Olkin Poisson-G (MOPG) family,
- for $\delta = a = 1$, Generalized Poisson (GP) family,
- As $\lambda \to 0$, the MOGGP family converges to MOGG family,
- As $\lambda \to 0$ and $\delta = 1$, the MOGGP family converges to GG family,
- As $\lambda \to 0$ and $a = 1$, the MOGGP family converges to MOG family,
- As $\lambda \to 0$ and $\delta = a = 1$, the MOGGP family converges to baseline G distribution.

Here, we concentrate on two special sub-models of the MOGGP family. These special models extend some well-known distributions appeared in the literature.

3.2 The MOG-Weibull-P (MOGWP) Distribution

As the first special sub-model, we consider the Weibull distribution with CDF $G(x;\theta,\gamma)=1-\exp\left[-(\theta x)^{\gamma}\right]$ for $x>0$ and $\theta,\gamma>0$. The CDF of the MOGWP distribution is given by

$$F(x;\delta,a,\lambda,\theta,\gamma)=\frac{1-\exp\left\{-\lambda\left[1-\frac{\delta(1-\exp\{-(\theta x)^{\gamma}\})^{a}}{1-(1-\delta)(1-\exp\{-(\theta x)^{\gamma}\})^{a}}\right]\right\}}{1-e^{-\lambda}}, x\geq 0.$$

We denote this CDF with MOGWP $(\delta,a,\lambda,\theta,\gamma)$. Some possible plots of the MOGWP density and HRF for selected parameter values are displayed in Figure 1.

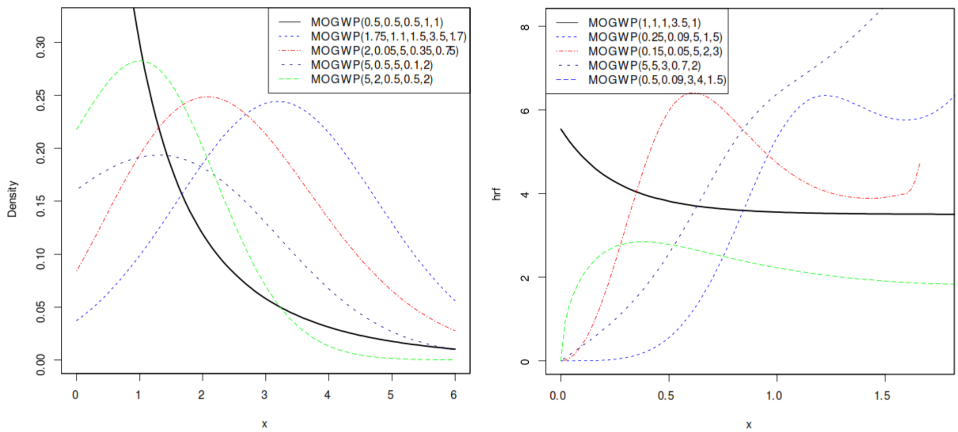

Figure 1: The PDF and HRF of the MOGWP Distribution for Selected Parameter Values

3.3 The MOG-Normal-P (MOGNP) Distribution

Secondly, we consider the MOGNP distribution by taking $G(x;\mu,\sigma)=\Phi\left(\frac{x-\mu}{\sigma}\right)$, where $x,\mu\in R$, $\sigma>0$ where $\Phi(\cdot)$ is the CDF of the standard normal distribution. Hence, the CDF of the MOGNP distribution is

$$F(x;\delta,a,\lambda,\mu,\sigma)=\frac{1-\exp\left\{-\lambda\left[1-\frac{\delta\left(1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right)^{a}}{1-(1-\delta)\left(1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right)^{a}}\right]\right\}}{1-e^{-\lambda}},\quad x\in\mathbb{R}.$$

We denote this CDF with MOGNP $(\delta,a,\lambda,\mu,\sigma)$. Some possible plots of the MOGNP density and HRF for selected parameter values are displayed in Figure 2.

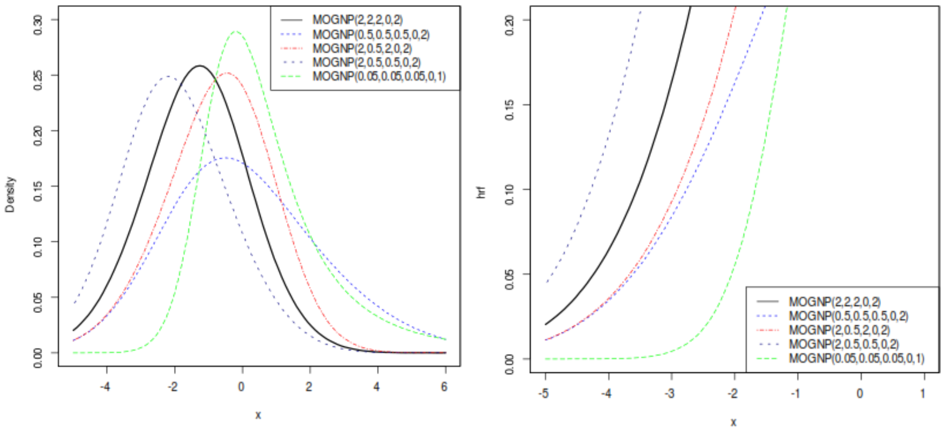

Figure 2: The PDF and HRF of the MOGNP Distribution for Selected Parameter Values

From these results above, we can say that the MOGGP model can generate very flexible distribution for data modelling.

3.4 Mixture Representation

Using the power series expansion, the PDF in (5) can be expressed as

$$f(x) = \delta a \frac{g(x)\bar{G}(x)^{a-1}}{\left(1-e^{-\lambda}\right)\left\{1-\bar{\delta}\bar{G}(x)^{a}\right\}^{2}} \sum_{w=0}^{\infty} \frac{(-1)^{w}\lambda^{w+1}}{w!} \overbrace{\left[1-\frac{\delta\bar{G}(x)^{a}}{1-\bar{\delta}\bar{G}(x)^{a}}\right]^{w}}^{A_i}, x \in \mathrm{R}.$$

Using the taylor series for z^{β} to the quantity A_i where

$$z^{\beta} = \sum_{i=0}^{\infty} \frac{(z-1)^{k}}{i!}(\beta)_{i},$$

and the index i is a positive integer and

$$(\beta)_{i} = \beta(\beta-1)\ldots(\beta-i+1)$$

is the descending factorial, we have

$$f(x) = \frac{ag(x)}{\left(1-e^{-\lambda}\right)} \sum_{w,i=0}^{\infty} \frac{(-1)^{w+i}\lambda^{w+1}\delta^{i+1}}{w!\,i!}(w)_{i}\,\bar{G}(x)^{a(i+1)-1} \overbrace{\left[1-\bar{\delta}\bar{G}(x)^{a}\right]^{-(i+2)}}^{B_i}, x \in \mathrm{R}.$$

Using the Taylor series for z^{β} again to the quantity B_i, we have

$$f(x) = \frac{g(x)}{\left(1-e^{-\lambda}\right)} \sum_{w,i,j=0}^{\infty} \frac{a\lambda^{w+1}(-1)^{w+i+j}}{w!\,i!\,j!\,\bar{\delta}^{-j}\delta^{-(i+1)}}(w)_{i}\left(-[i+2]\right)_{j} \overbrace{\bar{G}(x)^{a(i+j+1)-1}}^{C_i}, x \in \mathrm{R}.$$

Using the series expansion

$$\left(1-z\right)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{j!\Gamma(b-j)} z^j,$$

to the quantity C_i, the PDF can be expressed as a mixture of Exp-G distributions

$$f\left(x\right) = f\left(x;\lambda,a,b,\boldsymbol{\psi}\right) = \sum_{k=0}^{\infty} t_k \pi_{k+1}\left(x\right), \tag{6}$$

where

$$t_k = \frac{a\left(-1\right)^k}{\left(1-e^{-\lambda}\right)\left(k+1\right)} \sum_{w,i,j=0}^{\infty} \frac{\lambda^{w+1}\left(-1\right)^{w+i+j}\left(w\right)_i\left(-\left[i+2\right]\right)_j}{w!i!j!\bar{\delta}^{-j}\delta^{-(i+1)}} \binom{a\left(i+j+1\right)-1}{k}$$

and $\pi_{k+1}\left(x\right) = \left(k+1\right) g\left(x;\boldsymbol{\psi}\right) G\left(x;\boldsymbol{\psi}\right)^k$ is the Exp-G PDF with power parameter $k+1$. The CDF of the MOGGP family can also be expressed as a mixture of Exp-G distributions. By integrating (6), we obtain the same mixture representation

$$F\left(x\right) = F\left(x;\lambda,a,b,\boldsymbol{\psi}\right) = \sum_{k=0}^{\infty} t_k \, \Pi_{k+1}(x),$$

where $\Pi_{k+1}(x)$ is the CDF of the Exp-G family with power parameter $k+1$. Equation (6) can be used for deriving the statistical properties of the new family such as moments, incomplete moment, probability weighted moments, order statistics, among others.

4. ESTIMATION

In this Section, we consider the estimation of the unknown parameters of this family from complete samples only by maximum likelihood method. Let $x_1,\ldots,x_n$ be a random sample from the MOGGP distribution with parameters δ, a and $\boldsymbol{\psi}$. Let $\Theta = \left(\delta, a, \psi^T\right)^T$ be the $p\times 1$ parameter vector. The log-likelihood function is

$$\ell = \ell(\Theta) = n\log\delta + n\log a + n\log\lambda - n\log\left(1-e^{-\lambda}\right) + \sum_{i=1}^{n} \log g\left(x_i;\boldsymbol{\psi}\right)$$

$$+\left(a-1\right)\sum_{i=0}^{n} \log \bar{G}(x_i;\boldsymbol{\psi}) - 2\sum_{i=0}^{n} \log s_i - \lambda\sum_{i=0}^{n} z_i,$$

where $s_i = 1-\bar{\delta}\bar{G}(x_i;\boldsymbol{\psi})^a$ and $z_i = 1-\frac{\delta\bar{G}(x)^a}{s_i}$. The components of the score vector, $\mathbf{U}\left(\Theta\right) = \frac{\partial\ell}{\partial\Theta} = \left(\frac{\partial\ell}{\partial\delta}, \frac{\partial\ell}{\partial a}, \frac{\partial\ell}{\partial\lambda}, \frac{\partial\ell}{\partial\boldsymbol{\psi}}\right)^?$, are

$$U_\delta = \frac{n}{\delta} - 2\sum_{i=0}^{n}\frac{p_i}{s_i} - \lambda\sum_{i=0}^{n} m_i, U_\lambda = \frac{n}{\lambda} - \frac{ne^{-\lambda}}{\left(1-e^{-\lambda}\right)} - \sum_{i=0}^{n} z_i,$$

$$U_a = \frac{n}{a} + \sum_{i=0}^{n}\log\overline{G}(x_i;\boldsymbol{\psi}) - 2\sum_{i=1}^{n}\frac{q_i}{s_i} - \lambda\sum_{i=0}^{n} w_i$$

and

$$U_{\boldsymbol{\psi}} = \sum_{i=0}^{n}\frac{g'\left(x_i;\boldsymbol{\psi}\right)}{g\left(x_i;\boldsymbol{\psi}\right)} - \left(a-1\right)\sum_{i=0}^{\infty}\frac{G'\left(x_i;\boldsymbol{\psi}\right)}{\overline{G}(x_i;\boldsymbol{\psi})} - 2\sum_{i=0}^{n}\frac{t_i}{s_i} - \lambda\sum_{i=0}^{n} d_i,$$

where

$$q_i = -\bar{\delta} p_i \log\overline{G}(x_i;\boldsymbol{\psi}),\ p_i = \overline{G}(x_i;\boldsymbol{\psi})^a, G'\left(x_i;\boldsymbol{\psi}\right) = \partial G\left(x_i;\boldsymbol{\psi}\right)/\partial\boldsymbol{\psi},$$

$$m_i = -p_i\left[\delta p_i + s_i\right], t_i = a\bar{\delta}\overline{G}(x_i;\boldsymbol{\psi})^{a-1}G'\left(x_i;\boldsymbol{\psi}\right), g'\left(x_i;\boldsymbol{\psi}\right) = \partial g\left(x_i;\boldsymbol{\psi}\right)/\partial\boldsymbol{\psi},$$

$$w_i = -\delta p_i\left[s_i\log\overline{G}(x_i;\boldsymbol{\psi}) + q_i\right] \text{ and } d_i = -\delta\left[t_i p_i - a s_i G'\left(x_i;\boldsymbol{\psi}\right)\overline{G}(x_i;\boldsymbol{\psi})^{a-1}\right].$$

Setting the nonlinear system of equations

$$U_\delta = U_\lambda = U_a = 0 \text{ and } U_{\boldsymbol{\psi}} = \mathbf{0},$$

and solving them simultaneously yields the MLE

$$\Theta = (\widehat{\delta}, \widehat{\lambda}, \widehat{a}, \boldsymbol{\psi}^{?})^{?}.$$

To solve these equations, it is usually more convenient to use nonlinear optimization methods like the quasi-Newton algorithm to numerically maximize $\ell(\Theta)$. The elements of the observed information matrix $J(\Theta)$ are given in **Appendix B.**

5. REAL DATA MODELING

Here, we give a real data analysis for the MOGNP distribution. The real data set shows 63 observations of the strengths of 1.5 cm glass fibers, originally obtained by workers at the UK National Physical Laboratory (Smith and Naylor, 1987). The data are: 0.55, 0.93, 1.25, 1.360, 1.49, 1.52, 1.580, 1.61, 1.64, 1.68, 1.73, 1.810, 2.00, 0.740, 1.04, 1.27, 1.390, 1.490, 1.53, 1.590, 1.610, 1.660, 1.680, 1.760, 1.820, 2.010, 0.77, 1.110, 1.280, 1.420, 1.50, 1.540, 1.60, 1.620, 1.660, 1.690, 1.760, 1.840, 2.240, 0.81, 1.130, 1.290, 1.480, 1.50, 1.550, 1.610, 1.620, 1.660, 1.700, 1.770, 1.84, 0.840, 1.24, 1.30, 1.480, 1.510, 1.550, 1.61, 1.630, 1.67, 1.70, 1.78, 1.890. This data set is well-known in the literature. We compare the MOGNP model with beta normal (B-N) model (Eugene et al., 2002), Marshall-Olkin normal (MO-N) model (Garcia et al., 2010), Kumaraswamy normal (Kw-N) model (Cordeiro and Castro, 2011), McDonald-Normal (Mc-N) model (Cordeiro et al. 2012) and its sub-models under the estimated log-likelihood values $\hat{\ell}$, Kolmogorov-Smirnov (KS), Cramer von Mises (W^*) and Anderson-Darling (A^*) goodness of-fit statistics. The CDFs of the Mc-N, Kw-N, B-N and MO-N models are given by

$$F_{Mc-N}(x,\delta,a,\lambda,\mu,\sigma)=\left[B(a,\lambda)\right]^{-1}\int_0^{\Phi\left(\frac{x-\mu}{\sigma}\right)^{\delta}} w^{a-1}(1-w)^{\lambda-1}\,dw,$$
$$F_{Mc-N}(x,\delta,1,\lambda,\mu,\sigma),F_{Mc-N}(x,1,a,\lambda,\mu,\sigma)$$

and

$$F_{MO-N}(x,\delta,\mu,\sigma)=1-\left[\delta\bar{\Phi}\left(\frac{x-\mu}{\sigma}\right)\right]\left[1-(1-\delta)\,\bar{\Phi}\left(\frac{x-\mu}{\sigma}\right)\right]^{-1}$$

for $x,\mu\in\mathrm{R},a,\delta,\lambda,\sigma>0$ respectively, where $B(a,\lambda)$ is beta function. In addition, the A^* and W^* statistics are described by

$$A^*=-\sum_{i=1}^{n}\frac{2i-1}{n}\left[\ln\hat{F}\left(x_{(i)}\right)+\ln\hat{\bar{F}}\left(x_{(n+1-i)}\right)\right]-n$$

and

$$W^*=\sum_{i=1}^{n}\left(\hat{F}\left(x_{(i)}\right)-\frac{i-0.5}{n}\right)^2+\frac{1}{12n}$$

by Evans et al. (2008), where n is the sample size. In general, it can be chosen as the best model which has the smaller the values of the KS, W^* and A^* statistics and the larger the value of $\hat{\ell}$. All computations are performed by the maxLike routine in the R program. The results are the followings. The MLEs of all models parameters, their standard errors, $\hat{\ell}$, KS, W^* and A^* statistics are listed in Table 1 for data set. As it can be seen from Table 1, the MOGNP model could be considered as the best model under $\hat{\ell}$, W^* and A^* statistics among the other models. We note that the KS statistics of the MOG-N model, which is newly defined and sub-model of the MOGNP model, is smaller than the values of the MOGNP model with a very small difference. The plots of the fitted densities and CDFs of all models are displayed in Figure 3. These plots also show that the MOGNP model provides the better fit to data set than the other models and successfully captures the skewness and kurtosis of the data.

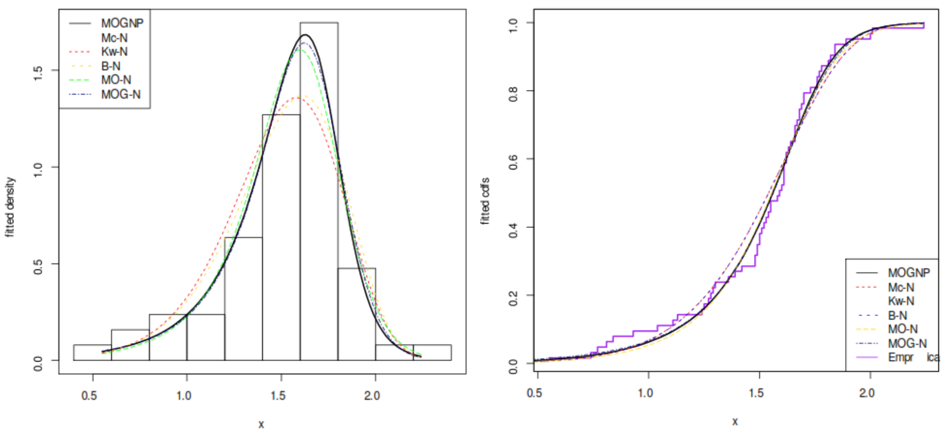

Figure 3: Fitted PDFs and CDFs for Data Set

6. SIMULATION STUDY

We generate $N = 1000$ samples of size $n = 20, 21, \ldots, 300$ from MOGWP distribution with true parameters values $a = 2$, $\lambda = 1$, $\delta = 8$, $\theta = 0.5$ and $\gamma = 1.5$. In this simulation study, we calculate the empirical mean, standard deviations (SD), bias and mean square error (MSE) of the MLEs. The bias and MSE are calculated by

$$Bias_h = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{h}_i - h\right)$$

and

$$MSE_h = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{h}_i - h\right)^2$$

for $h = a, \lambda, \delta, \theta, \gamma$. We give the results of this simulation study in Figure 4.

From Figure 4, we observe that when the sample size increases, the empirical means approach to true parameter value, whereas the all standard error decrease. The biases and MSEs approach 0 as sample size increases.

Table 1
MLEs, Standard Errors of the Estimates (in Parentheses),
$\hat{\ell}$, KS, W^* and A^* Statistics for the Application Models

Model	$\hat{\delta}$	$\hat{a}$	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\sigma}$	$-\hat{\ell}$	A^*	W^*	KS
MOGNP	59.9616	2.6667	4.0246	1.4005	0.5254	11.5287	0.4244	0.0662	0.0964
	(7.2655)	(2.3172)	(18.8010)	(1.2732)	(0.4455)				
Mc-N	0.5508	15.3888	0.7566	2.5357	0.3848	14.0652	0.9071	0.1635	0.1365
	(0.3898)	(1.2020)	2.7713	1.1386	0.5243				
B-N		0.5775	21.1174	2.5538	0.4611	14.0560	0.8977	0.1614	0.1356
		(0.3173)	(1.6854)	(0.1155)	(0.1214)				
Kw-N	0.0609		27.2249	3.3735	0.1733	14.2314	0.9484	0.1673	0.1375
	(0.7031)		(20.3318)	(0.9372)	(0.9503)				
MO-N	22.6351	1	0	0.9467	0.3529	12.3245	0.6190	0.0891	0.1011
	(12.9486)			(0.1453)	(0.0423)				
G-N	1	106.7995	0	3.3546	0.7299	14.1487	0.9176	0.1631	0.1360
		(25.1923)		(0.1941)	(0.0713)				
GP-N	1	22.1382	15.9909	3.7897	0.7815	13.8401	0.8771	0.1538	0.1338
		(6.6800)	(11.2714)	(0.4772)	(0.1100)				
MOP-N	79.1500	1	4.3288	0.9686	0.3861	11.5469	0.4557	0.0703	0.0981
	(22.6738)		(2.4567)	(0.1250)	(0.0473)				
MOG-N	17.7887	30.0514	0	0.3366	0.1585	12.2167	0.4396	0.0688	0.0946
	(10.1865)	(12.5749)		(0.3366)	(0.1585)				
N-P	1	1	160.3669	3.5361	0.0731	14.1070	0.8996	0.1596	0.1347
			(30.3076)	(0.2008)	(0.0731)				
N	1	1	0	1.5068	0.3216	17.9118	1.8993	0.3455	0.1811
				(0.0405)	(0.0286)				

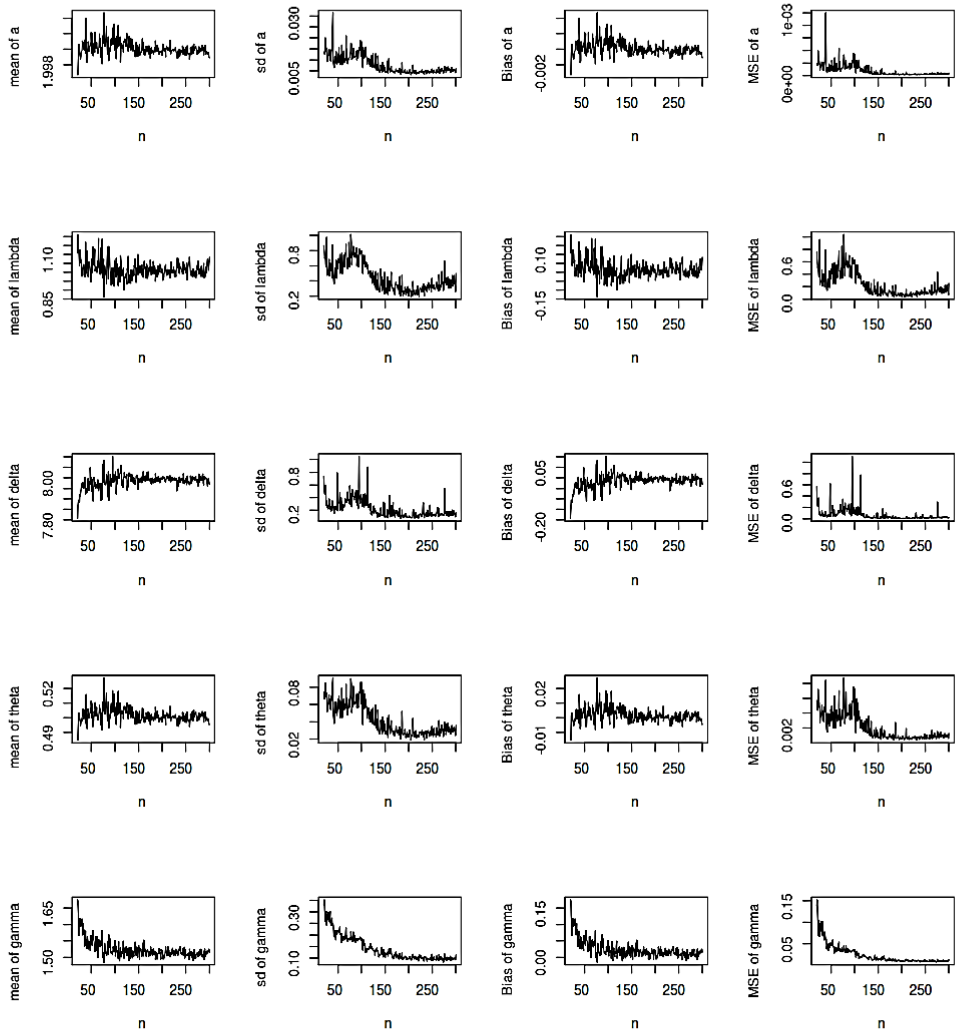

Figure 4: Simulation Results of the Special MOGWP Distribution

7. CONCLUSIONS

We propose a new class of lifetime distributions called the Marshall-Olkin Generalized G Poisson family. The proposed family of distributions is constructed by compounding the Marshall-Olkin Generalized distribution with the truncated Poisson distribution. It can provide better fits than some of the known lifetime distributions and this fact represents a good characterization of this new family. Some characterizations for the new family are presented. The maximum likelihood method is used for estimating the model parameters. The importance and flexibility of the new family are illustrated by means of an application to a real data set.

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APPENDIX A

Theorem 1.

Let $(\Omega, \mathbf{F}, \mathbf{P})$ be a given probability space and let $H = [a, b]$ be an interval for some $d < b$ $(a = -\infty,\ b = \infty$ might as well be allowed$)$. Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$\mathbf{E}\left[q_2(X) \mid X \geq x\right] = \mathbf{E}\left[q_1(X) \mid X \geq x\right]\eta(x), \quad x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^1(H)$, $\eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and η, particularly

$$F(x) = \int_a^x C\left|\frac{\eta'(u)}{\eta(u)q_1(u) - q_2(u)}\right| \exp(-s(u))\ du\ ,$$

where the function s is a solution of the differential equation $s' = \frac{\eta' q_1}{\eta q_1 - q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

APPENDIX B

$$U_{a\psi}=-\sum_{i=0}^{n}\frac{G'\left(x_i;\psi\right)}{\overline{G}(x_i;\psi)}-2\sum_{i=0}^{n}\frac{s_i\left(\partial q_i/\partial\psi\right)-t_i q_i}{s_i^2}-\lambda\sum_{i=0}^{n}w_i, U_{\lambda\lambda}=\frac{-n}{\lambda^2}-\frac{ne^{-\lambda}\left[1-2e^{-\lambda}\right]}{\left(1-e^{-\lambda}\right)^2},$$

$$U_{\delta\delta}=\frac{-n}{\delta^2}+2\sum_{i=0}^{n}\frac{p_i^2}{s_i^2}+\lambda\sum_{i=0}^{n}2p_i^2, U_{\lambda\psi}=-\sum_{i=0}^{n}d_i, U_{\delta\lambda}=-\sum_{i=0}^{n}m_i, U_{a\lambda}=-\sum_{i=0}^{n}w_i,$$

$$U_{\delta a}=\sum_{i=0}^{n}\frac{\overline{G}(x_i;\psi)^a\log\overline{G}(x_i;\psi)}{s_i\left[\lambda\left(2\delta p_i+s_i\right)s_i-2\right]^{-1}}+\sum_{i=0}^{n}\frac{\left(\lambda s_i^2-2\right)p_i q_i}{s_i^2},$$

$$U_{\delta\psi}=-2\sum_{i=0}^{n}\frac{-aG'\left(x_i;\psi\right)}{s_i\overline{G}(x_i;\psi)^{1-a}}-2\sum_{i=0}^{n}\frac{-t_i p_i}{s_i^2}-\lambda\sum_{i=0}^{n}\frac{\partial m_i}{\partial\psi},$$

$$U_{aa}=\frac{-n}{a^2}-2\sum_{i=0}^{n}\frac{s_i\left(\partial q_i/\partial a\right)-q_i^2}{s_i^2}+\lambda\delta\sum_{i=0}^{n}\frac{\partial w_i}{\partial a},$$

and

$$U_{\psi_k\psi_k}=\sum_{i=0}^{n}\frac{g\left(x_i;\psi\right)g''\left(x_i;\psi\right)-\left(g'\left(x_i;\psi\right)\right)^2}{g\left(x_i;\psi\right)^2}-2\sum_{i=0}^{n}\frac{s_i\left(\partial t_i/\partial\psi\right)-t_i^2}{s_i^2}$$
$$-\left(a-1\right)\sum_{i=0}^{n}\frac{\overline{G}(x_i;\psi)G''\left(x_i;\psi\right)-\left(G'\left(x_i;\psi\right)\right)^2}{\overline{G}(x_i;\psi)^2}-\lambda\sum_{i=0}^{n}\frac{\partial t_i}{\partial\psi},$$

where $g''\left(x_i;\psi\right)=\left[\partial^2 g\left(x_i;\psi\right)/\partial\psi^2\right], G''\left(x_i;\psi\right)=\left[\partial^2 G\left(x_i;\psi\right)/\partial\psi^2\right].$