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DISCRETE INVERSE WEIBULL BETA MODEL: PROPERTIES AND APPLICATIONS IN HEALTH SCIENCE

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ABSTRACT

The study deals with introducing a new discrete probability model which is obtained by compounding two parameter discrete inverse Weibull distribution [12] with Beta distribution of first kind. Structural properties of the distribution such as its unimodality, hazard rate behavior and index of dispersion are discussed. Model parameters are estimated by ML estimation method followed by Monte Carlo simulation procedure. Finally, a real data set is analyzed to investigate the suitability of the proposed model in modeling discrete data from health science.

KEYWORDS

Discrete Inverse Weibull Distribution, Monte Carlo simulation, Beta distribution, Hospital Stay Data.

1. INTRODUCTION

Probability models are commonly applied to describe real world phenomena. Due to the usefulness of probability models, their theory is widely studied and new distributions are developed [14,15]. Recently, there has been an increased interest in developing new probability models using compounding technique. Compound distributions are very flexible and can be used efficiently to model different types of data sets. Many probability distributions have been constructed by researchers using the technique of compounding. Dubey [4] studied a compound gamma, beta and F distribution by compounding a gamma distribution with another gamma distribution and reduced it to the beta first and second kind and to the F distribution by suitable transformations. Sankaran [18] studied a compound of PD (Poisson distribution) with that of LD (Lindley distribution) for modeling count data. Many compound distributions were proposed by Gerstenkorn [7,8], he studied compound of GD (gamma distribution) with ED (exponential distribution) by considering the parameter of GD (gamma distribution) as an exponential random variate and also he introduced compound of PD (Polya distribution) with BD (Beta distribution). Ghitany, Al-Mutairi and Nadarajah [9,10] proposed ZTPLD (zero truncated Poisson-Lindley distribution), the distribution was suitable for modeling count data in the case where the distribution has to be adjusted for the count of missing zeros. A new compound distribution was introduced by Zamani and Ismail [20] by compounding NBD (negative binomial distribution) with one parameter LD (Lindley distribution) that provides good fit for count data where the probability at zero has a hefty value. Kus et al. [13] introduced a new discrete distribution by compounding the binomial and discrete Lindley distributions. Recently, Ahmad, Rashid and Jan [1] introduced a new class of generalized complementary compound lifetime distributions which is obtained by compounding generalized Lindley distribution with power series distribution.

In this article, we introduce a new count data model named as Discrete Inverse Weibull Beta Distribution (DIWBD) by compounding two parameter DIWD (discrete inverse Weibull distribution) with BD (Beta distribution) of first kind, as there is a need to find more reasonable discrete probability models or survival models in medical science and other fields, to fit to various discrete data sets.

2. MATERIAL AND METHODS FOR DIWBD

DIWD (Discrete inverse Weibull distribution) was introduced by Jazi, Lai and Alamatsaz [12], which is a discrete alternative of the continuous inverse Weibull random variable with probability mass function (pmf) defined by:

$$f_1\left(x;\theta,\gamma\right)=\theta^{(x+1)^{-\gamma}}-\theta^{(x)^{-\gamma}}, \quad x=0,1,2,\ldots, \tag{1}$$

where $\gamma>0$ and $0<\theta<1$ are its parameters. The first and the second moments of the DIW random variable X are given by

$$E(X)=\sum_{x=1}^{\infty}(1-\theta^{x^{-\gamma}})$$

$$E(X^2)=2\sum_{x=1}^{\infty}x(1-\theta^{x^{-\gamma}})+E(X)$$

BD (Beta Distribution) of first kind provides the leading family of continuous distributions on bounded support. The probability density function of generalized beta distribution is given by

$$f(x;\tau,\alpha,\beta)=\frac{\tau x^{\tau\alpha-1}(1-x^{\tau})^{\beta-1}}{B(\alpha,\beta)}; \qquad 0<x<1;\tau>0;\alpha>0;\beta>0 \tag{2}$$

If we put $\tau=1,$ the equation (2) reduces to beta distribution of first kind with probability density function as:

$$f(x;\alpha,\beta)=\frac{1}{B(\alpha,\beta)}\,x^{\alpha-1}(1-x)^{\beta-1}; \qquad 0<x<1;\alpha>0;\beta>0, \tag{3}$$

where α and β are positive real quantities and the variable X satisfies $0<x<1$. The quantity $B(\alpha,\beta)$ is the beta function. Equation (3) is also known as the standard beta or classical beta distribution. The raw moments of Beta distribution of first kind (BD) are given by

$$E(X^r)=\int_0^1 x^r f_2(X;\alpha,\beta)dx$$

$$E(X^r)=\frac{\Gamma(\alpha+r)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+r)\Gamma(\alpha)} \tag{4}$$

Usually the parameters γ and θ in DIWD (Discrete Inverse Weibull Distribution) are fixed constants but here we are going to consider a problem in which the probability parameter θ in DIWD is itself a random variable following BD (Beta Distribution) with probability mass function (3).

3. DEFINITION OF THE PROPOSED MODEL

If $X|\theta\sim$ DIWD(θ,γ), where θ is itself a random variable following BD (α,β), then determining the distribution that results from marginalizing over θ will be known as a compound of discrete Weibull distribution with that of Beta distribution, which is designated by DIWBD (γ,α,β). It may be illustrious that proposed model will be a discrete since the parent distribution DIWD is discrete.

Theorem 3.1:

The probability mass function of a compound of DIWD (θ,γ) with BD (α,β) is given by

$$f_{DIWBD}(X;\gamma,\alpha,\beta)=\frac{1}{B(\alpha,\beta)}[B(\beta,(x+1)^{-\gamma}+\alpha)-B(\beta,x^{-\gamma}+\alpha)]$$

where $x=0,1,2,......$ and $\gamma,\alpha,\beta>0$

Proof:

Using the definition (3), the pmf of a compound of DIWD(θ,γ) with BD(α,β) can be obtained as

$$f_{DIWBD}(X;\gamma,\alpha,\beta)=\int_0^1 f_1(x|\theta)\,f_2(\theta)\,d\theta$$

$$f_{DIWBD}(X;\gamma,\alpha,\beta)=\frac{1}{B(\alpha,\beta)}\int_0^1(\theta^{(x+1)^{-\gamma}}-\theta^{x^{-\gamma}})\theta^{\alpha-1}(1-\theta)^{\beta-1}\,d\theta$$

$$f_{DIWBD}(X;\gamma,\alpha,\beta)=\frac{1}{B(\alpha,\beta)}[B(\beta,(x+1)^{-\gamma}+\alpha)-B(\beta,x^{-\gamma}+\alpha)]$$

$$f_{DIWBD}(X;\gamma,\alpha,\beta)=\frac{1}{B(\alpha,\beta)}\left[\frac{\Gamma(\beta)\,\Gamma\left((x+1)^{-\gamma}+\alpha\right)}{\Gamma\left(\beta+(x+1)^{-\gamma}+\alpha\right)}-\frac{\Gamma(\beta)\,\Gamma\left(x^{-\gamma}+\alpha\right)}{\Gamma\left(\beta+x^{-\gamma}+\alpha\right)}\right], \tag{5}$$

where $x=0,1,2,...$ and $\gamma,\alpha,\beta>0$. From here a random variable X following a compound of DIWD with BD will be symbolized by DIWBD$(X;\gamma,\alpha,\beta)$.

Fig. 1(a) to Fig. 1(i) provides a pmf plot of the proposed model DIWBD $\left(X;\gamma,\alpha,\beta\right)$ for different values of parameters. It is evident that the proposed model is right skewed with unimodel behavior.

The Cumulative distribution function of the DIWBD $\left(X;\gamma,\alpha,\beta\right)$ is given by

$$F(x)=\frac{1}{B(\alpha,\beta)}B\left(\ \beta,(x+1)^{-\gamma}+\alpha\right),\qquad x=0,1,2,\dots \text{ and } \gamma>0,\alpha>0,\beta>0,$$

where $B\left(\ \beta,(x+1)^{-\gamma}+\alpha\right)=\dfrac{\Gamma\left(\beta\right)\,\Gamma\left(\,(x+1)^{-\gamma}+\alpha\right)}{\Gamma\left(\beta+\,(x+1)^{-\gamma}+\alpha\,\right)}$

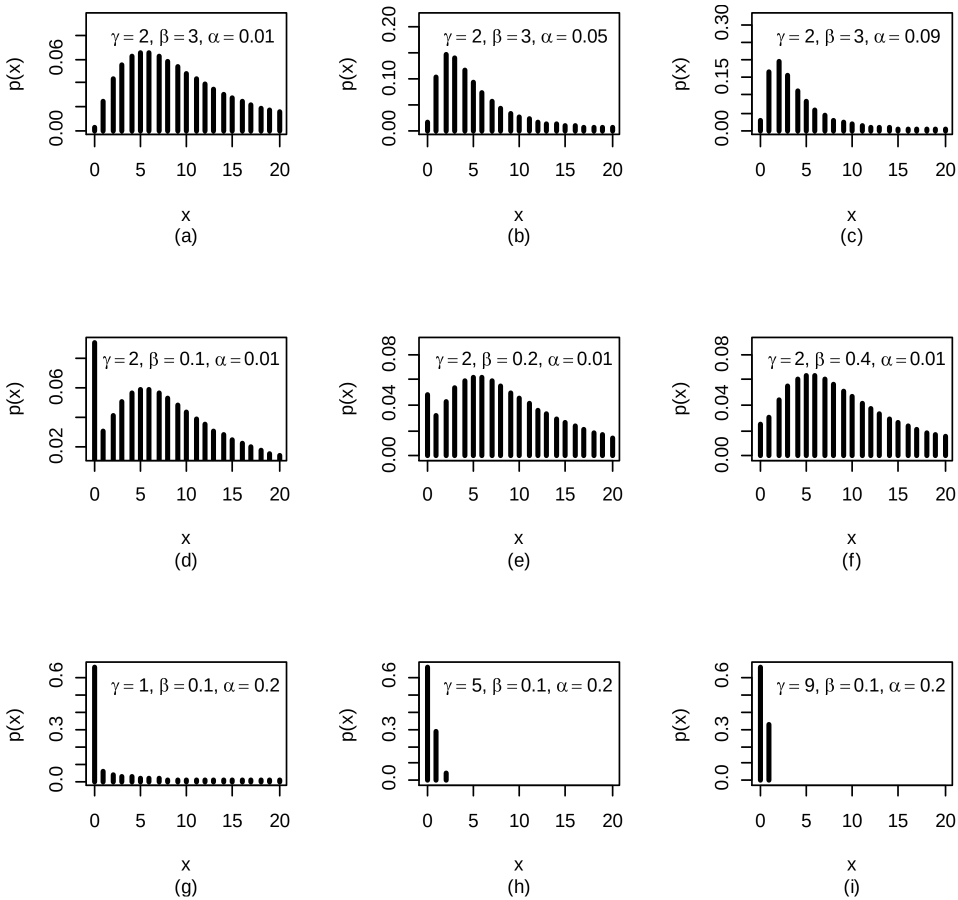

Fig. 1: pmf Plot of Discrete Inverse Weibull Beta Distribution

Fig. 2(a) to Fig. 2(i) provides a cdf plot of the proposed model $\mathrm{DIWBD}\left(X;\gamma,\alpha,\beta\right)$ for different values of parameters. The initial rise of the cdf plot increases as α and γ increases but as β increases, initial rise of the cdf plot also decreases.

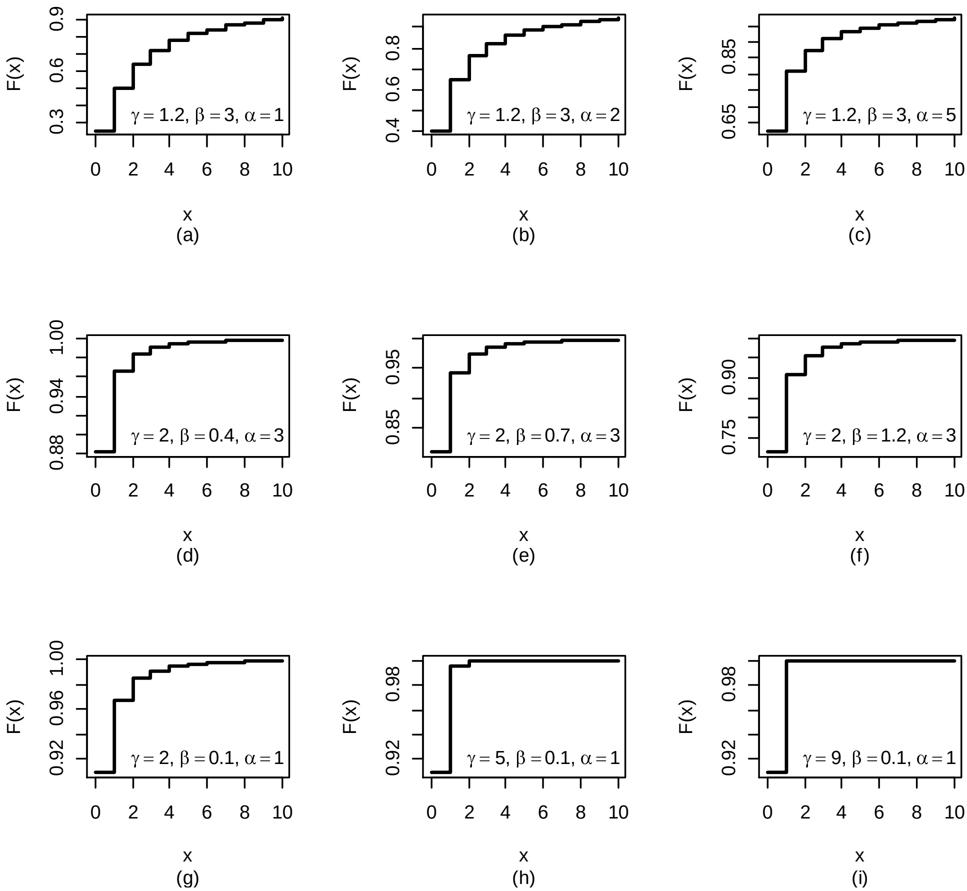

Fig. 2: CDF Plot of Discrete Inverse Weibull Beta Distribution

3.1 Random Data Generation from Discrete Inverse Weibull Beta Distribution (DIWBD)

For simulating a random number sequence $x_1, x_2, ..., x_n$ from the discrete inverse Weibull Beta random variable X with pmf $p(X = x_i) = \tau_i, \sum_{i=0}^{k} \tau_i = 1$ and a cdf F(x), where k may be finite or infinite can be described as

Step1: Generate a random number u from uniform distribution U(0,1).
Step2: Generate random number x_i based on

$$if\ u \le \tau_0 = F(x_0)\ then\ X = x_0$$
$$if\ \tau_0 < u \le \tau_0 + \tau_1 = F(x_1)\ then\ X = x_1$$
$$\vdots$$
$$.if\ \sum_{j=0}^{k-1} \tau_j < u \le \sum_{j=0}^{k} \tau_j = F(x_k)\ then\ X = x_k$$

In order to generate n random numbers from discrete inverse Weibull Beta distribution, $x_1, x_2, ..., x_n$, repeat step 1 to step 2 n times.

4. SUB-MODELS UNDER DIWBD

In this particular section we show that the proposed model can be nested to different models under specific parameter setting.

Proposition 4.1:

If $X \sim DIWBD(X; \gamma, \alpha, \beta)$ then by setting $\gamma = 1$, we get a compound of inverse geometric distribution with Beta distribution.

Proof:

For $\gamma = 1$ in (1), DIWD reduces to inverse geometric distribution (IGD), hence a compound of IGD with BD is followed from (5) by simply substituting $\gamma = 1$ in it.

$$f_{DIGBD}(X; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} [B(\beta, (x+1)^{-1} + \alpha) - B(\beta, (x^{-1} + \alpha))],$$

for $x = 0, 1, 2, ..., \alpha, \beta > 0$

where $B(\beta, x^{-1} + \alpha) = \dfrac{\Gamma(\beta)\, \Gamma\left(x^{-1} + \alpha\right)}{\Gamma\left(\beta + x^{-1} + \alpha\right)}$,

which is the probability mass function of a compound of IGD with BD.

Proposition 4.2:

If $X \sim DIWBD(X; \gamma, \alpha, \beta)$ then by setting $\alpha = \beta = 1$, we obtain a compound of DIWD distribution with uniform distribution.

Proof:

For $\alpha = \beta = 1$ in (3), BD reduces to Uniform (0,1) distribution, therefore a compound DIWD with Uniform distribution is followed from (5) by simply putting $\alpha = \beta = 1$ in it.

$$f_{DIWUD}(X; \gamma) = [B(1, (x+1)^{-\gamma} + 1) - B(1, x^{-\gamma} + 1)]$$

$$f_{DIWUD}(X;\gamma)=\left[\frac{x^{-\gamma}-(x+1)^{-\gamma}}{(x^{-\gamma}+1)[(x+1)^{-\gamma}+1]}\right] \text{ for } x=0,1,2,...,\gamma>0$$

which is probability mass function of a compound of DIWD with uniform distribution.

Proposition 4.3:

If $X \sim DIWBD(X;\gamma,\alpha,\beta)$ then by setting $\gamma=1$ and $\alpha=\beta=1$ we obtain a compound of inverse geometric distribution with uniform distribution.

Proof:

For $\gamma=1$ in (1), DIWD reduces to inverse geometric distribution and for $\alpha=\beta=1$, Beta distribution of first kind reduces to U(0,1) distribution hence a compound of inverse geometric distribution with uniform distribution can be obtained from (5) by simply substituting $\gamma=1$ and $\alpha=\beta=1$ in it.

$$f_{IGUD}(X)=\frac{1}{(x+1)(x+2)} \quad \text{for } x=0,1,2,...$$

Proposition 4.4:

If $X \sim DIWBD(X;\gamma,\alpha,\beta)$ then by setting $\gamma=2$ and $\alpha=\beta=1$ in (3), we obtain a compound of discrete inverse Rayleigh distribution with uniform distribution.

Proof:

For $\gamma=2$ in (1), DIWD reduces to DIRD (discrete inverse Rayleigh distribution) and for $\alpha=\beta=1$, BD (Beta distribution) of first kind reduces to U(0,1) distribution, hence a compound of (IRD) inverse Rayleigh distribution with UD (Uniform distribution) can be obtained from (5) by simply substituting $\gamma=2$ and $\alpha=\beta=1$ in it.

$$f_{DIRUD}(X)=\frac{2x+1}{(x^2+1)((x+1)^2+1)} \text{ for } x=0,1,2,...,$$

which is the pmf of DIRD (Discrete Inverse Rayleigh Distribution) with U(0,1) distribution.

Proposition 4.5:

If $X \sim DIWBD(X;\gamma,\alpha,\beta)$ then by setting $\gamma=2$, we get a compound of DRD (Discrete Rayleigh distribution) with BD (Beta distribution) of first kind.

Proof:

For $\gamma=2$ in (1), DIWD reduces to discrete inverse Rayleigh distribution (DIRD), hence a compound of DIRD with BD is followed from (5) by simply substituting $\gamma=2$ in it.

$$f_{DIRBD}(X;\alpha,\beta)=\frac{1}{B(\alpha,\beta)}[B(\beta,(x+1)^{-2}+\alpha)-B(\beta,x^{-2}+\alpha)]$$

for $x=0,1,2,...,\alpha,\beta>0$

where $B(\beta, x^{-2}+\alpha)=\dfrac{\Gamma(\beta)\,\Gamma\left(x^{-2}+\alpha\right)}{\Gamma\left(\beta+x^{-2}+\alpha\right)}$

which is the probability mass function of a compound of DIRD with BD.

5. RELIABILITY MEASURES OF COMPOUND DISCRETE INVERSE WEIBULL BETA DISTRIBUTION

If $X \sim DIWBD\left(X;\gamma,\alpha,\beta\right)$, then the various reliability measures of a random variable X are given by

(a) **Survival Function**

$$s(x)=1-\frac{1}{B(\alpha,\beta)}B\left((x+1)^{-\gamma}+\alpha,\beta\right) \quad x=0,1,2... \text{ and } \alpha>0,\beta>0$$

where $B\left((x+1)^{-\gamma}+\alpha,\beta\right)=\dfrac{\Gamma(\beta)\,\Gamma\left((x+1)^{-\gamma}+\alpha\right)}{\Gamma\left(\beta+(x+1)^{-\gamma}+\alpha\right)}$

(b) **Rate of Failure Function**

$$r(x)=\frac{p(x)}{s(x)}=\frac{[B(\beta,(x+1)^{-\gamma}+\alpha)-B(\beta,x^{-\gamma}+\alpha)]}{B(\alpha,\beta)-B\left((x+1)^{-\gamma}+\alpha,\beta\right)} \quad x=0,1,2...$$

and $\alpha>0,\beta>0,\gamma>0$

where $B\left((x+1)^{-\gamma}+\alpha,\beta\right)=\dfrac{\Gamma(\beta)\,\Gamma\left((x+1)^{-\gamma}+\alpha\right)}{\Gamma\left(\beta+(x+1)^{-\gamma}+\alpha\right)}$

(c) **Second Rate of Failure Function**

$$h(x)=\log\left(\frac{s(x)}{s(x+1)}\right)=\log\left(\frac{B(\alpha,\beta)-B\left((x+1)^{-\gamma}+\alpha,\beta\right)}{B(\alpha,\beta)-B\left((x+2)^{-\gamma}+\alpha,\beta\right)}\right)$$

$x=0,1,2...$ and $\alpha>0,\beta>0,\gamma>0$

where, B(.) refers to the beta function defined by $B(a,b)=\dfrac{\Gamma a\,\Gamma b}{\Gamma\left(a+b\right)}$

Fig. 3(a) to Fig. 3(i) provides a hazard rate function plot of the proposed model $DIWBD\left(X;\gamma,\alpha,\beta\right)$ for different values of parameters.

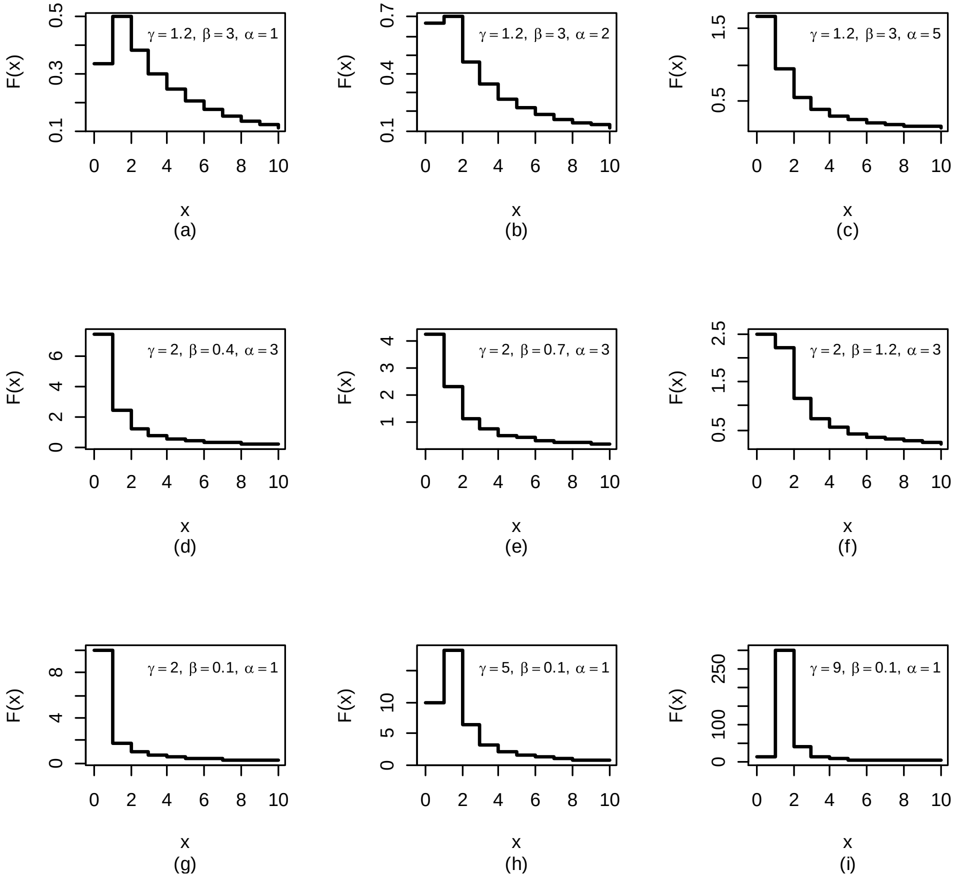

Fig. 3: r(x) Plot of Discrete Inverse Weibull Beta Distribution

6. MOMENT GENERATING AND PROBABILITY GENERATING FUNCTIONS OF $DIWBD(X;\gamma,\alpha,\beta)$

(a) The moment generating function of the Compound discrete Inverse Weibull Beta distribution is

$$M_x(t)=\sum_{x=0}^{\infty} e^{tx}\,p(x)$$

$$M_x(t)=\sum_{x=0}^{\infty} e^{tx}\,\frac{1}{B(\alpha,\beta)}[B(\beta,(x+1)^{-\gamma}+\alpha)-B(\beta,x^{-\gamma}+\alpha)]$$

$$M_x(t)=\sum_{x=0}^{\infty} e^{tx}[\{1-\frac{1}{B(\alpha,\beta)}B(\beta,x^{-\gamma}+\alpha)\}-\{1-\frac{1}{B(\alpha,\beta)}B(\beta,(x+1)^{-\gamma}+\alpha)\}]$$

$$M_x(t)=\sum_{x=0}^{\infty} e^{tx}[\psi(x;\gamma,\beta,\alpha)-\psi(x+1;\gamma,\beta,\alpha)]\ ,$$

where $\psi(x;\gamma,\beta,\alpha)=1-\frac{1}{B(\alpha,\beta)}B(\beta,x^{-\gamma}+\alpha)$

$$M_x(t)=\begin{pmatrix}\psi(0;\gamma,\beta,\alpha)+e^{t}\psi(1;\gamma,\beta,\alpha)+e^{2t}\psi(2;\gamma,\beta,\alpha)\\ +e^{3t}\psi(3;\gamma,\beta,\alpha)+....-\{\psi(1;\gamma,\beta,\alpha)\\ +e^{t}\psi(2;\gamma,\beta,\alpha)+e^{2t}\psi(3;\gamma,\beta,\alpha)+e^{3t}\psi(4;\gamma,\beta,\alpha)+......\}\end{pmatrix}$$

$$M_x(t)=\begin{pmatrix}\psi(0;\gamma,\beta,\alpha)+(e^{t}-1)\psi(1;\gamma,\beta,\alpha)\\ +(e^{2t}-e^{t})\psi(2;\gamma,\beta,\alpha)+(e^{3t}-e^{2t})\psi(3;\gamma,\beta,\alpha)+...\end{pmatrix}$$

$$M_x(t)=\left[1+\sum_{x=1}^{\infty}(e^{xt}-e^{(x-1)t})\psi(x;\gamma,\beta,\alpha)\right]$$

Differentiating $M_x(t)$ r times with respect to t

$$M_x^{(r)}(t)=\sum_{x=1}^{\infty}(x^{r}e^{xt}-(x-1)^{r}e^{(x-1)t})\psi(x;\gamma,\beta,\alpha)$$

First four moments of the proposed model are given by

$$\mu_1^{'}=\sum_{x=1}^{\infty}\ \psi(x;\gamma,\beta,\alpha)$$

$$\mu_2^{'}=\sum_{x=1}^{\infty}(2x-1)\psi(x;\gamma,\beta,\alpha)$$

$$\mu_3^{'}=\sum_{x=1}^{\infty}(3x^2-3x+1)\psi(x;\gamma,\beta,\alpha)$$

$$\mu_4^{'}=\sum_{x=1}^{\infty}(4x^3-6x^2+4x-1)\psi(x;\gamma,\beta,\alpha)$$

(b) Probability generating function of the Compound discrete Inverse Weibull Beta distribution is

$$G_{[x]}(t)=\sum_{x=0}^{\infty}t^{x}p(x)$$

$$G_{[x]}(t)=\sum_{x=0}^{\infty}t^{x}\frac{1}{B(\alpha,\beta)}[B(\beta,(x+1)^{-\gamma}+\alpha)-B(\beta,x^{-\gamma}+\alpha)]$$

$$G_{[x]}(t)=\sum_{x=0}^{\infty}t^{x}[\{1-\frac{1}{B(\alpha,\beta)}B(\beta,x^{-\gamma}+\alpha)\}-\{1-\frac{1}{B(\alpha,\beta)}B(\beta,(x+1)^{-\gamma}+\alpha)\}]$$

$$G_{[x]}(t)=\sum_{x=0}^{\infty}t^{x}[\psi(x;\gamma,\beta,\alpha)-\psi(x+1;\gamma,\beta,\alpha)]\ ,$$

where $\psi(x;\gamma,\beta,\alpha)=1-\frac{1}{B(\alpha,\beta)}B(\beta,x^{-\gamma}+\alpha)$

$$G_{[x]}(t)=\begin{pmatrix}\psi(0;\gamma,\beta,\alpha)+(t-1)\psi(1;\gamma,\beta,\alpha)\\ \qquad +t(t-1)\psi(2;\gamma,\beta,\alpha)+t^{2}(t-1)\psi(3;\gamma,\beta,\alpha)+...\end{pmatrix}$$

$$G_{[x]}(t)=\left[1+(t-1)\sum_{x=1}^{\infty}t^{x-1}\psi(x;\gamma,\beta,\alpha)\right]$$

Differentiating $G_{[x]}(t)$ with respect to t, we have

$$G_{[x]}^{'}(t)=\sum_{x=1}^{\infty}((t-1)(x-1)t^{x-2}+t^{x-1})\psi(x;\gamma,\beta,\alpha)$$

$$G_{[x]}^{'}(t)=\frac{1}{B(\alpha,\beta)}\sum_{x=1}^{\infty}(t^{x-2}(xt-x+1))\psi(x;\gamma,\beta,\alpha)$$

$$G_{[x]}^{''}(t)=\sum_{x=1}^{\infty}((x-1)t^{x-3}(xt-x+2))\psi(x;\gamma,\beta,\alpha)$$

At t=1, $G_{[x]}^{'}(t)$, $G_{[x]}^{''}(t)$ gives first and second factorial moments

$$E(x)=G_{[x]}^{'}(1)=\sum_{x=1}^{\infty}\ \psi(x;\gamma,\beta,\alpha)$$

$$E(x^{2})=G_{[x]}^{'}(1)+G_{[x]}^{''}(1)=\sum_{x=1}^{\infty}(2x-1)\psi(x;\gamma,\beta,\alpha)$$

Table 1 exhibits the index of dispersion, $IOD=\{E(X^{2})-(E(X))^{2}\}/E(X)$, mean and variance for different values of the parameters γ,α and β for three parameter discrete inverse Weibull Beta distribution. For DIWBD expressions for mean and variance are not in the closed form, hence we employed Wolfram mathematical language for calculation of index of dispersion. It can be seen that this variance to mean ratio indicates that discrete inverse Weibull Beta model is over dispersed as well as under-dispersed.

Table 1
Index of Dispersion, Mean and Variance of $DIWBD(X;\gamma,\alpha,\beta)$ for Different Values of Parameters

$\beta=0.5$											
γ	α	0.1	0.3	0.5	0.8	1.3	1.8	2.5	3.6	4.2	5
2.0	Mean	4.006	1.854	1.240	0.838	0.549	0.410	0.303	0.215	0.186	0.157
	Variance	204.913	61.780	34.564	20.243	11.762	8.242	5.794	3.947	3.360	2.805
	IOD	51.147	33.325	27.864	24.143	21.422	20.119	19.135	18.347	18.083	17.832
2.5	Mean	2.536	1.290	0.900	0.628	0.423	0.320	0.239	0.171	0.149	0.126
	Variance	14.900	5.540	3.420	2.182	1.372	1.005	0.733	0.516	0.444	0.375
	IOD	5.876	4.294	3.802	3.473	3.244	3.140	3.064	3.012	2.990	2.972
3.2	Mean	1.738	0.969	0.703	0.507	0.350	0.268	0.202	0.146	0.127	0.108
	Variance	3.119	1.494	1.043	0.742	0.514	0.396	0.302	0.220	0.192	0.164
	IOD	1.795	1.542	1.484	1.464	1.468	1.478	1.490	1.505	1.510	1.516
$\beta=1.2$											
γ	α	0.1	0.3	0.5	0.8	1.3	1.8	2.5	3.6	4.2	5
2.6	Mean	2.688	1.626	1.261	0.976	0.727	0.584	0.460	0.346	0.305	0.264
	Variance	11.845	5.271	3.658	2.625	1.860	1.466	1.143	0.857	0.755	0.652
	IOD	4.407	3.241	2.901	2.689	2.559	2.511	2.484	2.473	2.471	2.471
3.0	Mean	2.147	1.365	1.082	0.853	0.645	0.523	0.415	0.314	0.278	0.240
	Variance	4.557	2.287	1.691	1.292	0.976	0.801	0.646	0.500	0.445	0.389
	IOD	2.122	1.675	1.562	1.515	1.513	1.532	1.558	1.591	1.605	1.620
4.5	Mean	1.340	0.973	0.817	0.673	0.526	0.434	0.349	0.268	0.237	0.206
	Variance	0.722	0.493	0.446	0.413	0.372	0.336	0.293	0.242	0.221	0.197
	IOD	0.539	0.507	0.546	0.614	0.707	0.774	0.840	0.905	0.931	0.957
$\beta=2.5$											
γ	α	0.1	0.3	0.5	0.8	1.3	1.8	2.5	3.6	4.2	5
2.1	Mean	4.380	2.674	2.130	1.714	1.343	1.122	0.921	0.724	0.649	0.571
	Variance	120.894	50.078	34.230	24.345	17.111	13.437	10.449	7.820	6.892	5.959
	IOD	27.603	18.731	16.071	14.205	12.738	11.972	11.347	10.807	10.619	10.433
2.6	Mean	2.873	1.893	1.559	1.290	1.038	0.880	0.732	0.582	0.525	0.464
	Variance	12.438	5.992	4.396	3.352	2.547	2.114	1.737	1.379	1.244	1.102
	IOD	4.329	3.165	2.820	2.598	2.455	2.401	2.373	2.369	2.370	2.376
3.2	Mean	2.089	1.469	1.246	1.057	0.870	0.747	0.628	0.505	0.457	0.405
	Variance	3.238	1.788	1.408	1.159	0.962	0.849	0.741	0.624	0.576	0.523
	IOD	1.550	1.217	1.130	1.096	1.107	1.136	1.179	1.237	1.262	1.290

7. PARAMETER ESTIMATION OF DIWBD

In this section, we discuss the ML (Maximum Likelihood) and MM (Moments Method) methods of parameter estimation of $DIWBD\left(X;\gamma,\alpha,\beta\right)$

7.1 Moments Method of Estimation

For estimating three unknown parameters of $DIWBD\left(X;\gamma,\alpha,\beta\right)$ by the method of moments, we need to equate first three sample moments with their corresponding population moments.

$$\mu_1=\gamma_1;\ \mu_2=\gamma_2 \text{ and } \mu_3=\gamma_3$$

where γ_i is the ith sample moment and μ_i is the ith corresponding population moment and the solution for $\hat{\gamma},\hat{\alpha}$ and $\hat{\beta}$ may be obtained by solving above equations simultaneously through numerical methods like Newton Raphson method.

7.2 Maximum Likelihood Method of Estimation

The estimation of parameters of $DIWBD\left(X;\gamma,\alpha,\beta\right)$ model using ML (maximum likelihood) method of estimation requires the log likelihood function of $DIWBD\left(X;\gamma,\alpha,\beta\right)$

$$\pounds(X;\gamma,\alpha,\beta)=\log L(X;\gamma,\alpha,\beta)$$

$$=\sum_{i=1}^{n}\log(B(\beta,(x+1)^{-\gamma}+\alpha)-B(\beta,x^{-\gamma}+\alpha))-n\log(B(\beta,\alpha)) \tag{9}$$

The ML (maximum likelihood) estimate of $\Theta=\left(\hat{\gamma},\hat{\alpha},\hat{\beta}\right)^T$ can be obtained by differentiating (9) with respect to three unknown parameters γ,α and β respectively and then equating them to zero.

$$\frac{\partial}{\partial\gamma}\pounds(X;\gamma,\alpha,\beta)=\sum_{i=1}^{n}\left(\frac{\frac{\partial}{\partial\gamma}\left(B(\beta,(x+1)^{-\gamma}+\alpha)-B(\beta,x^{-\gamma}+\alpha)\right)}{\left(B(\beta,(x+1)^{-\gamma}+\alpha)-B(\beta,x^{-\gamma}+\alpha)\right)}\right), \tag{10}$$

where

$$B(\beta,(x+1)^{-\gamma}+\alpha)-B(\beta,x^{-\gamma}+\alpha)=\frac{\Gamma(\beta)\,\Gamma\left((x+1)^{-\gamma}+\alpha\right)}{\Gamma\left(\beta+(x+1)^{-\gamma}+\alpha\right)}-\frac{\Gamma(\beta)\,\Gamma\left(x^{-\gamma}+\alpha\right)}{\Gamma\left(\beta+x^{-\gamma}+\alpha\right)}$$

$$\frac{\partial}{\partial\beta}\pounds(X;\gamma,\alpha,\beta)=\sum_{i=1}^{n}\left(\frac{\frac{\partial}{\partial\beta}\left(B(\beta,(x+1)^{-\gamma}+\alpha)-B(\beta,x^{-\gamma}+\alpha)\right)}{\left(B(\beta,(x+1)^{-\gamma}+\alpha)-B(\beta,x^{-\gamma}+\alpha)\right)}\right)-n\frac{\frac{\partial}{\partial\beta}\left(B(\beta,\alpha)\right)}{B(\beta,\alpha)} \tag{11}$$

$$\frac{\partial}{\partial\alpha}\pounds(X;\gamma,\alpha,\beta)=\sum_{i=1}^{n}\left(\frac{\frac{\partial}{\partial\alpha}\left(B(\beta,(x+1)^{-\gamma}+\alpha)-B(\beta,x^{-\gamma}+\alpha)\right)}{\left(B(\beta,(x+1)^{-\gamma}+\alpha)-B(\beta,x^{-\gamma}+\alpha)\right)}\right)-n\frac{\frac{\partial}{\partial\alpha}\left(B(\beta,\alpha)\right)}{B(\beta,\alpha)} \tag{12}$$

Equations (10), (11) and (12) cannot be solved using analytical method, therefore $\hat{\gamma},\hat{\alpha}$ and $\hat{\beta}$ will be obtained by maximizing the loglikelihood function numerically and using Newton-Raphson method for solving equations iteratively and numerically. There are many statistical softwares like R studio, SAS etc., which helps in quick solution of the nonlinear equations. We can compute the second partial derivatives, which are useful to obtain the Fisher's information matrix or curvature matrix as follows.

$$I_y\left(\gamma,\alpha,\beta\right)=\begin{bmatrix} -E\left(\frac{\partial^2 l}{\partial\gamma^2}\right) & -E\left(\frac{\partial^2 l}{\partial\gamma\partial\alpha}\right) & -E\left(\frac{\partial^2 l}{\partial\gamma\partial\beta}\right) \\ -E\left(\frac{\partial^2 l}{\partial\alpha\partial\beta}\right) & -E\left(\frac{\partial^2 l}{\partial\alpha^2}\right) & -E\left(\frac{\partial^2 l}{\partial\alpha\partial\gamma}\right) \\ -E\left(\frac{\partial^2 l}{\partial\beta\partial\alpha}\right) & -E\left(\frac{\partial^2 l}{\partial\beta\partial\gamma}\right) & -E\left(\frac{\partial^2 l}{\partial\beta^2}\right) \end{bmatrix} \tag{13}$$

One can easily demonstrate that the discrete inverse Weibull Beta distribution satisfies the regularity conditions [5]. Hence, the MLE vector $\Theta=\left(\hat{\gamma},\hat{\alpha},\hat{\beta}\right)^T$ is consistent and asymptotically normal, that is, $\sqrt{n}\left[\left(\hat{\gamma},\hat{\alpha},\hat{\beta}\right)^T-\left(\gamma,\alpha,\beta\right)^T\right]$ converges in distribution to a normal distribution with the (vector) mean zero and the identity covariance matrix. Also, the Fisher's information matrix or curvature matrix can be computed using the approximation

$$I_y\left(\hat{\gamma},\hat{\alpha},\hat{\beta}\right)\approx\begin{bmatrix} -\left(\frac{\partial^2 l}{\partial\gamma^2}\right)\Big|_{\left(\hat{\gamma},\hat{\alpha},\hat{\beta}\right)} & -\left(\frac{\partial^2 l}{\partial\gamma\partial\alpha}\right)\Big|_{\left(\hat{\gamma},\hat{\alpha},\hat{\beta}\right)} & -\left(\frac{\partial^2 l}{\partial\gamma\partial\beta}\right)\Big|_{\left(\hat{\gamma},\hat{\alpha},\hat{\beta}\right)} \\ -\left(\frac{\partial^2 l}{\partial\alpha\partial\beta}\right)\Big|_{\left(\hat{\gamma},\hat{\alpha},\hat{\beta}\right)} & -\left(\frac{\partial^2 l}{\partial\alpha^2}\right)\Big|_{\left(\hat{\gamma},\hat{\alpha},\hat{\beta}\right)} & -\left(\frac{\partial^2 l}{\partial\alpha\partial\gamma}\right)\Big|_{\left(\hat{\gamma},\hat{\alpha},\hat{\beta}\right)} \\ -\left(\frac{\partial^2 l}{\partial\beta\partial\alpha}\right)\Big|_{\left(\hat{\gamma},\hat{\alpha},\hat{\beta}\right)} & -\left(\frac{\partial^2 l}{\partial\beta\partial\gamma}\right)\Big|_{\left(\hat{\gamma},\hat{\alpha},\hat{\beta}\right)} & -\left(\frac{\partial^2 l}{\partial\beta^2}\right)\Big|_{\left(\hat{\gamma},\hat{\alpha},\hat{\beta}\right)} \end{bmatrix},$$

where $\hat{\gamma},\hat{\alpha}$ and $\hat{\beta}$ are the MLEs of γ,α and β, respectively (see, e.g., [11]). Using this approximation, we may build confidence intervals for three parameters of the DIWB (discrete Inverse Weibull Beta) model.

8. SIMULATION STUDY

In this section, we explore and investigate the performance of the ML estimators for a finite sample size n. Simulation study is carried out on different $DIWBD\left(X;\gamma,\alpha,\beta\right)$ random samples. The random observations are generated by using the inverse cdf method presented in section 3.1 from $DIWBD\left(X;\gamma,\alpha,\beta\right)$. A simulation study was carried out for each triplet $\left(\gamma,\alpha,\beta\right)$ taking four parameter combinations as $\left(\gamma=1.5,\alpha=0.1,\beta=0.2\right)$, $\left(\gamma=2.0,\alpha=0.2,\beta=0.5\right)$, $\left(\gamma=2.5,\alpha=0.3,\beta=0.8\right)$, $\left(\gamma=2.7,\alpha=0.5,\beta=1.0\right)$ and the process was repeated 500 times by taking different sample sizes n = (25, 50, 100, 200, 300, 500). The simulated results are presented in Table 2. We observe in table 2 that the correspondence between theory and practice improves as the sample size *n* increases. MSE and Variance of the estimators suggest us that the estimators are consistent and the maximum likelihood method performs quite well in estimating the model parameters of the proposed distribution.

Table 2
Average Bias, MSE and Variance for Simulated Results of ML Estimates

Sample Size (n)		$(\gamma=1.5, \alpha=0.1, \beta=0.2)$			$(\gamma=2.0, \alpha=0.2, \beta=0.5)$		
		Bias	Variance	MSE	Bias	Variance	MSE
25	γ	-0.05237	0.089404	0.092147	0.76995	0.999647	1.59247
	α	0.047245	0.005477	0.007709	0.043465	0.129973	0.131862
	β	0.161686	0.085246	0.111388	0.037023	0.696992	0.698363
50	γ	0.054339	0.058164	0.061116	0.084853	0.145532	0.152733
	α	0.005111	0.004037	0.004063	0.102675	0.090958	0.1015
	β	0.056593	0.057523	0.060726	0.395751	0.999225	1.155844
100	γ	0.045463	0.093662	0.095729	-0.01579	0.053704	0.053953
	α	0.009942	0.004344	0.004443	0.070709	0.023332	0.028332
	β	0.041693	0.023937	0.025675	0.290298	0.219924	0.304197
200	γ	0.003947	0.02543	0.025445	0.049671	0.070071	0.072538
	α	0.013296	0.002165	0.002342	0.03507	0.018085	0.019314
	β	0.023296	0.007959	0.008502	0.119934	0.163446	0.17783
300	γ	-0.00564	0.020097	0.020129	0.012333	0.030391	0.030543
	α	0.013208	0.001638	0.001812	0.012486	0.005029	0.005185
	β	0.031505	0.006758	0.00775	0.018789	0.031123	0.031476
500	γ	0.028181	0.008378	0.009172	0.003443	0.021585	0.021597
	α	-0.00443	0.000396	0.000416	0.02087	0.003716	0.004151
	β	-0.00438	0.002083	0.002102	0.071375	0.026797	0.031891
	γ	$(\gamma=2.5, \alpha=0.3, \beta=0.8)$			$(\gamma=2.7, \alpha=0.5, \beta=1.0)$		
25	γ	0.837009	2.050667	2.751252	1.380358	0.745384	2.650773
	α	0.167986	0.391168	0.419387	-0.36505	0.008077	0.141336
	β	0.522155	2.77025	3.042896	-0.75465	0.019952	0.589446
50	γ	0.529862	0.475773	0.756526	0.790837	1.61162	2.237043
	α	-0.01487	0.101754	0.101975	-0.17955	0.087763	0.120002
	β	0.032262	0.745712	0.746753	-0.1992	0.455729	0.495411
100	γ	0.312454	0.228102	0.325729	0.596811	0.217568	0.573752
	α	-0.04081	0.03975	0.041416	-0.25177	0.019547	0.082935
	β	0.063326	0.701241	0.705251	-0.43886	0.121267	0.313869
200	γ	0.074099	0.062332	0.067823	0.176442	0.101304	0.132436
	α	-0.01366	0.017515	0.017702	-0.0537	0.047559	0.050443
	β	-0.04251	0.160012	0.161819	-0.08833	0.221322	0.229125
300	γ	0.181963	0.048056	0.081167	0.133997	0.110248	0.128203
	α	-0.04954	0.007649	0.010103	-0.03942	0.033157	0.034711
	β	-0.09947	0.049858	0.059753	-0.05592	0.137677	0.140803
500	γ	-0.00485	0.03947	0.039493	-0.18114	0.021311	0.054124
	α	0.016939	0.008946	0.009233	-0.18654	0.007013	0.04181
	β	0.041038	0.092865	0.094549	-0.17893	0.053977	0.085992

9. APPLICATION OF DISCRETE INVERSE WEIBULL BETA DISTRIBUTION IN HEALTH SCIENCE

Plethora of discrete models (for example: [16,17]) are available in the statistical literature to analyse count data from different applied fields. Due to complex nature of the data, the classical models fail sometimes to analyse the data much accurately. Here we used discrete data set for this part of the analysis to illustrate the applications of the DIWBD (discrete inverse Weibull Beta distribution) in comparison to various well known discrete models already in the statistical literature. The data set has the number of hospital stays by United States residents aged 66 and over (see [3,6]), which is shown in table 3(a). This data has 80.37% of zeros and the sample index of dispersion is 1.882.The goodness of fit test (chi-squared test) is used to compare between observed and expected values of data. We compared the proposed model with some standard zero inflated models and some well-known classical discrete models for model significance purpose. The p-values of Pearson's Chi-square statistic are given in table 4. This reveals that discrete inverse Weibull Beta distribution is a good approximation for the hospital stay data set given in table 3(a). The null hypothesis that data come from discrete inverse Weibull Beta distributions is strongly accepted as p-value for the proposed model is 0.5683 (>0.05). Even though the data set has zero inflation of 80.37% but zero inflated models does not perform well in this case. This proves that discrete inverse Weibull Beta distribution to be good competitive model for zero inflated models as well.

Table 3(a)
Number of Hospital Stays by United States Residents Aged 66 and Over

Number of hospital stays	0	1	2	3	4	5	6	7	8
Observed	3541	599	176	48	20	12	5	1	4

Table 3(b) provides the parameter estimates along with their standard errors and model functions of the fitted distributions to the data set in table 2. The parameter estimation was employed using ML technique with the help of R studio statistical software.

Table 3(b)
Estimated Parameters by ML Method alongwith Standard Errors in Braces for Fitted Distributions for the Number of Hospital Stays of United States Residents Aged 66 and Over

Distribution	Parameter Estimates	Model Function
Discrete Inverse Weibull Beta (DIWBD)	$\hat{\gamma} = 3.16\ (0.21)$ $\hat{\beta} = 0.07\ (0.01)$ $\hat{\alpha} = 0.31\ (0.07)$	$p(x) = \frac{1}{B(\alpha,\beta)}[B(\beta,(x+1)^{-\gamma}+\alpha) - B(\beta, x^{-\gamma}+\alpha)]$ $x = 0,1,2,...,$ for $\gamma > 0, \beta > 0, \alpha > 0$
Poisson (PD)	$\hat{\lambda} = 0.29\ (0.008)$	$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}\ \ \lambda > 0;\ x = 0,1,2,...$
Poisson Akasha (PAD)	$\hat{\theta} = 4.104\ (0.09)$	$p(x) = \frac{\theta^3(x^2+3x+(\theta^2+2\theta+2))}{(\theta^2+2)(\theta+1)^{x+3}}$, $x = 0,1,2,....\quad \theta > 0$
Poisson Lindley (PLD)	$\hat{\theta} = 4.07 (0.11)$	$p(x) = \frac{\theta^2(x+\theta+2)}{(\theta+1)^{x+3}}, \qquad x = 0,1,2,...\ \theta > 0$
Discrete Rayleigh (DRD)	$\hat{q} = 0.42\ (0.006)$	$p(x) = q^{x^2} - q^{(x+1)^2}\ \ 0 < q < 1; x = 0,1,2,...$
Geometric (GD)	$\hat{\pi} = 0.77\ (0.005)$	$p(x) = \pi^x - \pi^{(x+1)}\ \ 0 < \pi < 1, x = 0,1,2,...$
Zero Inflated Poisson (ZIP)	$\hat{\alpha} = 0.67\ (0.01)$ $\hat{\lambda} = 0.88\ (0.03)$	$p(x) = \begin{cases} \alpha + (1-\alpha)\frac{e^{-\lambda}\lambda^x}{x!}, \lambda > 0; x = 0 \\ (1-\alpha)\frac{e^{-\lambda}\lambda^x}{x!}, \lambda > 0; x = 0,1,2,.. \end{cases}$ $0 < \alpha < 1\ ;\ \lambda > 0$
Zero Inflated Negative Binomial (ZINB)	$\hat{\phi} = 0.604\ (0.01)$ $\hat{r} = 3.97\ (0.21)$ $\hat{p} = 0.60\ (0.11)$	$p(x) = \begin{cases} \phi + (1-\phi)p^r; x = 0 \\ (1-\phi)\binom{x+r-1}{x}p^r q^x;\ x = 0,1,2,... \end{cases}$ $0 < \phi < 1, 0 < p < 1,\ r > 0$
Discrete Generalized Inverse Weibull (DGIW)	$\hat{q} = 0.08 (0.4)$ $\hat{\alpha} = 2.04\ (0.07)$ $\hat{\beta} = 0.3\ (0.32)$	$p(x) = q^{\left(\frac{\beta}{x+1}\right)^\alpha} - q^{\left(\frac{\beta}{x}\right)^\alpha}\ \ \alpha > 0,\ \beta > 0,$ $0 < q < 1; x = 0,1,2,...$

Table 4
Observed and Expected Frequencies of the Number of Hospital Stays of United States Residents Aged 66 and Over

Observed	DGIW	PD	DRD	GD	PLD	PAD	DIWBD	ZIP	ZINB
3541.00	3538.12	3277.25	2511.65	3399.79	3398.93	3409.92	3540.98	1816.50	3541.10
599.00	639.37	969.94	1743.79	776.42	781.19	763.29	598.57	1609.50	533.40
176.00	127.18	143.53	148.35	177.31	175.95	177.53	176.37	712.90	217.70
48.00	44.73	14.16	2.21	40.49	39.02	42.09	51.84	210.60	68.70
20.00	20.63	1.05	0.01	9.25	8.55	10.04	18.99	46.70	18.90
12.00	11.15	0.06	0.00	2.11	1.85	2.39	8.34	8.40	4.40
5.00	6.69	0.00	0.00	0.48	0.40	0.56	4.18	1.30	1.30
1.00	4.32	0.00	0.00	0.11	0.09	0.13	2.31	0.10	0.50
4.00	13.81	0.00	0.00	0.03	0.02	0.04	4.42	0.00	0.00
P-value	0.0001	<0.01	<0.01	<0.01	<0.01	<0.01	0.5683	<0.01	<0.01

Table 5
AIC, BIC and Loglikelihood Values for Fitted Models to the Data Set of Number of Hospital Stays of United States Residents Aged 66 and Over

Distributions	Loglikelihood Values	AIC	BIC
Discrete Generalized Weibull	-3024.917	6055.835	6075.007
Poisson	-3304.509	6611.019	6617.409
Discrete Rayleigh	-4439.711	8881.422	8887.813
Zero Inflated Poisson	-3059.418	6122.836	6135.617
Zero Inflated Negative Binomial	-3036.000	6078.000	6079.000
Geometric	-3067.988	6137.976	6144.366
Poisson Lindley	-3074.004	6150.008	6156.399
Poisson Akasha	-3060.730	6123.460	6129.851
Discrete Inverse Weibull Beta	-3008.112	6022.223	6041.395

For model selection in this study, we also compare the proposed model with other related models using AIC (Akaike Information Criterion) given by Akaike [2] and BIC (Bayesian information criterion) given by Schwarz [19]. Generic functions calculating AIC and BIC for the model having p number of parameters are given by

$$AIC = 2p - 2\log(l)$$

$$BIC = p\log(n) - 2\log(l)$$

Table 5 presents AIC, BIC and Loglikelihood values for the discrete models fitted to the data in table 3(a). It is clear that AIC and BIC criterion favors DIWBD (discrete inverse Weibull Beta distribution) in comparison with ZIPD (Zero Inflated Poisson distribution), ZINBD (Zero Inflated Negative Binomial distribution), GD (Geometric distribution), PD (Poisson distribution), PAD (Poisson Akasha distribution), PLD (Poisson Lindley distribution), DRD (discrete Rayleigh distribution), DGIWD (discrete generalized inverse Weibull distribution). Thus, the DIWBD (discrete Weibull Beta distribution) can be chosen as the best model for the discussed data.

CONCLUSION

In this paper, a new discrete probability model is studied by compounding DIWD (discrete inverse Weibull distribution) with BD (Beta distribution) of first kind and it has been shown that introduced probability distribution can be nested to different compound distributions. Some important properties and the problem of estimation of its parameters by ML and MM method are studied. In addition, the DIWBD (discrete inverse Weibull Beta distribution) is appropriate for modeling both over and under dispersed data.

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