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A NOTE ON PRODUCT GENERALIZED PEARSON TYPE VII DISTRIBUTION

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ABSTRACT

In this paper a product generalized Pearson (PGP) type VII distribution is proposed by taking the product of two generalized Pearson type VII probability densities functions (pdfs). Some properties of PGP type VII distribution are investigated including its pdf, cumulative distribution function (cdf), moments, hazard rate function, cumulative hazard rate function, survival function, mean residual function and their graphs, maximum likelihood estimators and their asymptotic variances and mixed random variables are derived. The characterization of PGP type VII distribution is presented through conditional expectation. A simulation study is used to find the estimates of parameters of PGP type VII distribution. Finally, an application of PGP type VII distribution to a real data set is shown.

KEY WORDS

Hazard rate function; cumulative function; maximum likelihood estimation; entropy; asymptotic variance.

1. INTRODUCTION

Pearson (1895, 1905, 1916) introduced a system of probability distributions useful in flood frequency analysis, engineering and biological sciences, econometrics, survey sampling and in life-testing. Pearson type VII distribution was first studied by Pearson (1916) and developed some of its properties. Pearson and Hartrey (1954), Merrington and Pearson (1958), Pearson (1963) and Shenton and Carpenter (1964) applied Pearson type VII distribution in the research of, engineering and biological sciences.

Ahmad (1985) proposed a class of inverted distributions and presented some of its properties. Cobb (1980) discussed a differential equation of the form $\frac{df}{dx}=\frac{g(y)}{h(y)}f(y)dy$, $h(y)>0$ where $g(y)$ and $h(y)$ are polynomials such that the degree of $h(y)$ is one higher than the degree of $g(y)$. Ahmad (1985) used $\frac{d}{dx}\left[\ln g(x)\right]=-\frac{\left(a_2x^2+a_1x+a_0\right)}{x\left(B_0x^2+B_1x+B_2\right)}$ where the coefficients a_0, a_1, a_2 are given by $a_0=2B_2-1$, $a_1=2B_2-a$, $a_2=2B_0$ and generated the inverted Pearson system of probability distributions. This class includes the inverted normal as well as the inverted I, II, III, V distribution. See also Ahmad and Sheikh (1986), Ali and Ahmad (1985), Ahmad and Kazi (1987) and Ahmad (2007). In recent years, there are many efforts to develop new statistical distribution with more flexibility which can be fitted well to complex data.

The *t*-distribution which is heavier tails in nature and approaches to normal distribution when degrees of freedom going to indefinitely large and *t*-distribution is more inclined to generating data that lie far from its arithmetic mean than a normal distribution. The nature of *t*-distribution makes it suitable for understanding and checking the statistical behavior of random quantities of ratio type when denominator of the ratio approaches to zero.

Nadarajah and Kotz (2008) introduced a new Pearson-type VII distribution by taking the product of two Pearson-type VII pdfs and found some of its structural properties with its application to a real data set.

Habibullah (2009) introduced and discussed various properties of log-Pearson type VII distribution (See also Iqbal and Ahmad, 2015).

Takano (2003) studied the product of Cauchy pdfs and found some of its properties and applications in physics. Jacobs (2005) discussed that observers made their measurements through a product of densities formula for combined states of knowledge (pooling knowledge) where the states of knowledge are probability densities and he observed lower entropy of combined state of knowledge if respective observers agree.

In this paper, we propose a more flexible model i.e. a product generalized Pearson (PGP) type VII distribution.

$$f(x) = \frac{k}{x} a\left(x\right)^{-\upsilon} b\left(x\right)^{-\delta} \tag{1}$$

Let $f(x)$ and $g(x)$ be the two independent density function such that $f(x)g\left(x\right) > 0$ on a set of positive measure and for any constant $k > 0$, $f(x)g\left(x\right)$ is a probability function, if

$$k \int_R f(x)g\left(x\right) dx = 1, \;\; x > 0$$

where

$$f(x) = \frac{k_1}{x} a\left(x\right)^{-\upsilon} \tag{2}$$

$$g(x) = k_2 \; b\left(x\right)^{-\delta} \tag{3}$$

$$a(x) = 1 + \alpha\left(\ln x\right)^{2p} \;\; b(x) = 1 + \beta\left(\ln x\right)^{2p}$$

$$x, \alpha, \beta, p, \upsilon, \delta > 0 \; and \; \delta + \nu > \frac{1}{2p},$$

It is easy to show that

$$k^{-1} = p^{-1}\alpha^{-\frac{1}{2p}}\; B\left(\frac{1}{2p}, \nu+\delta-\frac{1}{2p}\right)\; {}_2F_1\left(\frac{1}{2p}, \delta; \nu+\delta; 1-\frac{\alpha}{\beta}\right)$$

where $B(a,b)$ is a beta function and ${}_2F_1(a,b;c;x)$ is a hyper-geometric function.

The flexibility of the distribution can be seen through properties and graphs of the new proposed distribution. The beauty of PGP type VII model is the fact that it not only generalize the GLP type VII distribution but also show limiting behavior of some averages of the mixture random variables containing algebraic, logarithmic and trigonometric functions of PGP type VII distribution. The graph of entropy measures of PGP type VII for different parameters values shows more randomness or disorderness in the random variable than the randomness of r.v contained by its competitors' models.

This proposed distribution has many sub models such as GLP type VII distribution (Iqbal and Ahmad, 2015), log-Pearson type VII distribution (Habibullah, 2009) and many others like the product of Pearson type VII distribution (Nadarajah and Kotz, 2008) after some suitable transformation and fixing the parameter values.

The following graphs are shown for different values of parameters.

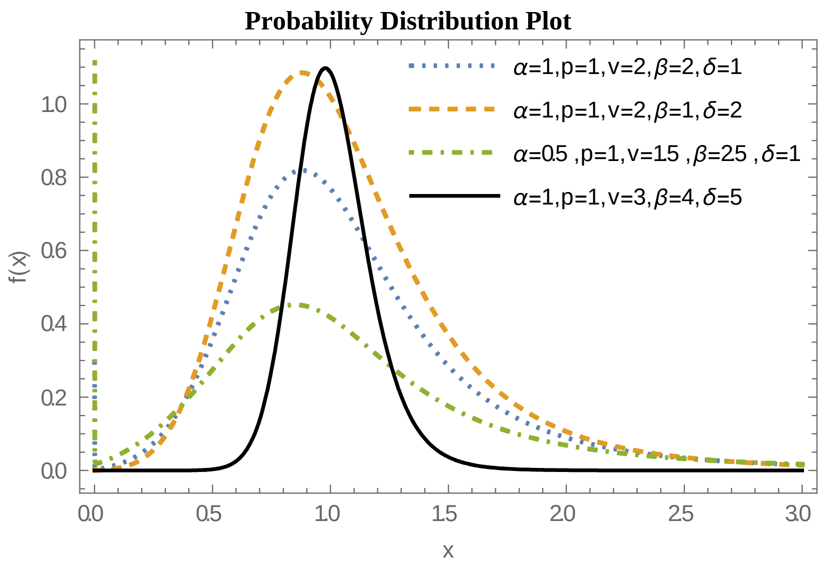

Fig. 1.1: pdf of PGP Type VII Distribution for the Indicated Values of the Parameters

The distribution function of PGP type VII distribution is

$$F(x)=\begin{cases}\frac{1}{2}-A(x), & 0<x\le 1,\alpha,\beta>0,\, p\in Z^{+},\nu+\delta>\frac{1}{2p},\\ \frac{1}{2}+A(x), & x>1,\alpha,\beta>0,\, p\in Z^{+},\nu+\delta>\frac{1}{2p}.\end{cases} \tag{4}$$

where

$$A(x)=\frac{\ln x}{2}\frac{B\left(\frac{1}{2p},1\right){}_3F_1\left(\frac{1}{2p},\nu,\delta;\frac{1}{2p}+1;-\alpha\ln^{2p}x,-\beta\ln^{2p}x\right)}{p^{-1}\alpha^{-\frac{1}{2p}}B\left(\frac{1}{2p},\nu+\delta-\frac{1}{2p}\right){}_2F_1\left(\frac{1}{2p},\delta;\nu+\delta;1-\frac{\alpha}{\beta}\right)}$$

and

$${}_3F_1(a,b,c;d;x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_n\,x^m y^n}{(d)_{m+n}\,m!\,n!}.$$

Figure 1.2 shows the graphs of distribution function for various values of p,ν and δ for $\alpha=1,\beta=1$

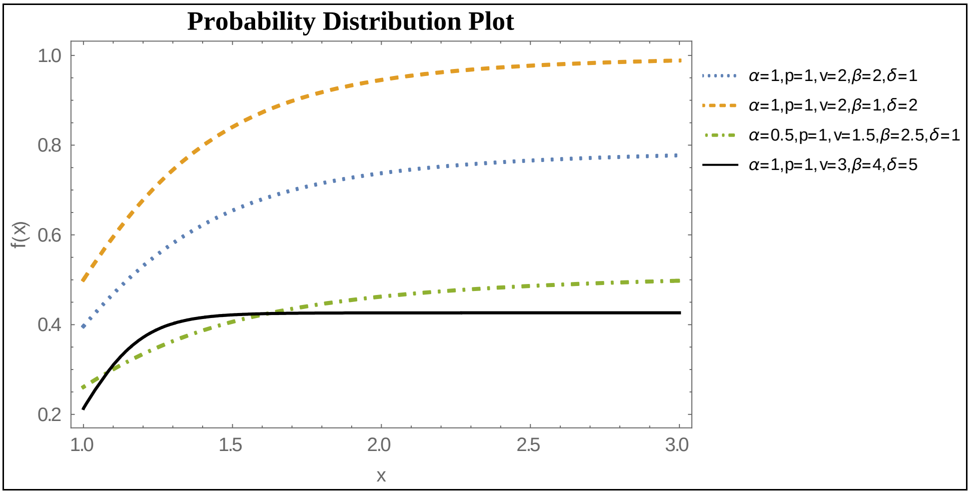

Fig. 1.2: CDF of PGP Type VII Distribution for the Indicated Values of the Parameters

Hazard rate function and survival function with graphs are discussed in Section 2. Shannon entropy of PGP type VII distribution is found in Section 3. In Section 4 characterization of PGP type VII distribution is presented in terms of conditional expectation. Maximum likelihood estimators and asymptotic variances are found in section 5. In section 6, another pdf corresponding to PGP type VII distribution is found in section 7. A simulation study is carried out in Section 8 and an application on real data is provided in Section 9. Finally, Some concluding remarks are provided in section 10.

2. THE HAZARD RATE

Hazard rate function arises in the situation of the analysis of the time to the event and it describes the current chance of failure for the population that has not yet failed. This function plays a pivotal role in reliability analysis, survival analysis, actuarial sciences and demography, in extreme value theory and in duration analysis in economics and sociology. This is very important for researchers and practitioners working in areas like engineering statistics and biomedical sciences. Hazard rate function is very useful in defining and formulating a model when dealing with lifetime data.

The survival function of the PGP type VII distribution is

$$S(x)=\begin{cases}\frac{1}{2}+A(x), & 0< x\le 1, \alpha,\beta>0, p\in Z^{+}, \nu+\delta>\frac{1}{2p},\\ \frac{1}{2}-A(x), & x>1, \alpha,\beta>0, p\in Z^{+}, \nu+\delta>\frac{1}{2p}.\end{cases} \tag{5}$$

The hazard rate function the PGP type VII distribution is given by

$$h(x)=\begin{cases}\dfrac{2k\left[a(x)\right]^{-\nu}\left[b(x)\right]^{-\delta}}{kx+x\ln xB\left(\frac{1}{2p},1\right){}_3F_1}, & 0< x\le 1, \alpha,\beta>0, p\in Z^{+}, \nu+\delta>\frac{1}{2p},\\ \dfrac{2k\left[a(x)\right]^{-\nu}\left[b(x)\right]^{-\delta}}{k\,x+x\ln x\,B\left(\frac{1}{2p},1\right){}_3F_1}, & x>1, \alpha,\beta>0, p\in Z^{+}, \nu+\delta>\frac{1}{2p}.\end{cases} \tag{6}$$

where ${}_3F_1$ is defined in (4).

Figure 2.1 shows the graphs of *hrf* defined in (6) for various values of p and v.

The cumulative hazard rate function of the PGP type VII distribution is

$$H_g(x)=\begin{cases}-\ln\left(\frac{1}{2}+A(x)\right), & if\ \ 0< x\le 1, \alpha,\beta>0, p\in Z^{+}, \nu+\delta>\frac{1}{2p},\\ -\ln\left(\frac{1}{2}-A(x)\right), & if\ \ x>1, \alpha,\beta>0, p\in Z^{+}, \nu+\delta>\frac{1}{2p}.\end{cases}$$

The mean residual function of the PGP type VII distribution is

$$\mu(x)=\begin{cases}\dfrac{1}{\bar{F}(t)}\displaystyle\int_t^{\infty}\left(\frac{1}{2}+A(x)\right)dx, & if\ \ 0<x,t\le 1,\alpha>0, p\in Z^{+},\nu+\delta>\dfrac{1}{2p},\\ \dfrac{1}{\bar{F}(t)}\displaystyle\int_t^{\infty}\left(\frac{1}{2}-A(x)\right)dx, & if\ \ x,t>1,\alpha>0, p\in Z^{+},\nu+\delta>\dfrac{1}{2p}.\end{cases}$$

The graphs of hazard rate function for various values of p

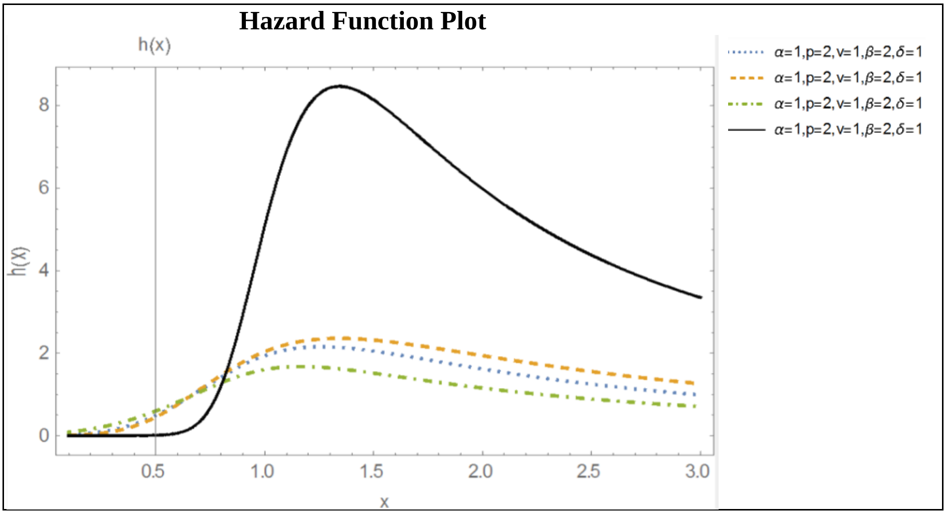

Fig 2.1: Hazard Rate Function of PGP Type VII Distribution for the Indicated Values of the Parameters

Fig 2.1 shows that the failure rate function is upside-down bathtub shaped. A function is termed as upside-down bathtub shaped if it is first increasing and then decreasing, with a single maximum.

The mean residual function $\mu(x)$ of the PGP type VII distribution is

$$\mu(x)=\begin{cases}\dfrac{1}{\bar{F}(t)}\displaystyle\int_t^{\infty}\left(\frac{1}{2}+A(x)\right)dx, & if\ \ 0<x,t\le 1,\alpha>0, p\in Z^{+},\nu+\delta>\dfrac{1}{2p},\\ \dfrac{1}{\bar{F}(t)}\displaystyle\int_t^{\infty}\left(\frac{1}{2}-A(x)\right)dx, & if\ \ x,t>1,\alpha>0, p\in Z^{+},\nu+\delta>\dfrac{1}{2p}.\end{cases}$$

The mean residual function gives an interpretable measure of how much more time to be expected to survive for an individual, given that one already reached the time point X. This can be readily estimated from right censored data in which, even the mean, cannot be estimated without additional assumptions.

3. SHANNON ENTROPY OF PGP TYPE VII DISTRIBUTION

In the probability theory, entropy is a statistical measure of uncertainty and this term is used as outcome's ignorance in random experiment Shannon's (1948) used entropy in information theory as a tool and nowadays, it is used in almost every branch of engineering and science.

Soofi et al. (1995), Paninski (2003), Lesne and Benecke (2008), Shams (2011), Hasnain (2013) and Iqbal (2013) estimated the entropies of continuous probability distributions and calculated the results numerically for different parameters through software.

The Shannon entropy $h(x)$ of a continuous *r.v* X with pdf $f(x)$ is defined as

$$h(x) = -\int_S f(x)\ \ln f(x)\, dx,$$

where S is the support set of *r.v*. Thus,

$$h(x) = -2\ln k + v\ k + v\ k\int_0^\infty \frac{1}{x}\left(\ln a(x)\right) a^{-v}(x)\, b^{-\delta}(x)\, dx$$
$$+\delta\ k\int_0^\infty \frac{1}{x}\left(\ln b(x)\right) a^{-v}(x)\, b^{-\delta}(x)\, dx,$$

Assuming $\beta = \alpha$ and applying transformation $\left(1+\alpha(\ln x)^{2p}\right) = \frac{1}{t}$, we obtain

$$h(x) = -\ln k + \frac{(v+\delta)\ k}{2p\alpha^{1/2p}}\int_0^1 \ln t\ \ t^{v+\delta-1/2p-1}(1-t)^{1/2p-1}\, dt.$$

Following (See 4.253 Gradshteyn and Ryzhik 2007).

$$\int_0^1 t^{\mu-1}(1-t)^{\nu-1}\ln t\ dt = \mathrm{B}(\mu,\nu)\left[\psi(\mu)-\psi(\mu+\nu)\right].$$

We have

$$h(x) = -\ln k + \frac{(v+\delta)\ k}{2p\ \alpha^{1/2p}}\mathrm{B}\left(\nu+\delta\text{-}1/2p, 1/2p\right)\left[\psi\left(\nu+\delta\text{-}1/2p\right)-\psi(\nu+\delta)\right],$$

where $\mathrm{B}(\mathrm{a},\mathrm{b})$ is a beta function and $\psi(x)$ is defined as $\psi(x) = \frac{d\ln\Gamma(x)}{dx}$.

It can be seen that the value of $h(x)$ changes as v, δ and p changes, and it measures the average uncertainty or disorderness in the random variable.

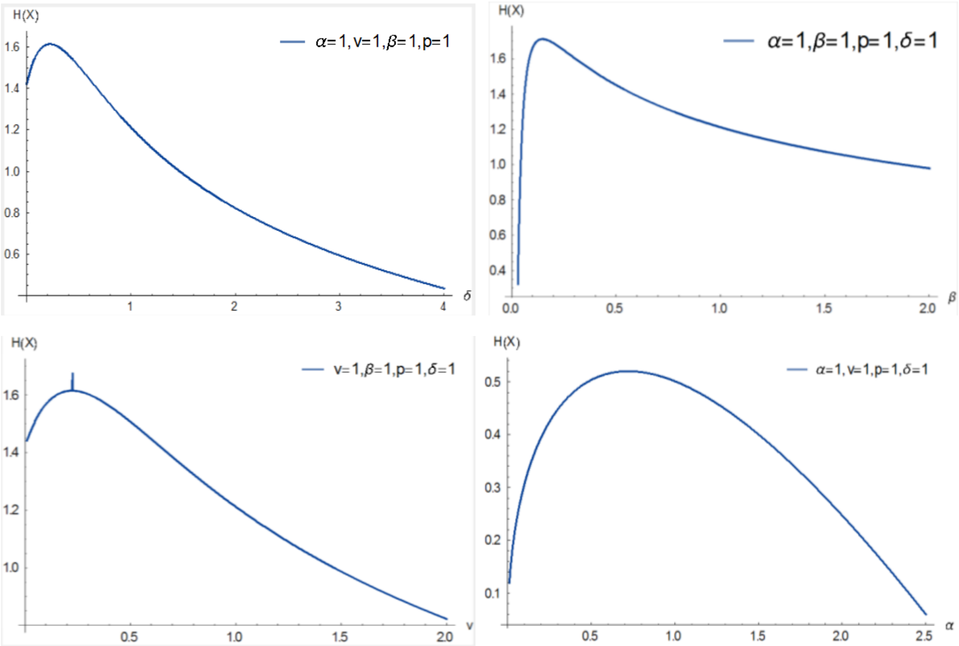

Fig. 3.1: Shannon Entropy of PGP Type VII Distribution for the Indicated Values of the Parameters

4. CHARACTERIZATION OF PGP TYPE VII DISTRIBUTION

Characterizations of probability distributions have a position of prominence in statistical theory. A number of methods have been developed by researchers for characterizing both discrete and continuous probability distributions [See Ruiz and Navarro (1996), Wu and Ouyang (1996), Su and Huang (2000), Gupta and Ahsanullah (2004), Gupta and Kirmani (2004) and Sunoj et al. (2009) and Ahsanullah and Hamedani (2012)].

Theorem 4.1

Let X be a r.v. with differentiable density, f(x) given at equation (1) then the

$$E\left((\ln x)^{2p-1}\left[b(x)\right]^{\delta} \mid X > y\right) = \frac{\left[b(x)\right]^{1+\delta}}{2p\alpha(\nu-1)} y r_X(y), \tag{7}$$

holds, if and only it is a PGP type VII distribution
where

$$r_X(y) = \frac{f(y)}{\bar{F}(y)}, \;\; \bar{F}(y) = 1 - F(y).$$

Proof:

Suppose X is the PGP type VII distribution.

Then,

$$E\left[(\ln x)^{2p-1}\left[b(x)\right]^{\delta} \mid X > y\right]$$

$$= \frac{k}{\overline{F}(y)}\int_{y}^{\infty}(\ln x)^{2p-1}\left[b(x)\right]^{\delta} f(x)dx .$$

The result is trivial
where k is defined in (1)

$$= \frac{k}{\overline{F}(y)}\int_{y}^{\infty}\frac{1}{x}(\ln x)^{2p-1}\, b^{\delta}(x)a^{-\nu}(x)b^{-\delta}(x)dx,$$

$$a(x) = 1+\alpha(\ln x)^{2p} \quad b(x) = 1+\beta(\ln x)^{2p}$$

$$a'(x) = \frac{2\alpha p}{x}(\ln x)^{2p-1} \quad b'(x) = \frac{2\beta p}{x}(\ln x)^{2p-1}$$

$$= \frac{2p}{x}\,\frac{1+\alpha(\ln x)^{2p}-1}{\ln x} \quad = \frac{2p}{x}\,\frac{1+\beta(\ln x)^{2p}-1}{\ln x}$$

$$a'(x) = \frac{2p}{x}\,\frac{a(x)-1}{\ln x} \quad b'(x) = \frac{2p}{x}\,\frac{b(x)-1}{\ln x}$$

$$= \frac{k\left[a(x)\right]^{1-\nu}}{2p\alpha(\nu-1)\overline{F}(y)}.$$

Since

$$f(y) = \frac{k}{y}\left[1+\alpha(\ln y)^{2p}\right]^{-\nu}\left[1+\beta(\ln y)^{2p}\right]^{-\delta},$$

$$yf(y) = k\left[1+\alpha(\ln y)^{2p}\right]^{-\nu}\left[1+\beta(\ln y)^{2p}\right]^{-\delta},$$

$$m(y) = \frac{\left[1+\beta(\ln y)^{2p}\right]\left[1+\beta(\ln y)^{2p}\right]^{\delta} yf(y)}{2p\alpha(\nu-1)\overline{F}(y)},$$

$$= \frac{\left[1+\beta(\ln y)^{2p}\right]^{1+\delta} yf(y)}{2p\alpha(\nu-1)\overline{F}(y)},$$

$$m(y) = \frac{\left[1+\beta(\ln y)^{2p}\right]^{1+\delta}}{2p\alpha(\nu-1)}\, yr_X(y).$$

Conversely, let $f(x)$ be an unknown pdf of a random variable X defined on $(0,\infty)$ such that (7) hold, then

$$\frac{1}{\overline{F}(y)}\int_y^\infty (\ln x)^{2p-1} b^{\delta}(x)\, f(x)\,dx = \frac{a^{1+\delta}(x)\, y f(y)}{2p\alpha(\nu-1)\overline{F}(y)}$$

$$-\frac{2p\alpha(\nu-1)(\ln y)^{2p-1}}{y\left[1+\alpha(\ln y)^{2p}\right]} - \frac{2p\alpha(1+\delta)(\ln y)^{2p-1}}{y\left[1+\alpha(\ln y)^{2p}\right]} - \frac{1}{y} = \frac{f'(y)}{f(y)},$$

$$f(y) = k\frac{a^{-\nu}(x) b^{-\delta}(x)}{y}, \qquad y,\alpha,\beta,\delta,\nu > 0, \delta+\nu > \frac{1}{2p}. \tag{8}$$

and normalizing constant is

$$k^{-1} = p^{-1}\alpha^{-\frac{1}{(2p)}} B\left(\frac{1}{2p}, \nu+\delta-\frac{1}{2p}\right) {}_2F_1\left(\frac{1}{2p}, \delta; \nu+\delta; 1-\frac{\alpha}{\beta}\right).$$

which is a product generalized Pearson (PGP) type VII distribution where

$$r_X(y) = \frac{f(y)}{\overline{F}(y)}, \ \overline{F}(y) = 1 - F(y).$$

5. MAXIMUM LIKELIHOOD ESTIMATION OF THE PARAMETERS OF THE PGP TYPE VII DISTRIBUTION

In this section, we find the estimation of the log- likelihood class parameters. Let $X_1,\ldots,X_n$ be an r.v with observed values $x_1,\ldots,x_n$ from all distributions. Let $\Theta = (v,\delta,\alpha,\beta)$ be the vector of parameters. The log–likelihood function of (1) based on the observed random sample size of n is obtained by

$$\ell(\nu,\delta,\alpha,\beta;\underline{x}_{obs}) = n\ln k - \nu\sum_{i=1}^{n}\ln a(x) - \delta\sum_{i=1}^{n}\ln b(x) - \sum_{i=1}^{n}\ln x_i,$$

$$\frac{\partial\ell(\nu,\delta,\alpha,\beta;\underline{x}_{obs})}{\partial\nu} = \frac{n}{k}\frac{\partial k}{\partial\nu} - \sum_{i=1}^{n}\ln a(x),$$

$$\frac{\partial\ell(\nu,\delta,\alpha,\beta;\underline{x}_{obs})}{\partial\delta} = \frac{n}{k}\frac{\partial k}{\partial\delta} - \sum_{i=1}^{n}\ln b(x),$$

$$\frac{\partial\ell(\nu,\delta,\alpha,\beta;\underline{x}_{obs})}{\partial\alpha} = \frac{n}{k}\frac{\partial k}{\partial\alpha} - \nu\sum_{i=1}^{n}\frac{\ln^{2p} x}{a(x)}$$

and

$$\frac{\partial\ell(\nu,\delta,\alpha,\beta,\underline{x}_{obs})}{\partial\beta} = \frac{n}{k}\frac{\partial k}{\partial\beta} - \delta\sum_{i=1}^{n}\frac{\ln^{2p} x}{b(x)},$$

The ML estimates can be obtained from the following equations

$$\frac{n}{k}\frac{\partial k}{\partial \nu}=\sum_{i=1}^{n}\ln a\left(x\right), \tag{9}$$

$$\frac{n}{k}\frac{\partial k}{\partial \delta}=\sum_{i=1}^{n}\ln b\left(x\right), \tag{10}$$

$$\frac{n}{k}\frac{\partial k}{\partial \alpha}=\nu\sum_{i=1}^{n}\frac{\ln^{2p}x}{a\left(x\right)} \tag{11}$$

and

$$\frac{n}{k}\frac{\partial k}{\partial \beta}=\delta\sum_{i=1}^{n}\frac{\ln^{2p}x}{b\left(x\right)}. \tag{12}$$

The maximum likelihood estimates of $\Theta=\left(\alpha,v,\beta,\delta\right)$, say $\hat{\Theta}$ is obtained by solving the nonlinear equations (9 to 12). The software MATHEMATICA and R can be used to solve numerically this nonlinear system of equations.

After solving (9)-(12) we get maximum likelihood estimators of the PGP type VII distribution of these parameters.

The numerical calculations of the second-order derivatives of $\ell\left(\nu,\delta,\alpha,\beta;x_{obs}\right)$ take the form

$$\frac{\partial^2 \log L}{\partial\theta_i\partial\theta_j}=-\frac{n}{K^2}\frac{\partial K}{\partial\theta_i}\frac{\partial K}{\partial\theta_j}+\frac{n}{K}\frac{\partial^2 K}{\partial\theta_i\partial\theta_j}$$

Except for

$$\frac{\partial^2\ell\left(\nu,\delta,\alpha,\beta;\underline{x}_{obs}\right)}{\partial\nu^2}=-\frac{n}{k^2}\left(\frac{\partial k}{\partial\nu}\right)^2+\frac{n}{k}\frac{\partial^2 k}{\partial\nu^2},$$

$$\frac{\partial^2\ell\left(\nu,\delta,\alpha,\beta;\underline{x}_{obs}\right)}{\partial\delta^2}=-\frac{n}{k^2}\left(\frac{\partial k}{\partial\delta}\right)^2+\frac{n}{k}\frac{\partial^2 k}{\partial\delta^2},$$

$$\frac{\partial^2\ell\left(\nu,\delta,\alpha,\beta;\underline{x}_{obs}\right)}{\partial\alpha^2}=-\frac{n}{k^2}\left(\frac{\partial k}{\partial\alpha}\right)^2+\frac{n}{k}\frac{\partial^2 k}{\partial\alpha^2}+\nu\sum_{i=1}^{n}\frac{\ln^{2p}x}{a\left(x\right)},$$

$$\frac{\partial^2\ell\left(\nu,\delta,\alpha,\beta;\underline{x}_{obs}\right)}{\partial\beta^2}=-\frac{n}{k^2}\left(\frac{\partial k}{\partial\beta}\right)^2+\frac{n}{k}\frac{\partial^2 k}{\partial\beta^2}+\delta\sum_{i=1}^{n}\frac{\ln^{2p}x}{b\left(x\right)}$$

$$\frac{\partial^2\ell\left(\nu,\delta,\alpha,\beta;\underline{x}_{obs}\right)}{\partial\nu\,\partial\alpha}=-\frac{n}{k^2}\left(\frac{\partial k}{\partial\nu}\right)\left(\frac{\partial k}{\partial\alpha}\right)+\frac{k}{k}\frac{\partial^2 k}{\partial\nu\,\partial\alpha}+\sum_{i=1}^{n}\frac{\ln^{2p}x}{a\left(x\right)}$$

and

$$\frac{\partial^2\ell\left(\nu,\delta,\alpha,\beta;\underline{x}_{obs}\right)}{\partial\delta\,\partial\beta}=-\frac{n}{k^2}\left(\frac{\partial k}{\partial\delta}\right)\left(\frac{\partial k}{\partial\beta}\right)+\frac{k}{k}\frac{\partial^2 k}{\partial\delta\,\partial\beta}+\sum_{i=1}^{n}\frac{\ln^{2p}x}{b\left(x\right)}.$$

where the x's denote the observations from a *r.s* of size n. Equations (9) to (12) for a $\hat{\alpha},\hat{\beta},\hat{\nu}$ and $\hat{\delta}$ are solved simultaneously through Mathematica's command Find Root. The estimates of $\hat{\alpha}$, $\hat{\beta}$, $\hat{\nu}$ and $\hat{\delta}$ obtained for the case $p=1,\ 2$ can be used as initial values for determining the parameters of the PGP type VII distribution.

5.1 Asymptotic Variances for $\hat{\nu}$, $\hat{\delta}$, $\hat{\alpha}$ and $\hat{\beta}$ when p is Fixed

For interval estimation of model parameters and their hypothesis testing, we need information matrix containing second partial derivatives as derived in (9) to (12). Under the regularity conditions, the asymptotic distribution of the estimators of the parameters is multivariate normal $N_4\left(0,k^{-1}(\Theta)\right)$, where $k(\Theta)=\lim_{n\to\infty} n^{-1}I_n(\Theta)$ is the information matrix.

6. A PDF CORRESPONDING TO THE PGP TYPE VII DISTRIBUTION

The following theorem provide us another pdf corresponding to PGP type VII distribution

Theorem 6.1

If X follows the PGP type VII distribution in (1) and X_δ, $|\delta|\le 1$, is the r.v. with pdf $f_\delta(x)=f(x)\left[1+\delta\ \sin(2\pi\ \ln x)\right]$, then $f_\delta(x)$ is also a pdf.

Proof:

We obtain

$$k\int_{-\infty}^{\infty}\left(1+\alpha t^{2p}\right)^{-\nu}\ \left(1+\beta t^{2p}\right)^{-\delta}\sin(2\pi t)\,dt,$$ after applying transformation $\ln x = t$,

in the expression $\int_0^\infty \frac{k}{x}a^{-\nu}(x)b^{-\delta}(x)\sin(2\pi\ln x)\,dx,$ which is an odd function for $p\in Z^+$ therefore its value is zero. And hence this complete the proof.

7. MIXED RANDOM VARIABLES OF THE PGP TYPE VII DISTRIBUTION

In the next two theorems we show some averages of the mixture random variables containing algebraic, logarithmic and trigonometric functions with respect to limiting distribution of PGP type VII distribution are found. We use here some identities from Gradshteyn and Ryzhik (2007).

Theorem 7.1

The following results can be deduced from PGP type VII distribution when $p=1$ and $v\to\infty,\beta\to 0$. Expectations of the integreble function, $g(x)$ w.r.t. (1) are given below:

$g(x)$	$\lim\limits_{\substack{v\to\infty \\ \alpha\to 1/(2v) \\ \beta\to 0 \\ p=1}} \int_0^\infty g(x)\, f(x) dx$
1) $\sin(a\ \ln x)\ \sin(b\ \ln x)$	1) $\frac{1}{2}\sqrt{2\pi}\left(\exp\left(-\frac{(a-b)^2}{2}\right)-\exp\left(-\frac{(a+b)^2}{2}\right)\right)$ (See 3.898(1) Gradshteyn and Ryzhik 2007)
2) $\cos(a\ \ln x)\ \cos(b \ln x)$	2) $\frac{1}{2}\sqrt{2\pi}\left(\exp\left(-\frac{(a-b)^2}{2}\right)+\exp\left(-\frac{(a+b)^2}{2}\right)\right)$ (See 3.899(2) Gradshteyn and Ryzhik 2007)
3) $\frac{\cos((2n+1)\ln x)}{\cos(\ln x)}$	3) $\sqrt{2\pi}\left(1+2\sum_{k=1}^{2n}(-1)^{k-n}\exp\left(-2k^2\right)\right)$ (See 3.899(2) Gradshteyn and Ryzhik(2007)
4) $\sin\left(a\ (\ln x)^2\right)$	4) $\frac{\sqrt{\pi}}{6\left(\sqrt[4]{1+4a^2}\right)}\sin\left(\frac{1}{2}\tan^{-1}(2a)\right)$ (See 3.922(1) Gradshteyn and Ryzhik 2007)
5) $\cos\left(a(\ln x)^2\right)$	5) $\frac{\sqrt{\pi}}{6\left(\sqrt[4]{1+4a^2}\right)}\cos\left(\frac{1}{2}\tan^{-1}(2a)\right)$ (See 3.922(2) Gradshteyn and Ryzhik 2007)
6) $\ln x\ \sin(a(\ln x))$	6) $1-\frac{a}{\sqrt{2}}\sum_{k=0}^{\infty}\frac{(-1)^k\, k!}{(2k+1)!}\left(\sqrt{2}a\right)^{2k+1}$ (Gradshteyn and Ryzhik (2007) 3.952(1))
7) $(\ln x)^2\cos(a(\ln x))$	7) $\sqrt{2\pi}\left(1-a^2\right)\exp\left(-\frac{a^2}{2}\right)$ (See 3.952(4) Gradshteyn and Ryzhik(2007)
8) $(\ln x)^3\sin(a(\ln x))$	8) $\sqrt{2\pi}\left(3-a^3\right)\exp\left(-\frac{a^2}{2}\right)$ (See 3.952(5) Gradshteyn and Ryzhik 2007)
9) $(\ln x)^{2n}\cos(a(\ln x))$	9) $(-1)^n\sqrt{2\pi}\exp\left(-\frac{a^2}{4}\right)D_{2n}(a)$ (See 3.952(9) Gradshteyn and Ryzhik 2007)
10) $(\ln x)^{2n+1}\sin(a(\ln x))$	10) $(-1)^n\sqrt{2\pi}\exp\left(-\frac{a^2}{4}\right)D_{2n+1}(a)$ (See 3.952(10)) Gradshteyn and Ryzhik 2007)

where $D_\nu(z)$ is the parabolic cylinder functions.

Some other results are given for $p = 2$

1) $$\lim_{\substack{\nu\to\infty \\ \alpha\to 1/(2\nu) \\ \beta\to 0}} \int_0^\infty f(x)\sin\left(b(\ln x)^2\right) dx = \frac{\pi}{4}\sqrt{b}\, \exp\left(-\frac{b^2}{4}\right) I_{\frac{1}{4}}\left(\frac{b^2}{4}\right).$$

(See 3.924(1) Gradshteyn and Ryzhik (2007))

2) $$\lim_{\substack{\nu\to\infty \\ \alpha\to 1/(2\nu) \\ \beta\to 0}} \int_0^\infty f(x)\cos\left(b(\ln x)^2\right) dx = \frac{\pi}{4}\sqrt{b}\, \exp\left(-\frac{b^2}{4}\right) I_{-\frac{1}{4}}\left(\frac{b^2}{4}\right)$$

(See 3.924(2) Gradshteyn and Ryzhik(2007))

where $I_\nu(z)$ is the modified Bessel functions.

8. SIMULATION STUDY

In this section we generate artificial population from PGP type VII distribution to examine the performance of PGP distribution. We generate different sample of size 50,100,200,500 and 1000 and to obtain convergence of the parameters' estimates we repeat it for 5000 times. From the 5000 repetition minimum of log likelihood and its parameter estimates are listed below for each sample sizes. Different initial values are taken to generate the artificial population.

Table 1
p=1, k=5000

Sample Size	**Parameter Estimates**				**-2 Log Likelihood**
	$\hat{\alpha}$	$\hat{\nu}$	$\hat{\beta}$	$\hat{\delta}$	
	Initial Values				
	α =1	ν =1	β =1	δ =1	
50	1.7137	0.3139	1.1548	1.2315	426.517
100	1.6415	0.3658	1.1503	1.1543	228.712
200	1.4293	0.4385	1.1235	1.1548	91.8278
500	1.2188	0.5241	1.1148	1.1148	41.6947
1000	1.1409	0.7630	1.0996	1.0995	23.8739
	Initial Values				
	α =2	ν =2	β =2	δ =2	
50	2.2676	1.5442	1.8489	1.7498	317.8976
100	2.1689	1.5943	1.8567	1.8566	215.25
200	2.0813	1.7486	1.9457	1.9475	95.7604
500	2.0783	1.8330	1.9791	1.9953	44.357
1000	2.0132	1.9153	2.0105	2.0166	24.0715

9. APPLICATION

We use fracture toughness MPa $m^{1/2}$ data from the material Alumina (Al_2O_3) (Nadarajah and Kotz, 2007) to illustrate the method of fitting the PGP VII distribution. We compare the results PGP VII distribution to its sub-model. The data is available online at http://www.ceramics.nist.gov/srd/summary/ftmain.htm.

The descriptive statistics of data set is

Table 2
A Descriptive Statistics of Alumina (Al_2O_3)

Data	A.M	Med	S.D	Var	S.K	Kurtosis
Fracture Toughness (in the unit MPa $m^{1/2}$)	4.33	4.38	1.012	1.026	-0.42	3.093

Table 3
The ML Estimates and -2 log-likelihood

Distribution	MLE Estimates					-2 Log-likelihood
	α	β	δ	ν	**P**	
PGP VII	4.120	0.225	0.456	2.565	1	155.221
GLP VII	$4.573\ 10^{-6}$	-	-	1.902	9.180	327.175
LP VII	$3.8\ 10^{-4}$	-	-	606.700	-	427.364

10. CONCLUDING REMARKS

In this paper we have introduced a Product generalized Pearson type VII distribution and developed its different properties including characterization through conditional expectation. A family of density functions have developed corresponding to this density which shows the flexibility of the PGP distribution over the other related distributions. A simulation study is carried out to find the numerical estimates of the parameters of PGP type VII distribution. Since PGP type VII distribution has number of distributions as its sub models, therefore, we hope PGP type VII distribution will be useful in different areas of related research such as statistics and probability.

DECLARATION

We all the authors of this paper declare that there is no conflict of interests regarding the publication of this article.

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