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A GENERALIZED WEIGHTED MAXWELL DISTRIBUTION: PROPERTIES AND ESTIMATION

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ABSTRACT

The concept of weighted distribution is commonly used for efficient modeling of real life data in various fields including medicine, ecology, reliability, physics, etc. A Generalized Weighted Maxwell Distribution (GWMD) using weight function $w(x) = x^m$ is proposed here for better fitting of physical sciences data. Various properties and parameter estimates for GWMD are discussed. An example demonstrating the application of the proposed distribution in real life phenomenon is presented.

KEYWORDS

Weighted distribution, Maxwell distribution, Hazard rate, Maximum likelihood estimate, Characterizations.

1. INTRODUCTION

The concept of weighted distributions was introduced by Fisher (1934). The weighted density function is $f_w(x) = \frac{w(x)f(x)}{\int_{-\infty}^{\infty} w(x)f(x)dx}$, where $f(x)$ is a density function and $w(x)$ is a weight function. Clearly, the choice of $w(x)$ results in different weighted models. For example, if $w(x) = x$, the resulting weighted distribution is called length-biased distribution. Rao (1965) developed this concept as an approach to dealing with model specification and data interpretation problems. He identified that various situations can be modelled by weighted distributions. The weighted distributions occur frequently in studies related to reliability, survival analysis, analysis of family data, Meta analysis, analysis of intervention data, biomedicine, ecology and several other areas (cf. Stene (1981), Gupta and Keating (1985), Patil and Taillie (1989), and Oluyede and George (2002)).

Zelen (1974) introduced weighted distribution to represent what he broadly perceived as a length-biased sampling. Patil and Ord (1976) studied a size biased sampling and related invariant weighted distributions. Statistical applications of weighted distributions related to the human population and ecology can be found in Patil and Rao (1978). Gupta and Tripathi (1996) studied the weighted version of the bivariate logarithmic series distribution, which has applications in many fields such as ecology, social and behavioural sciences and species abundance studies. Castillo and Perez-Casany (1998) introduced new exponential families that come from the concept of weighted distribution, that include and generalized the Poisson distribution. For other recent works on the weighted distributions see Shi et al. (2012), Broderick et al. (2012), Ahmed et al. (2013), Mahdy (2013), Idowu and Adebayo (2014), Mahdavi (2015), Nasiru (2015), Asgharzadeh et al. (2016), Fatima and Ahmed (2017), Acitas (2018), Zamani et al. (2018), among others. A brief review on the weighted distribution and their perspectives can be found in Saghir et al. (2017a).

The Maxwell (or Maxwell-Boltzmann) distribution has wide applications in Physics and Chemistry. The Maxwell distribution forms the basis of the kinetic energy of gases, which explains many fundamental properties of gases, including pressure and diffusion. This distribution is sometimes referred to as the distribution of velocities, energy and magnitude of momenta of molecules. The probability density function (pdf) and cumulative distribution function (cdf) of the one-parameter Maxwell distribution are:

$$f(x;\alpha) = \sqrt{\frac{2}{\pi}}\alpha^{-3}x^2 e^{-\frac{x^2}{2\alpha^2}},\ x > 0, \alpha > 0, \tag{1}$$

and

$$F(x;\alpha) = \frac{\gamma\left(\frac{3}{2},\frac{x^2}{2\alpha^2}\right)}{\Gamma\left(\frac{3}{2}\right)}, x \geq 0, \tag{2}$$

where $\gamma(b,x) = \int_x^\infty x^{b-1}e^{-x}dx$ is the lower incomplete gamma function.

It was Tyagi and Bhattacharya (1989a-b) who considered the Maxwell distribution as a lifetime model for the first time and discussed the Bayes and minimum variance unbiased estimation procedures for its parameter and reliability function. Chaturvedi and Rani (1998) obtained classical and Bayes estimators for the Maxwell distribution, after generalizing it via introducing one more parameter. Empirical Bayes estimation for the Maxwell distribution was studied by Bekker and Roux (2005). Kazmi et al. (2012) carried out the Bayesian estimation for two component mixture of Maxwell distribution, assuming type I censored data. Modi (2015) and Saghir and Khadim (2016) proposed a length-biased weighted Maxwell distribution (LBWMD) in separate studies. Saghir et al. (2017b) introduced a new class of Maxwell length-biased distribution with application in real life cases. However, no study on the Weighted Maxwell distribution has been reported in the literature. This is an important gap and this article attempts to cover it.

In view of the importance of weighted distribution, the present work introduces a generalized Weighted Maxwell distribution. The pdf, cdf and other useful statistical properties of GWMD are studied in the following sections. The sub-models of GWMD are identified. The characterizations of the GWMD is also studied. The application of this distribution for a real life data set is presented.

2. GENERALIZED WEIGHTED MAXWELL DISTRIBUTION (GWMD)

In this section we derive the generalized weighted Maxwell distribution and discuss the shape of its pdf and cdf for various selected parametric values.

Taking the weight function $w(x) = x^m, m \in \mathbb{R}$, the Maxwell pdf (1) and the basic definition of weighted distribution of Fisher (1934), we have the following weighted pdf

$$f_w(x;\alpha,m)=\frac{w(x)f(x)}{\int_{-\infty}^{\infty}w(x)f(x)dx}=\sqrt{\frac{2}{\pi}}\frac{x^{m+2}\alpha^{-3}e^{-\frac{x^2}{2\alpha^2}}}{E[w(X)]}$$

or

$$f_{w(X;\,\alpha,m)=}\frac{x^{m+2}e^{-\frac{x^2}{2\alpha^2}}}{2^{\frac{(m+2)}{2}}\alpha^{m+3}\Gamma(\frac{3+m}{2})},\ x>0. \quad (3)$$

The distribution with pdf (3) is called a generalized weighted Maxwell distribution with two parameters α and m. The cdf of GWMD is denoted by $F_w(X;\ \alpha,m)$ and is given by

$$F_w(x;a,m)=\frac{1}{2^{\frac{(m+2)}{2}}\alpha^{m+3}\Gamma(\frac{3+m}{2})}\int_0^x u^{m+2}\,e^{-\frac{u^2}{2\alpha^2}}du,$$

or

$$F_w(x,a,m)=\frac{\gamma(\frac{m+3}{2},\frac{x^2}{2\alpha^2})}{\Gamma\left(\frac{3+m}{2}\right)},x\geq 0. \quad (4)$$

The shape of the pdf and cdf of the GWMD for selected parameter values are shown in Figures 1 and 2.

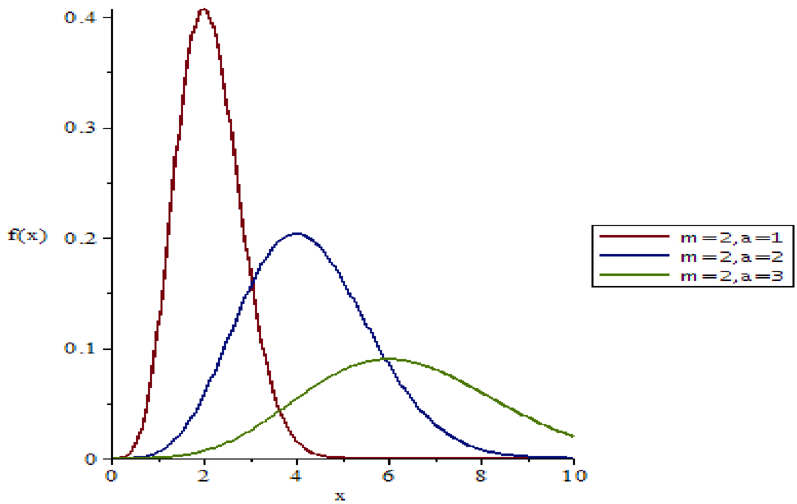

Figure 1: The Probability Density Function of GWMD for Different Values of α at $m=2$.

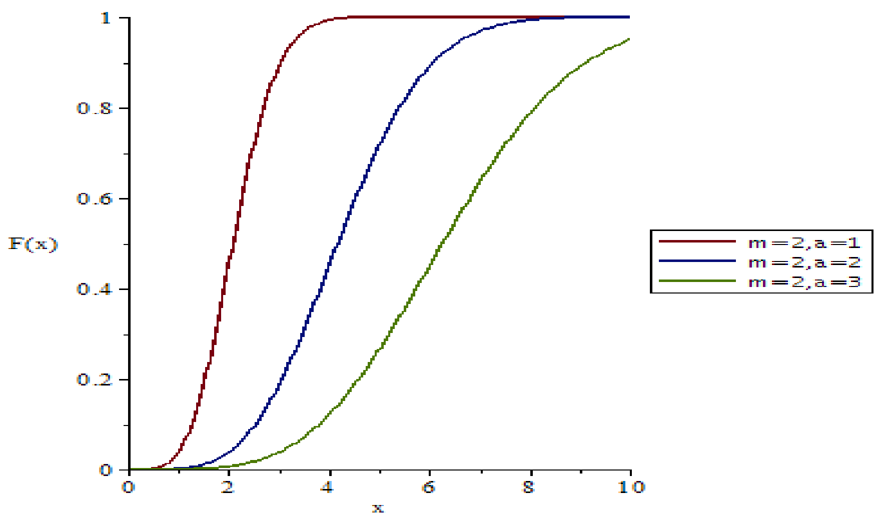

Figure 2: The Cumulative Distribution Function of GWMD for Different Values of α at $m = 2$.

The pdf decreases when αincreases and cdf increases from zero to 1 after long run for various values of α.

Sub-Models

For different choices of the parameter values in (3), we have;

i) the Rayleigh pdf $f(x) = \frac{x\, e^{-\frac{x^2}{2\alpha^2}}}{\alpha^2}$ when $m = -1$;

ii) the half-normal pdf $f(x) = \frac{2e^{-\frac{x^2}{2\alpha^2}}}{\alpha\sqrt{2\pi}}$ when $m = -2$;

iii) the length biased weighted Rayleigh pdf $f(x) = \frac{2x^3\beta^2 e^{-\frac{x^2}{2\alpha^2}}}{\theta^2}$ when $m = 1$;

iv) the length biased Maxwell pdf $f_{(X;\,\alpha\,)=}\frac{x^3 e^{-\frac{x^2}{2\alpha^2}}}{2^{\frac{3}{2}}\alpha^4}$ when $m = 1$.

The pdf of the proposed weighted model deduced the existing probability distribution as its special cases; therefore we named it a generalized weighted Maxwell distribution.

3. STATISTICAL PROPERTIES AND RELIABILITY ANALYSIS

This section derived and studied the statistical properties of generalized weighted Maxwell distribution and reliability analysis.

3.1 Mode

The logarithm of the pdf (3) is

$$\ln\big(f(x;\alpha,m)\big) = (m+2)\ln(x) - \frac{x^2}{2\alpha^2} - (m+3)\ln(\alpha) - \left(\frac{m+1}{2}\right)\ln(2) - \ln\left(\Gamma\left(\frac{m+3}{2}\right)\right), \quad (5)$$

Taking the derivative of both sides of (5) with respect to x, we arrive at

$$\frac{\partial}{\partial x}\ln\big(f(x)\big) = \frac{m+2}{x} - \frac{x}{\alpha^2},$$

which has positive root $x = \alpha\sqrt{m+2}$.

The following Table 1 shows mode value of GWMD for different parameter values. The value of mode increases for different choices of values for α and m. We can also conclude that GWMD is a unimodal distribution.

Table 1
The Mode of GWMD for Different Parameters Choices

α	m	Mode
1	1	1.73205
1	2	2.00000
1	3	2.23606
1	1	1.73205
2	1	3.46410
3	1	5.19620
1	-1	1
1	-2	0
3	-1	3
3	-2	0

3.2 Moments of GWMD

The k^{th} moments of the random variable X with pdf (3) is

$$E[X^k] = \frac{\int_0^\infty x^{m+k+2} e^{-\frac{x^2}{2\alpha^2}} dx}{\alpha^{m+3}\, 2^{\frac{m+2}{2}}\, \Gamma\left(\frac{m+3}{2}\right)},$$

$$E[X^k] = \frac{\alpha^k 2^{k/2}\, \Gamma\left(\frac{m+k+3}{2}\right)}{\Gamma\left(\frac{m+3}{2}\right)}. \quad (6)$$

Now, using (6) we can compute the mean, variance, coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) as follows:

$$E[X] = \frac{\alpha\, 2^{\frac{1}{2}}\Gamma\left(\frac{m+4}{2}\right)}{\Gamma\left(\frac{m+3}{2}\right)};$$

$$Var(X) = \frac{2\alpha^2\Gamma\left(\frac{m+3}{2}\right)\Gamma\left(\frac{(m+5)}{2}\right)-2\alpha^2\left(\Gamma\left(\frac{m+4}{2}\right)\right)^2}{\left(\Gamma\left(\frac{m+3}{2}\right)\right)^2};$$

$$CV = \frac{\left[\Gamma\left(\frac{m+3}{2}\right)\Gamma\left(\frac{(m+5)}{2}\right)-\left(\Gamma\left(\frac{m+4}{2}\right)\right)^2\right]^{\frac{1}{2}}}{\Gamma\left(\frac{m+4}{2}\right)};$$

$$CS = \frac{[\Gamma\left(\frac{m+6}{2}\right)\left(\Gamma\left(\frac{m+3}{2}\right)\right)^2-3\Gamma\left(\frac{m+3}{2}\right)\Gamma\left(\frac{m+4}{2}\right)\Gamma\left(\frac{m+5}{2}\right)+2\left(\Gamma\left(\frac{m+4}{2}\right)\right)^3\Big]}{\left[\Gamma\left(\frac{m+3}{2}\right)\Gamma\left(\frac{m+5}{2}\right)-\left(\Gamma\left(\frac{m+4}{2}\right)\right)^2\right]^{\frac{3}{2}}};$$

$$CK = \frac{\left(\Gamma\left(\frac{m+3}{2}\right)\right)^3\Gamma\left(\frac{m+7}{2}\right)-4\left(\Gamma\left(\frac{m+3}{2}\right)\right)^2\Gamma\left(\frac{m+4}{2}\right)\Gamma\left(\frac{m+6}{2}\right)+6\Gamma\left(\frac{m+5}{2}\right)\Gamma\left(\frac{m+3}{2}\right)\left(\Gamma\left(\frac{m+4}{2}\right)\right)^2-3\left(\Gamma\left(\frac{m+4}{2}\right)\right)^4}{\left[\Gamma\left(\frac{m+3}{2}\right)\Gamma\left(\frac{m+5}{2}\right)-\left(\Gamma\left(\frac{m+4}{2}\right)\right)^2\right]^{\frac{1}{2}}}.$$

Table 2 below, provides the numerical values of mean, variance, CV, CS and CK for various combinations of α and m.

Table 2
The Mean, Variance, CV, CS and CK of GWMD for Various Choices of *m* and α

α	m	Mean	Variance	CV	CS	CK
1	1	1.8799	0.4658	0.3630	69.1014	-38.3145
1	2	2.1277	0.4729	0.3232	98.1348	-83.8102
1	3	2.3499	0.4777	0.2941	240.28035	-678.7686
1	1	1.8799	0.4658	-	-	-
2	1	3.7599	1.8632	-	-	-
3	1	5.6399	4.1922	-	-	-

Table 2 reveals that the values of mean, variance and CS are increasing when *m* increases for fixed value of α, while the values of CV and CK decreasing. The values of mean and variance are increasing when α increases for fixed value of m. In the later case CV, CS and CK do not depend on α when m is fixed.

3.3 Reliability; Hazard and Reverse (Reversed) Hazard Functions

The above mentioned functions are given, respectively, by

$$R_F(x) = R_F(x;\alpha,m) = 1 - F(x;\alpha,m) = \frac{\Gamma\left(\frac{3+m}{2}\right)-\gamma\left(\frac{m}{2}+\frac{3}{2},\frac{x^2}{2\alpha^2}\right)}{\Gamma\left(\frac{3+m}{2}\right)}, x \geq 0;$$

$$h_F(x) = h_F(x;\alpha;m) = \frac{x^{m+2}e^{-\frac{x^2}{2\alpha^2}}}{2^{\frac{m+1}{2}}\alpha^{m+3}\left(\Gamma\left(\frac{3+m}{2}\right)-\gamma\left(\frac{m}{2}+\frac{3}{2},\frac{x^2}{2\alpha^2}\right)\right)}, x > 0;$$

$$r_F(x)_= r_F(x;\alpha,m) = \frac{x^{m+2}e^{-\frac{x^2}{2\alpha^2}}}{2^{\frac{m+1}{2}}\alpha^{m+3}\square(\frac{m}{2}+\frac{3}{2},\frac{x^2}{2\alpha^2})}, x > 0.$$

Figures 3-5 below give the graphical behavior of Reliability, Hazard and Reverse Hazard functions at m=2 and α=1, 2, 3. However, these can be easily constructed for any other choice of α and m.

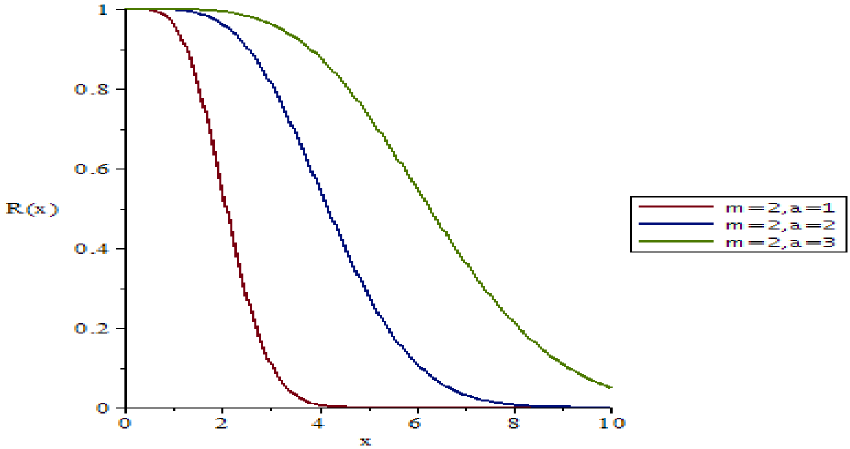

Figure 3: The Reliability Function of GWMD for Different Values of α

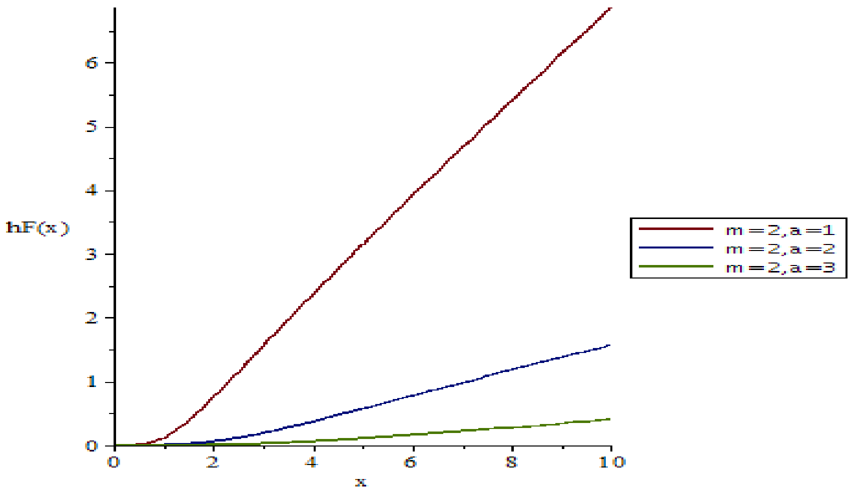

Figure 4: The Hazard Function of GWMD for Different Values of α

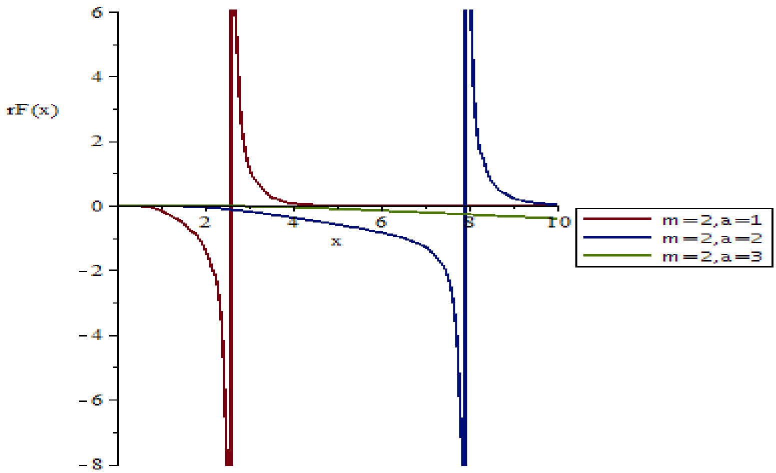

Figure 5: The Reverse Hazard Function of GWMD for Different Values of α

Figure 3 shows that the behavior of $R_F(x)$ for m = 2 and α= 1,2,3, which is increasing when αincreases, while Figure 4 shows that $h_F(x)$ starts from maximum value and approaches zero after a long run. From Figure 5, it can be seen that $r_F(x)$ varies with $\alpha$for fixed value of m.

4. ESTIMATION OF PARAMETERS

In this section, estimates of the parameters α and m of GWMD are obtained via method of moment (MOM) as well as maximum likelihood estimation (MLE).

4.1 Method of Moments (MOM)

Let $x_1, x_2, \dots, x_n$ be observed values of a random sample of size n from GWMD with parameters α and m. In view of (6), MOM results in the following equations:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{\alpha\, 2^{\frac{1}{2}} \Gamma\left(\frac{m+4}{2}\right)}{\Gamma\left(\frac{m+3}{2}\right)}, \frac{\sum_{i=1}^{n} x_i^2}{n} = \frac{2\,\alpha^2 \Gamma\left(\frac{m+5}{2}\right)}{\Gamma\left(\frac{m+3}{2}\right)}.$$

These equations do not provide close form solution. However, for the given data, the estimates can be obtained by simultaneous solution of the above equations.

4.2 Method of Maximum Likelihood Estimation

The maximum likelihood estimates $\hat{\alpha}_{ML}$ and $\widehat{m}_{ML}$, of α and m can be obtained by maximizing the log-likelihood function. Consider the likelihood function

$$L(x_1, x_2, \dots, x_n;\ \alpha, m) = \frac{\prod_{i=1}^{n} x_i^{m+2} e^{-\frac{\sum_{i=1}^{n} x_i^2}{2\alpha^2}}}{\alpha^{n(m+3)} 2^{\frac{n(m+2)}{2}} \left(\Gamma\left(\frac{3+m}{2}\right)\right)^{n}},$$

from which, we have

$$\ln[L(x_1, x_2, \dots, x_n;\ \alpha, m)] = (m+2)\sum_{i=1}^{n}\ln(x_i) - \frac{\sum_{i=1}^{n}x_i^2}{2\alpha^2} - n(m+3)\log(\alpha) - n\left(\frac{m+2}{2}\right)\ln(2) - n ln\left(\Gamma\left(\frac{m+3}{2}\right)\right).$$

Taking partial derivatives of $\ln[L(x_1, x_2, \dots, x_n;\ \alpha, m)]$ with respect to α and m, we arrive at

$$\frac{\partial lnL(x_1, x_2, \dots, x_n, \alpha; m)}{\partial\alpha} = \frac{\sum_{i=1}^{n}x_i^2}{\alpha^3} - \frac{n(m+3)}{\alpha},$$

and

$$\frac{\partial lnL(x_1, x_2, \dots, x_n, \alpha; m)}{\partial m} = \sum_{i=1}^{n} ln(x_i) - n ln(\alpha) - \frac{n}{2} ln(2) - \frac{n}{2}\psi\left(\frac{m+3}{2}\right),$$

where $\psi\left(\frac{m+3}{2}\right) = 2\frac{\partial}{\partial m} ln\left(\Gamma\left(\frac{m+3}{2}\right)\right)$.

The equations $\frac{\partial lnL(x_1, x_2, \dots, x_n, \alpha; m)}{\partial\alpha} = 0$ and $\frac{\partial lnL(x_1, x_2, \dots, x_n, \alpha; m)}{\partial m} = 0$ must be solved simultaneously to obtain MLEs of α and m. Since a closed form solution is not known, an iterative technique is required to compute the estimators $\hat{\alpha}$ and $\hat{m}$. The system of equations is solved by Newton-Raphson iteration method.

5. CHARACTERIZATION RESULTS

Characterizations of distributions is an important research area which has recently attracted the attention of many researchers. This section deals with various characterizations of GWMD. These characterizations are based on: (*i*) a simple relationship between two truncated moments; (*ii*) the hazard function; (*iii*) reverse hazard function. It should be mentioned that for the characterization (*i*) the cdf need not have a closed form and depends on the solution of a first order differential equation, which provides a bridge between probability and differential equation.

5.1 Characterizations based on Two Truncated Moments

This subsection deals with the characterizations of the above mentioned distributions based on the ratio of two truncated moments. Our first characterization employs a theorem of Glänzel (1987); see Theorem 1 of Appendix A. The result, however, holds also when the interval H is not closed.

Proposition 1:

Let $X : \Omega \rightarrow (0;\infty)$ be a continuous random variable and let $q_1(x) = x^{-(m+1)}$ and $q_2(x) = q_1(x)\, e^{-\frac{x^2}{2\alpha^2}}$ for $x > 0$. Then, the random variable X has pdf (3) if and only if the function ξ defined in Theorem 1 is of the form

$$\xi(x) = \frac{1}{2} e^{-\frac{x^2}{2\alpha^2}},\ x > 0.$$

Proof:

Suppose the random variable Xhas pdf (3), then

$$(1 - F(x))\, E[q_1(x)|X \geq x] = K\alpha^2 e^{-\frac{x^2}{2\alpha^2}}, x > 0,$$

and

$$(1 - F(x))\, E[q_2(x)|X \geq x] = \frac{K\alpha^2}{2} e^{-\frac{x^2}{2\alpha^2}}, x > 0,$$

where K is the appropriate constant.

Further,

$$\xi(x)q_1(x) - q_2(x) = -\frac{1}{2} q_1(x)\, e^{-\frac{x^2}{2\alpha^2}} < 0 \; for \; x > 0.$$

Conversely, if ξ is of the above form, then

$$\acute{s}(x) = \frac{\acute{\xi}(x)q_1(x)}{\xi(x)q_1(x) - q_2(x)} = \frac{x}{\alpha^2}, x > 0,$$

and hence

$$s(x) = \frac{x^2}{2\alpha^2}, x > 0.$$

Now, in view of Theorem 1, X has density (3).

Corollary 1:

Let $X : \Omega \rightarrow (0; \infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 1. The pdf of X is (3) if and only if there exist q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\acute{\xi}(x)q_1(x)}{\xi(x)q_1(x) - q_2(x)} = \frac{x}{\alpha^2}, x > 0. \tag{7}$$

Upon multiplying both sides of (7) by $e^{-\frac{x^2}{2\alpha^2}}$, it is easy to show that, the general solution of the differential equation (7) is

$$\xi(x) = e^{\frac{x^2}{2\alpha^2}} \left[-\int \frac{x^2}{\alpha^2} e^{-\frac{x^2}{2\alpha^2}} \left(q_1(x)\right)^{-1} q_2(x) dx + D \right],$$

where D is a constant. Note that a set of functions satisfying the differential equation (7) is given in Proposition 1 with $D = 0$*:* However, it should also be noted that there are other triplets ($q1; q2; \eta$) satisfying the conditions of Theorem 1.

5.2 Characterization based on Hazard Function

It is well known that the hazard function, h_F, of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{\acute{f}(x)}{f(x)} = \frac{\acute{h_F}(x)}{h_F(x)} - h_F(x). \tag{8}$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following Proposition establish non-trivial characterization of GWMD in terms of the hazard function which is not of the trivial form given in (8).

Proposition 2:

Let $X : \Omega \to (0;\infty)$ be a continuous random variable. The pdf of X is (3) if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h'_F(x) - \frac{\eta+2}{x} h_F(x) = -\frac{x^{\eta+3} e^{-\frac{x^2}{2\alpha^2}}}{\alpha^2 K}, x > 0. \ (9)$$

with the boundary condition $h_F(0) = 0$.

Proof:

If X has pdf (3), then clearly (9) holds. Now, if (9) holds, then

$$\frac{d}{dx}\left\{x^{-(\eta+3)} h_F(x)\right\} = -\frac{x\, e^{-\frac{x^2}{2\alpha^2}}}{\alpha^2 K} = \frac{1}{K}\frac{d}{dx}\left\{e^{-\frac{x^2}{2\alpha^2}}\right\},$$

or equivalently,

$$h_F(x) = \frac{x^{\eta+2}\, e^{-\frac{x^2}{2\alpha^2}}}{K},$$

which is the hazard function of GWMD.

5.3 Characterization in terms of the Reverse (or reversed) Hazard Function

The reverse hazard function, r_F, of a twice differentiable distribution function, F, is defined as

$$r_F(x) = \frac{f(x)}{F(x)}, x \, \epsilon \; support \; og \; F.$$

In this subsection we present characterization of GWMD distribution in terms of the reverse hazard function.

Proposition 3:

Let $X : \Omega \to (0;\infty)$ be a continuous random variable. The random variable X has pdf (3) if and only if its reverse hazard function $r_F(x)$ satisfies the differential equation

$$r'_F(x) - \frac{\eta+2}{x} r_F(x) = \frac{x^{\eta+3}}{K}\frac{d}{dx}\left\{\frac{e^{-\frac{x^2}{2\alpha^2}}}{\gamma\left(\frac{\eta+3}{2}, \frac{x^2}{2\alpha^2}\right)}\right\}, x > 0.$$

Proof:

It is similar to that of Proposition 2.

6. APPLICATION OF GWMD

The data set employed here is taken from Lawless (2003), which represent the number of revolution before failure for each of the 23 ball bearing. The Maxwell distribution and the proposed GWMD are fitted on this data set by finding the MLEs of the parameters. The Kolmogorov-Smirnov (KS) test is used for the goodness of fit distribution. The parameters estimates and the KS test values are shown in Table 3 below. The obtained results are also compared with the LBMD (Saghir et al., 2017) and given in

Table 3. The comparison of p-values of K.S statistic for both the under study distributions reveals that the GWMD is best fitted for ball bearing data as its p-value is higher than the value of the Maxwell distribution and the length biased Maxwell distribution. Therefore, it is concluded that the generalized weighted Maxwell distribution is more suitable for the modeling of Physical data sets than the Maxwell distribution.

Table 3
The Parameters Estimates and the K.S Statistic Values for Real Data

Distribution	MD	GWMD	LBMD
Parameter estimate	α = 46.76	α = 38.42, m = 0.10	α = 40.50
K.S statistic	0.1810	0.0706	0.0895
p-value	0.4550	0.690	0.650

7. SUMMARY AND CONCLUSIONS

Weighted distributions are widely used in many real life fields including medicine, ecology, and reliability and so on. In this work, a generalized weighted Maxwell distribution is derived. At first, the pdf of the GWMD have been obtained considering weight $w(x) = x^m$ by the idea proposed by Rao (1965). To characterize the distribution of a random variable X of the GWMD three functions have been introduced; namely the survival function, the probability density or mass function and the failure rate function or hazard function. The moments, the coefficient of variation, the coefficient of skewness and the coefficient of kurtosis of GWMD are derived.

For estimating the parameters of GWMD, maximum likelihood estimation method along with method of moments have been used. The pdf of the proposed weighted model deduced some existing probability distributions as its special cases; therefore we named it a generalized weighted Maxwell distribution. An application to a real data set shows that the fit of GWMD is best fit to the physical science data with highest p-values. Further work on the double weighted version of the Maxwell distribution will be carried out in a future study.

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APPENDIX A

Theorem 1:

Let (Ω, F, P) be a given probability space and let $H = [d; e]$ be an interval for some $d < e$ ($d = -\infty$; $e = \infty$ might as well be allowed). Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$E[q_2(X)|X \geq x] = E[q_1(X)|X \geq x]\xi(x), x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \epsilon C^1(H), \xi \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and ξ, particularly

$$F(x) = \int_a^x C \left| \frac{\acute{\xi}(u)}{\xi(u)q_1(u) - q_2(u)} \right| \exp(-s(u))\, du,$$

where the function s is a solution of the differential equation $s' = \frac{\acute{\xi} q_1}{\xi q_1 - q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence (see, Glänzel 1990), in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions q_{1n}, q_{2n} and ξ_n ($n \in \mathbb{N}$) satisfy the conditions of Theorem 1 and let $q_{1n} \to q_1$, $q_{2n} \to q_2$ for some continuously differentiable real functions q_1 and q_2. Let, finally, X be a random variable with distribution F : Under the condition that $q_{1n}(X)$ and $q_{2n}(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if ξ_n converges to ξ, where

$$\xi(x) = \frac{E[q_2(X)|X \geq x]}{E[q_1(X)|X \geq x]}.$$

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions q_1, q_2 and ξ respectively. It guarantees, for instance, the ‘convergence’ of characterization of the Wald distribution to that of the Lévy-Smirnov distribution if $\alpha \to \infty$, as was pointed out in Glänzel and Hamedani (2001).

A further consequence of the stability property of Theorem 1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions q_1, q_2 and ξ speciallyshould be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose ξ as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics.

In some cases, one can take $q_1(X) \equiv 1$which reduces the condition of Theorem 1 to $E[q_2(X)|X \geq x] = \xi(x), x \in H$. We, however, believe that employing three functions q_1, q_2 and ξ will enhance the domain of applicability of Theorem 1.