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TWO PARAMETRIC NEW GENERALIZED AVERAGE CODE-WORD LENGTH AND ITS BOUNDS IN TERMS OF NEW GENERALIZED INACCURACY MEASURE AND THEIR CHARACTERIZATION

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ABSTRACT

In this research article we develop a new two parametric new generalized inaccuracy measure $I_{\alpha}^{\beta}(P:Q)$ and a new two parametric generalized average code-word length $L_{\alpha}^{\beta}(P)$. These measures are the generalizations of some familiar measures existing in the subjects of information and coding theory. Also we develop the noiseless coding theorems for discrete noiseless channel. (i.e., we find the lower and upper bounds of $L_{\alpha}^{\beta}(P)$ in terms of $I_{\alpha}^{\beta}(P:Q)$). The mathematical results obtained in this research article are verified by using empirical data. Also the monotonic behavior of $I_{\alpha}^{\beta}(P:Q)$ and $L_{\alpha}^{\beta}(P)$ with respect to parameters α and β have been discussed.

KEYWORDS

Shannon's entropy, Kerridge's inaccuracy measure, Mean code-word length, Kraft's inequality, Holder's inequality, Huffman codes, Shannon-Fano codes, Noiseless coding theorem.

AMS Classification 94A17, 94A24

1. INTRODUCTION

The branch of Information theory is new to statistics and probability theory, and has wide applications in communication science. Like several other branches of mathematics, information theory has a physical origin. It was initiated by communication scientist Shannon (1948) who studied the statistical structure of communication system. The subject followed by a flood of research papers speculating upon the possible applications to a broad spectrum of research areas, such as pure mathematics, semantics, physics, management, thermodynamics, botany, econometrics, operations research, psychology, epidemiological studies, disease management and related disciplines. For measuring information, a general approach is provided in a statistical framework based on information entropy introduced by Shannon (1948).

For uniquely decodable codes, Shannon (1948) found the bounds for the average code-word length in terms of his own entropy. Later, Campbell (1965) defined his own average code-word length and found bounds for his average code-word length in terms of Renyi's (1961) entropy by applying Kraft's (1949) inequality.

In the last few decades researchers develop various generalized noiseless coding theorems for discrete channel under the condition of uniquely decipherability by taking different generalized information measures, Nath (1968), inaccuracy and coding theory, Longo (1976), also develop noiseless coding theorems for useful mean code-word length in terms of weighted entropy given by Belis and Guiasu (1968), Guiasu and Picard (1971), Gurdial (1977), extended the noiseless coding theorem for useful mean code-word length of order α, also various authors like Jain and Tuteja (1989), Taneja et al. (1985), Bhatia (1995), Hooda and Bhaker (1997), Khan et al. (2005), Bhat and Baig (2016a; 2016b 2016c; 2017a; 2017b), also develop various generalized coding theorems under the condition of uniquely decipherability.

2. SELF INFORMATION AND ENTROPY

Shannon (1948), introduced the idea of self-information as:

Suppose we have an event X, where x_i represents a particular outcome of the event and p_i be the probability of outcome x_i, then according to Shannon the self information of x_i is defined as

$$I(x_i) = log\frac{1}{p_i} = -\,log\, p_i \tag{1}$$

Suppose a random variable X taking values $X = (x_1, x_2, \dots, x_n)$ with respective probabilities $P = (p_1, p_2, \dots, p_n), p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$. Shannon (1948) has defined the following measure of information and call it as entropy.

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i \tag{2}$$

Suppose the probabilities of transmission of n code-words be $p_1, p_2, p_3, \dots, p_n$ and let their lengths be $l_1, l_2, \dots, l_n$ and these lengths satisfies Kraft's (1949) inequality.

$$\sum_{i=1}^{n} D^{-l_i} \leq 1 \tag{3}$$

where D is considered to be the size of code alphabet. Shannon (1948) showed that for all uniquely decipherable codes satisfying inequality (3), the lower bound of the mean code-word length.

$$L = \sum_{i=1}^{n} p_i l_i \tag{4}$$

lies between $H(P)$ and $H(P) + 1$.

Campbell (1965) considered the more general exponentiated mean code-word length as

$$L_\alpha = \frac{\alpha}{1-\alpha} \log_D \left[\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right], \alpha > 0, \alpha \neq 1 \tag{5}$$

and showed that for all uniquely decodable codes, satisfying constraint (3), the lower bound of (5) lies between $R_\alpha(P)$ and $R_\alpha(P)+1$, where

$$R_\alpha(P)=\frac{1}{1-\alpha}\log_D\left[\sum_{i=1}^{n}p_i^{\alpha}\right],\alpha>0,\alpha\neq 1 \tag{6}$$

is Renyi's (1961) entropy.

3. CONCEPT OF INACCURACY

Kerridge (1961), introduces the concept of inaccuracy and is considered as a generalization of Shannon's (1948) entropy. It is a useful tool for measuring the error in experimental results. Suppose that an experimenter states the probabilities of the various possible outcomes of an experiment. His statement can lack precision in two ways: he may not have enough information and so his statement is vague, or some of the information he has may be incorrect. Kerridge (1961) proposed the measure of inaccuracy and this measure is suitable for these two types of errors. Suppose that the experimenter asserts the probability distribution of a random variable X as $Q=(\,q_1,\,q_2,\ldots,\,q_n)$ whereas the true probability distribution of a random variable X is $P=(\,p_1,\,p_2,\ldots,\,p_n)$, then the inaccuracy of the observer, as proposed by Kerridge (1961) can be measured as:

$$I(P,Q)=-\sum_{i=1}^{n}p_i\log q_i \tag{7}$$

In this communication we present a new two parametric generalized inaccuracy measure $I_\alpha^\beta(P{:}\,Q)$ and a new two parametric generalized average code-word length $L_\alpha^\beta(P)$. Then we characterize these two measures in different aspects.

4. NOISELESS CODING THEOREMS

Define a new two parametric generalized inaccuracy measure as:

$$I_\alpha^\beta(P{:}\,Q)=\frac{\beta}{1-\alpha}log_D\left[\sum_{i=1}^{n}{p_i}^{\beta}{q_i}^{\beta(\alpha-1)}\right] \tag{8}$$

where, $0<\alpha<1, 0<\beta\leq 1, p_i\geq 0\,, q_i\geq 0, \forall i=1,2,\ldots,n\,, \sum_{i=1}^{n}p_i\,\sum_{i=1}^{n}\,q_i=1\,.$

The parameters α and β can be interpreted in the following way, if we consider the ensemble of a message x_i with respective probabilities p_i as a cybernetic system $[x_i,p_i]$, then α and β can be considered as flexibility parameters or as pre-assigned numbers associated with different cybernetic systems. For example, consider two cybernetic systems, with the same set of x_i, p_i may have different informations (with respect to the same goal) for different values of α and β. The parameters α and β may represent the environment factors, such as temperature, humidity etc. More-over, there are many factors like temperature, humidity etc. which affect the diversity in cost. Let α and β represent such factors upon which the information regarding such a cybernetic system $[x_i,p_i]$, depends.

Particular Cases for (8):

i) When $\beta = 1$, the measure defined in (8) becomes $I_\alpha(P:Q) = \frac{1}{1-\alpha} log_D\left[\sum_{i=1}^{n} p_i q_i{}^{(\alpha-1)}\right]$, this is inaccuracy measure due to Nath (1968).

ii) When $\beta = 1$ and $\alpha \to 1$, the measure defined in (8) becomes $I(P:Q) = -\sum_{i=1}^{n} p_i \, log\, q_i$, this is inaccuracy measure due to Kerridge (1961).

iii) When $p_i = q_i \forall\, i$ the measure defined in (8) becomes $H_\alpha^\beta(P) = \frac{\beta}{1-\alpha} log_D\left[\sum_{i=1}^{n} p_i^{\alpha\beta}\right]$, this is the information measure due to Bhat and Baig (2017a).

iv) When $\beta = 1$ and $p_i = q_i \forall\, i$ the measure defined in (8) becomes $H_\alpha(P) = \frac{1}{1-\alpha} log_D[\sum_{i=1}^{n} p_i^\alpha]$, this is Renyi's (1961) entropy.

v) When $\beta = 1$, $\alpha \to 1$ and $p_i = q_i \forall\, i$ the measure defined in (8) becomes $H(P) = -\sum_{i=1}^{n} p_i log p_i$, this is Shannon (1948) entropy.

vi) When $\alpha \to 1, \beta = 1$ and $p_i = q_i = \frac{1}{n} \forall i$ the measure defined in (8) becomes $H\left(\frac{1}{n}\right) = \log_D n$, this is maximum entropy.

Further we propose a new two parametric generalized average code-word length as:

$$\mathrm{L}_\alpha^\beta(\mathrm{P}) = \frac{\alpha\beta}{1-\alpha} \log_\mathrm{D}\left[\sum_{i=1}^{n} p_i^\beta D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right], 0 < \alpha < 1, 0 < \beta \le 1, \tag{9}$$

where D is considered to be the size of code alphabet.

Particular cases for (9):

i) For $\beta = 1$, the measure defined in (9) becomes $L_\alpha(P) = \frac{\alpha}{1-\alpha} \log_D \left[\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right]$, this is code-word length due Campbell (1965).

ii) For $\beta = 1$ and $\alpha \to 1$, the measure defined in (9) becomes $\sum_{i=1}^{n} p_i l_i$, this is optimal code-word length due to Shannon (1948).

iii) For $\beta = 1$ and $l_1 = l_2 = \cdots = l_n = 1$, then the measure defined in (9) reduces to 1. i.e., $L_\alpha = 1$.

Now we characterize measure defined in (9) in terms of the measure defined in (8) under the condition

$$\sum_{i=1}^{n} p_i{}^\beta q_i{}^{-\beta} D^{-l_i} \le 1 \tag{10}$$

This is generalization of Kraft's inequality. It is easy to see that when $p_i = q_i \forall i$, then the inequality (10) becomes Kraft's (1949) inequality, where D is considered to be the size of code alphabet.

Theorem 1:

For all integers $(D > 1)$ if the code-word lengths $l_1, l_2, \dots, l_n$ satisfies the condition (10), then the inequality

$$L_{\alpha}^{\beta}(P) \geq I_{\alpha}^{\beta}(P:Q), 0 < \alpha < 1, 0 < \beta \leq 1, \tag{11}$$

is fulfilled. Where $I_{\alpha}^{\beta}(P:Q)$ and $L_{\alpha}^{\beta}(P)$ are defined in (8) and (9) respectively. Furthermore, the equality holds good iff

$$l_i = -\log_D \left[\frac{q_i^{\alpha\beta}}{\sum_{i=1}^{n} p_i{}^{\beta} q_i{}^{\beta(\alpha-1)}}\right] \tag{12}$$

Proof:

We know that for all $x_i, y_i > 0, i = 1, 2, 3, \dots, n$ and $\frac{1}{\gamma} + \frac{1}{\delta} = 1, \gamma < 1 (\neq 0), \delta < 0$ or $\delta < 1 (\neq 0), \gamma < 0$, then the Holder's inequality

$$\left(\sum_{i=1}^{n} x_i{}^{\gamma}\right)^{\frac{1}{\gamma}} \left(\sum_{i=1}^{n} y_i{}^{\delta}\right)^{\frac{1}{\delta}} \leq \sum_{i=1}^{n} x_i y_i \tag{13}$$

holds, and equality holds in (13) iff there exists a positive constant μ such that

$$x_i^{\gamma} = \mu y_i^{\delta} \tag{14}$$

Substitute $x_i = p_i^{\frac{\alpha\beta}{\alpha-1}} D^{-l_i}, y_i = p_i^{\frac{\beta}{1-\alpha}} q_i{}^{-\beta}, \gamma = \frac{\alpha-1}{\alpha}$, and $\delta = 1 - \alpha$, in (13) we get,

$$\left[\sum_{i=1}^{n} p_i^{\beta} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right]^{\frac{\alpha}{\alpha-1}} \left[\sum_{i=1}^{n} p_i{}^{\beta} q_i{}^{\beta(\alpha-1)}\right]^{\frac{1}{1-\alpha}} \leq \sum_{i=1}^{n} p_i{}^{\beta} q_i{}^{-\beta} D^{-l_i} \tag{15}$$

Now using the inequality (10), we get

$$\left[\sum_{i=1}^{n} p_i^{\beta} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right]^{\frac{\alpha}{\alpha-1}} \left[\sum_{i=1}^{n} p_i{}^{\beta} q_i{}^{\beta(\alpha-1)}\right]^{\frac{1}{1-\alpha}} \leq 1 \tag{16}$$

or the inequality (16) can be written as

$$\left[\sum_{i=1}^{n} p_i^{\beta} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right]^{\frac{\alpha}{\alpha-1}} \leq \left[\sum_{i=1}^{n} p_i{}^{\beta} q_i{}^{\beta(\alpha-1)}\right]^{\frac{1}{\alpha-1}} \tag{17}$$

Taking logarithms with base D throughout to the inequality (17), we get

$$\frac{\alpha}{\alpha-1} \log_D \left[\sum_{i=1}^{n} p_i^{\beta} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right] \leq \frac{1}{\alpha-1} \log_D \left[\sum_{i=1}^{n} p_i{}^{\beta} q_i{}^{\beta(\alpha-1)}\right] \tag{18}$$

or we can write the inequality (18), as

$$\frac{\alpha}{1-\alpha} \log_D \left[\sum_{i=1}^{n} p_i^{\beta} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right] \geq \frac{1}{1-\alpha} \log_D \left[\sum_{i=1}^{n} p_i{}^{\beta} q_i{}^{\beta(\alpha-1)}\right] \tag{19}$$

As $0 < \beta \leq 1$, multiply inequality (19) throughout by β, we get

$$\frac{\alpha\beta}{1-\alpha}\log_D\left[\sum_{i=1}^{n} p_i^{\beta} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right] \geq \frac{\beta}{1-\alpha}\log_D\left[\sum_{i=1}^{n} {p_i}^{\beta} {q_i}^{\beta(\alpha-1)}\right] \tag{20}$$

This implies

$L_\alpha^\beta(P) \geq I_\alpha^\beta(P:Q)$. Hence the result for $0 < \alpha < 1, 0 < \beta \leq 1$.

Now we will show that the equality in (11) holds if and only if

$$l_i = -\log_D\left[\frac{q_i^{\alpha\beta}}{\sum_{i=1}^{n} {p_i}^{\beta} {q_i}^{\beta(\alpha-1)}}\right], 0 < \alpha < 1, 0 < \beta \leq 1. \tag{21}$$

or equivalently, we can write the equation (21) as

$$D^{-l_i} = \left[\frac{q_i^{\alpha\beta}}{\sum_{i=1}^{n} {p_i}^{\beta} {q_i}^{\beta(\alpha-1)}}\right]$$

or equivalently, we can write the above equation as

$$D^{-l_i} = q_i^{\alpha\beta}\left[\sum_{i=1}^{n} {p_i}^{\beta} {q_i}^{\beta(\alpha-1)}\right]^{-1} \tag{22}$$

Raise power $\frac{\alpha-1}{\alpha}$, throughout to equation (22), and after suitable simplification, we get

$$D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)} = q_i^{\beta(\alpha-1)}\left[\sum_{i=1}^{n} {p_i}^{\beta} {q_i}^{\beta(\alpha-1)}\right]^{\frac{1-\alpha}{\alpha}} \tag{23}$$

Multiply equation (23) throughout by p_i^β, and summing over $i = 1,2,\dots,n$, throughout to the resulted expression and after suitable simplification, we get

$$\sum_{i=1}^{n} p_i^{\beta} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)} = \left[\sum_{i=1}^{n} {p_i}^{\beta} {q_i}^{\beta(\alpha-1)}\right]\left[\sum_{i=1}^{n} {p_i}^{\beta} {q_i}^{\beta(\alpha-1)}\right]^{\frac{1-\alpha}{\alpha}}$$

or we can write the above equation as

$$\sum_{i=1}^{n} p_i^{\beta} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)} = \left[\sum_{i=1}^{n} {p_i}^{\beta} {q_i}^{\beta(\alpha-1)}\right]^{\frac{1}{\alpha}} \tag{24}$$

Take logarithms with base D throughout to equation (24), then multiply throughout by $\frac{\alpha\beta}{(1-\alpha)}$, we get

$$\frac{\alpha\beta}{1-\alpha}log_D\left[\sum_{i=1}^{n} p_i^{\beta} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right] = \frac{\beta}{1-\alpha}log_D\left[\sum_{i=1}^{n} {p_i}^{\beta} {q_i}^{\beta(\alpha-1)}\right] \tag{25}$$

This implies that

$L_\alpha^\beta(P;U) = H_\alpha^\beta(P:U)$. Hence the result

***Theorem 2*:**

If for every code with lengths $l_1, l_2, \dots, l_n$ satisfies the condition (10), then $L_\beta^\alpha(P)$ satisfy the inequality

$$L_\beta^\alpha(P) < I_\alpha^\beta(P:Q) + \beta, \text{where } 0 < \alpha < 1, 0 < \beta \leq 1. \tag{26}$$

Proof:

From the theorem 1 we have

$L_\alpha^\beta(P) = I_\alpha^\beta(P:Q)$, holds if and only if

$$l_i = -\log_D \left[\frac{q_i^{\alpha\beta}}{\sum_{i=1}^n {p_i}^\beta {q_i}^{\beta(\alpha-1)}}\right], 0 < \alpha < 1, 0 < \beta \leq 1.$$

or equivalently the above equation can be written as

$$l_i = -log_D q_i^{\alpha\beta} + log_D \left[\sum_{i=1}^n {p_i}^\beta {q_i}^{\beta(\alpha-1)}\right]$$

Now the code-word lengths $l_1, l_2, \dots, l_n$, can be choosen in such a manner that they satisfy the inequality,

$$-log_D q_i^{\alpha\beta} + log_D \left[\sum_{i=1}^n {p_i}^\beta {q_i}^{\beta(\alpha-1)}\right] \leq l_i < -log_D q_i^{\alpha\beta} + log_D \left[\sum_{i=1}^n {p_i}^\beta {q_i}^{\beta(\alpha-1)}\right] + 1$$

Consider the intervals δ_i of length unity as

$$\delta_i = \left[-log_D q_i^{\alpha\beta} + log_D \left[\sum_{i=1}^n {p_i}^\beta {q_i}^{\beta(\alpha-1)}\right], -log_D q_i^{\alpha\beta} + log_D \left[\sum_{i=1}^n {p_i}^\beta {q_i}^{\beta(\alpha-1)}\right] + 1\right]$$

In every interval δ_i, there is exactly one positive integer l_i, such that,

$$0 < -log_D q_i^{\alpha\beta} + log_D \left[\sum_{i=1}^n {p_i}^\beta {q_i}^{\beta(\alpha-1)}\right] \leq l_i < -log_D q_i^{\alpha\beta} + log_D \left[\sum_{i=1}^n {p_i}^\beta {q_i}^{\beta(\alpha-1)}\right] + 1 \tag{27}$$

Now we will first prove that the sequence of code-words $l_1, l_2, \dots, l_n$, satisfies the inequality (10), which is generalization of Kraft's (1949) inequality.

First consider the left inequality of (27), we have

$$-log_D q_i^{\alpha\beta} + log_D \left[\sum_{i=1}^n {p_i}^\beta {q_i}^{\beta(\alpha-1)}\right] \leq l_i$$

or we can write the above inequality as

$$D^{-l_i} \leq \left[\frac{q_i^{\alpha\beta}}{\sum_{i=1}^{n} {p_i}^{\beta} {q_i}^{\beta(\alpha-1)}}\right] \tag{28}$$

Multiply inequality (28) throughout by ${p_i}^{\beta} {q_i}^{-\beta}$ and summing over $i = 1,2,\ldots,n$, throughout to the resulted expression, and after suitable simplification, we get the required result (10), i.e.,

$$\sum_{i=1}^{n} {p_i}^{\beta} {q_i}^{-\beta} D^{-l_i} \leq 1$$

Now consider the last inequality of (27), we have

$$l_i < -log_D q_i^{\alpha\beta} + log_D \left[\sum_{i=1}^{n} {p_i}^{\beta} {q_i}^{\beta(\alpha-1)}\right] + 1$$

or equivalently, we can be write the above inequality as

$$D^{l_i} < q_i^{-\alpha\beta} \left[\sum_{i=1}^{n} {p_i}^{\beta} {q_i}^{\beta(\alpha-1)}\right] D \tag{29}$$

As $0 < \alpha < 1$, then $\left(\frac{1-\alpha}{\alpha}\right) > 0$, raise power $\left(\frac{1-\alpha}{\alpha}\right) > 0$, throughout to the inequality (29), and after suitable simplification, we get

$$D^{l_i\left(\frac{1-\alpha}{\alpha}\right)} < q_i^{\beta(\alpha-1)} \left[\sum_{i=1}^{n} {p_i}^{\beta} {q_i}^{\beta(\alpha-1)}\right]^{\frac{1-\alpha}{\alpha}} D^{\frac{1-\alpha}{\alpha}}$$

or equivalently, we can write the above inequality as

$$D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)} < q_i^{\beta(\alpha-1)} \left[\sum_{i=1}^{n} {p_i}^{\beta} {q_i}^{\beta(\alpha-1)}\right]^{\frac{1-\alpha}{\alpha}} D^{\frac{1-\alpha}{\alpha}} \tag{30}$$

Multiply inequality (30) throughout by ${p_i}^{\beta}$, and summing over $i = 1,2,\ldots,n$, throughout to the resulted expression and after suitable simplification, we get

$$\sum_{i=1}^{n} p_i^{\beta} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)} < \left[\sum_{i=1}^{n} {p_i}^{\beta} {q_i}^{\beta(\alpha-1)}\right] \left[\sum_{i=1}^{n} {p_i}^{\beta} {q_i}^{\beta(\alpha-1)}\right]^{\frac{1-\alpha}{\alpha}} D^{\frac{1-\alpha}{\alpha}}$$

or equivalently, the above inequality can be written as

$$\sum_{i=1}^{n} p_i^{\beta} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)} < \left[\sum_{i=1}^{n} {p_i}^{\beta} {q_i}^{\beta(\alpha-1)}\right]^{\frac{1}{\alpha}} D^{\frac{1-\alpha}{\alpha}} \tag{31}$$

Taking logarithms with base D throughout to the inequality (31), we get

$$log_D\left[\sum_{i=1}^{n} p_i^{\beta} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right] < \frac{1}{\alpha} log_D\left[\sum_{i=1}^{n} {p_i}^{\beta} {q_i}^{\beta(\alpha-1)}\right] + \frac{1-\alpha}{\alpha} \tag{32}$$

As $0 < \alpha < 1, 0 < \beta \leq 1$ then $\frac{\alpha\beta}{1-\alpha} > 0$, multiply inequality (32), throughout by $\frac{\alpha\beta}{1-\alpha} > 0$, we get

$$\frac{\alpha\beta}{1-\alpha} log_D\left[\sum_{i=1}^{n} p_i^{\beta} D^{-l_i\left(\frac{\alpha-1}{\alpha}\right)}\right] < \frac{\beta}{1-\alpha} log_D\left[\sum_{i=1}^{n} {p_i}^{\beta} {q_i}^{\beta(\alpha-1)}\right] + \beta$$

This implies that

$L_\alpha^\beta(P) < I_\alpha^\beta(P{:}Q) + \beta$. Hence the result for $0 < \alpha < 1, 0 < \beta \leq 1$.

Thus on combining theorem 1 and theorem 2 we conclude that

$I_\alpha^\beta(P{:}Q) \leq L_\alpha^\beta(P) < I_\alpha^\beta(P{:}Q) + \beta$, where $0 < \alpha < 1, 0 < \beta \leq 1$.

In this section the coding theorems proved above are verified by taking an empirical data.

5. ILLUSTRATION

In this section the coding theorems proved above are verified by taking an empirical data. Let X be a discrete random variable taking finite values as:

$X = (x_1, x_2, x_3, x_4, x_5, x_6)$

Suppose that the experimenter asserts the probability distribution of this random variable as:

$Q = (q_1, q_2, q_3, q_4, q_5, q_6) = (0.4, 0.2, 0.15, 0.12, 0.08, 0.05)$

Whereas the true probability distribution of this random variable is:

$P = (p_1, p_2, p_3, p_4, p_5, p_6) = (0.41, 0.18, 0.15, 0.13, 0.1, 0.03)$

The values of $I_\alpha^\beta(P{:}Q)$, $I_\alpha^\beta(P{:}Q) + \beta$, $L_\alpha^\beta(P)$ and η for different values of α and β using Huffman coding are shown in the following table:

Table 1
Values of $I_\alpha^\beta(P:Q)$, $I_\alpha^\beta(P:Q)+\beta$, $L_\alpha^\beta(P)$ and η for different values of α and β using Huffman Coding

p_i	q_i	*Huffman Code-word*	l_i	α	β	$I_\alpha^\beta(P:Q)$	$L_\alpha^\beta(P)$	$\eta = \frac{I_\alpha^\beta(P:Q)}{L_\alpha^\beta(P)} \times 100$	$I_\alpha^\beta(P:Q)+\beta$
0.41	0.4	1	1	0.9	1	2.290	2.3596	97.050%	3.290
0.18	0.2	000	3	0.9	0.9	3.950	4.031	97.990%	4.850
0.15	0.15	001	3	0.8	1	2.3182	2.4208	95.761%	3.3182
0.13	0.12	010	3						
0.1	0.08	0110	4						
0.03	0.05	0111	4						

Now the values of $I_\alpha^\beta(P:Q)$, $I_\alpha^\beta(P:Q)+\beta$, $L_\alpha^\beta(P)$ and η for different values of α and β using Shannon-Fano coding are shown in the following table:

Table 2
Values of $I_\alpha^\beta(P:Q)$, $I_\alpha^\beta(P:Q)+\beta$, $L_\alpha^\beta(P)$ and η for different values of $\boldsymbol{\alpha}$ and β using Shannon-Fano Coding

p_i	q_i	*Shannon Fano Code-word*	l_i	α	β	$I_\alpha^\beta(P:Q)$	$L_\alpha^\beta(P)$	$\eta = \frac{I_\alpha^\beta(P:Q)}{L_\alpha^\beta(P)} \times 100$	$I_\alpha^\beta(P:Q)+\beta$
0.41	0.4	00	2	0.9	1	2.290	2.4193	94.655%	3.290
0.18	0.2	01	2	0.9	0.9	3.9502	4.0461	97.629%	4.8502
0.15	0.15	100	3	0.8	1	2.3182	2.4311	95.356%	3.3182
0.13	0.12	101	3						
0.1	0.08	110	3						
0.03	0.05	111	3						

From the tables 1 and 2 we infer the following:

i) Theorems 1 and 2 holds in both the cases of Shannon-Fano and Huffman coding i.e.
$$I_\alpha^\beta(P:Q) \le L_\alpha^\beta(P) < I_\alpha^\beta(P:Q)+\beta.\text{where } 0<\alpha<1, 0<\beta\le 1.$$

ii) Also from above two tables we see that our new generalized mean code-word length has less code-word length in case of Huffman coding as compared to Shannon-Fano coding.

iii) Also we see that the efficiency (η) of our generalized mean code-word length is greater in case of Huffman coding scheme as compared to Shannon-Fano coding scheme, so Huffman coding is most effective than Shannon-Fano coding.

In the next section we study the monotonic behavior of our proposed measures $I_\alpha^\beta(P:Q)$ and $L_\alpha^\beta(P)$ with respect to the parameters α and β.

6. MONOTONIC BEHAVIOR OF OUR PROPOSED MEASURES $\mathbf{I_\alpha^\beta(P:Q)}$ AND $\mathbf{L_\alpha^\beta(P)}$

In this section the monotonic behavior of our proposed Measures $I_\alpha^\beta(P:Q)$ and $L_\alpha^\beta(P)$ given in (8) and (9) respectively are studied with respect to the parameters α and β.

Let X be a discrete random variable taking finite values as $X = (x_1, x_2, x_3, x_4, x_5, x_6)$.

Suppose that the experimenter asserts the probability distribution of this random variable as:

$$Q = (q_1, q_2, q_3, q_4, q_5, q_6) = (0.4, 0.2, 0.15, 0.12, 0.08, 0.05)$$

Whereas the true probability distribution of this random variable is:

$$P = (p_1, p_2, p_3, p_4, p_5, p_6) = (0.41, 0.18, 0.15, 0.13, 0.1, 0.03)$$

Assuming $\beta = 1$, we calculate the values of new generalized inaccuracy measure $I_\alpha^\beta(P:Q)$ and new generalized average code-word length $L_\alpha^\beta(P)$ for various values of α as shown in the table below:

Table 3
For Fixed $\boldsymbol{\beta = 1}$, the Monotonic Behavior of $\mathbf{I_\alpha^\beta(P:Q)}$ and $\mathbf{L_\alpha^\beta(P)}$ with respect to $\boldsymbol{\alpha}$

$\boldsymbol{\alpha}$	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
$I_\alpha^\beta(P:Q)$	2.521	2.492	2.462	2.433	2.404	2.375	2.346	2.318	2.290
$L_\alpha^\beta(P)$	3.674	3.336	3.076	2.875	2.718	2.594	2.497	2.420	2.359

Next we plot the Table 3 and see from Figures 1 and 2 that both $I_\alpha^\beta(P:Q)$ and $L_\alpha^\beta(P)$ are monotonic decreasing with increasing values of α.

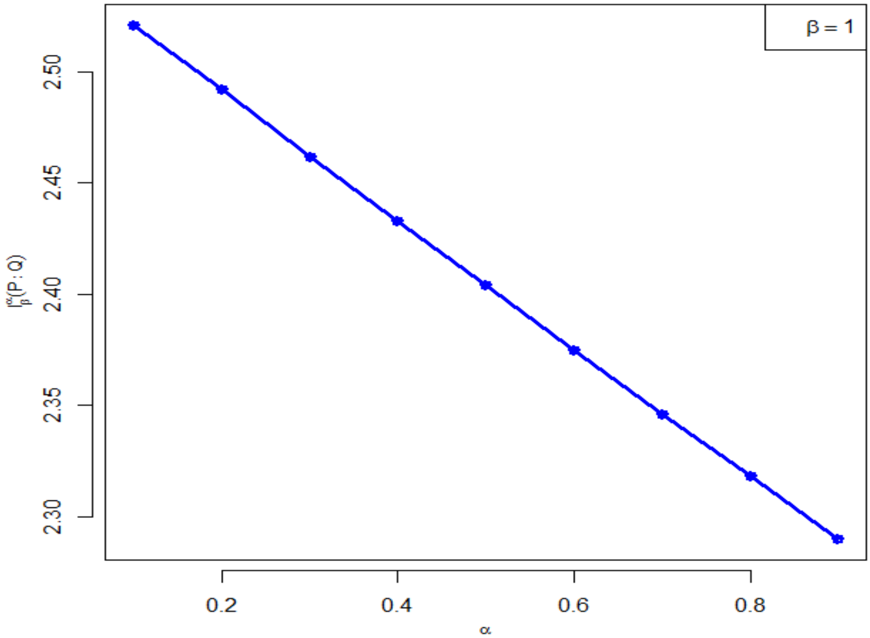

Fig. 1: Monotonic Behavior of $I^{\beta}_{\alpha}(P:Q)$ with respect to α for Fixed $\beta = 1$

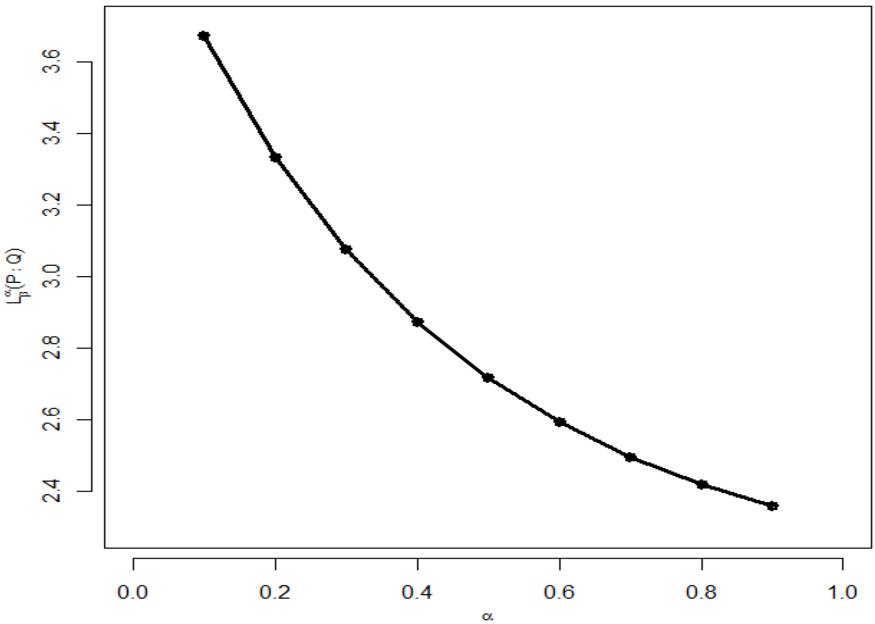

Fig. 2: Monotonic Behavior of $L^{\beta}_{\alpha}(P)$ with respect to α for Fixed $\beta = 1$

Now for fixed $\alpha = 0.5$ we find the values of new generalized inaccuracy measure $I_\alpha^\beta(P:Q)$ and new generalized average code-word length $L_\alpha^\beta(P)$ for various values of β as shown in table below:

Table 4

For Fixed $\alpha = 0.5$, the Monotonic Behavior of $I_\alpha^\beta(P:Q)$ and $L_\alpha^\beta(P)$ with respect to β

β	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$I_\alpha^\beta(P:Q)$	0.486	0.914	1.285	1.600	1.862	2.070	2.228	2.335	2.393	2.404
$L_\alpha^\beta(P)$	0.551	1.035	1.454	1.809	2.104	2.339	2.516	2.637	2.704	2.718

Next we plot the Table 4 and see from Figures 3 and 4 that for increasing values of β, both $I_\alpha^\beta(P:Q)$ and $L_\alpha^\beta(P)$ are monotonic increasing.

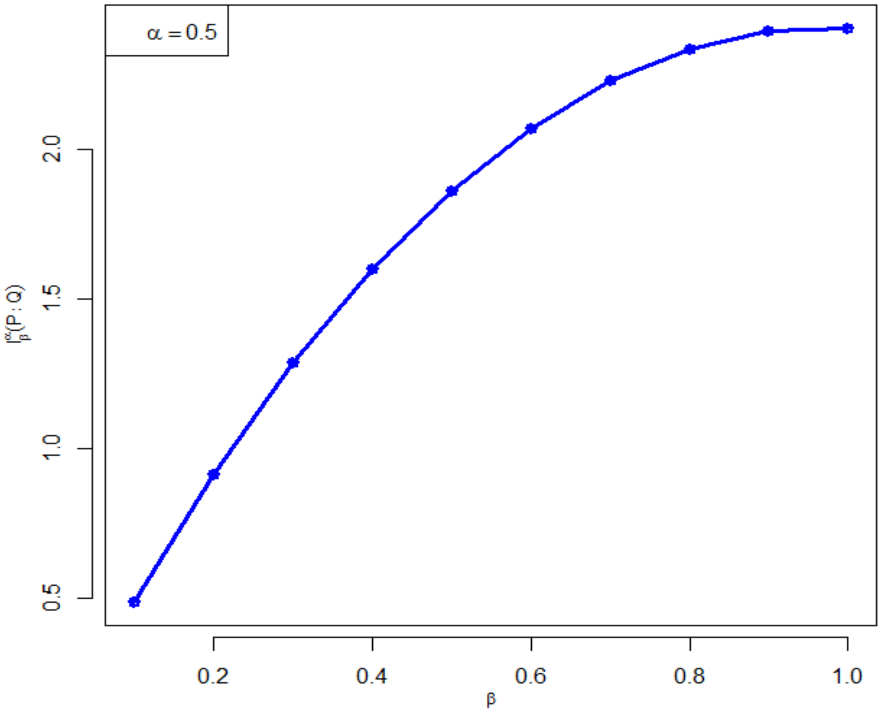

Fig. 3: Monotonic Behavior of $I_\alpha^\beta(P:Q)$ with respect to β for Fixed $\alpha = 0.5$

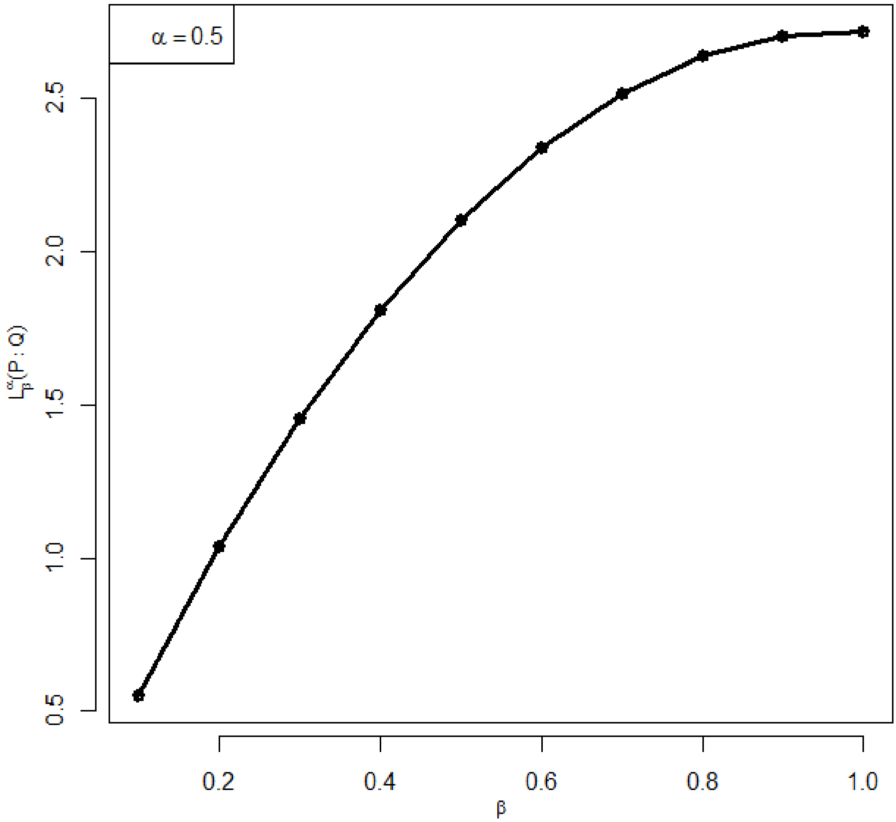

Fig. 4: Monotonic Behavior of $\mathbf{L}_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}(\mathbf{P})$ with respect to $\boldsymbol{\beta}$ for Fixed $\boldsymbol{\alpha} = \mathbf{0.5}$.

7. CONCLUSION

In this communication we present a new two parametric generalized inaccuracy measure $I_\alpha^\beta(P:Q)$ and a new two parametric generalized average code-word length $L_\alpha^\beta(P)$, and show that these measures are the generalizations of some familiar measures that already exists in the subject of information and coding theory. Also we develop the noiseless coding theorems for a discrete noiseless channel. (i.e., we found the lower and upper bounds of $L_\alpha^\beta(P)$ in terms of $I_\alpha^\beta(P:Q)$). The results obtained in this paper are empirically verified by using Huffman and Shannon-Fano coding schemes and see that our generalized mean code-word length has less mean code-word length in case of Huffman coding as compared to Shannon-Fano coding scheme. We have also studied the monotonic behavior of $I_\alpha^\beta(P:Q)$ and $L_\alpha^\beta(P)$ with respect to parameters α and β, and conclude that for fixed values of β both $I_\alpha^\beta(P:Q)$ and $L_\alpha^\beta(P)$ are monotonic decreasing with increasing values of α, and for fixed values of α both $I_\alpha^\beta(P:Q)$ and $L_\alpha^\beta(P)$ are monotonic increasing with increasing values of β.

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