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GENERALIZED MULTIVARIATE EXPONENTIAL TYPE (GMET) ESTIMATOR USING MULTI-AUXILIARY INFORMATION UNDER TWO-PHASE SAMPLING

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ABSTRACT

In this paper, a generalized multivariate exponential type estimator is proposed in two-phase sampling using multi-auxiliary variables when population information on some auxiliary variables is not available. The variance covariance matrix of the proposed estimator is derived by using simple random sampling without replacement scheme. Some special cases of proposed estimator are also deduced. Finally, empirical as well as simulation study is conducted to assess the performance of proposed estimator.

KEY WORDS AND PHRASES

Multivariate exponential estimator; multi-auxiliary variables; two-phase sampling; variance covariance matrix, partial information case.

1. INTRODUCTION

Over the years, authors of survey sampling are interested to construct estimators to improve precision of estimate using auxiliary information. Cochran (1940) proposed classical ratio method of estimation for estimating the population mean by considering the positive linear relation between study and auxiliary variable. The contribution of Cochran (1977) in ratio method of estimation under two-phase sampling attracts a lot of attention for upcoming survey statisticians. Chand (1975) proposed a chain ratio estimator based on information of two auxiliary variables for the estimation of population mean in two-phase sampling. Srivastava et al. (1990) extended the Chand (1975) estimator to the generalized chain ratio estimator for the estimation of population mean in two-phase sampling. Bahl and Tuteja (1991) was the first who suggested the exponential-type ratio and product estimators for estimating the population mean in single-phase sampling design and thereafter Singh and Vishwakarma (2007) extended this work in two-phase sampling. Further contribution is done by Samiuddin and Hanif (2007), Singh and Solanki (2014) and Sanaullah et al. (2012, 2014) etc.

Olkin (1958) was the first to suggest an extension of classical ratio estimator to the multivariate ratio estimator taking several auxiliary variables to increase precision of the estimate. John (1969) proposed two multivariate generalizations of ratio and product estimators. Tripathi and Khattree (1989) estimated means of several study variables using multi-auxiliary variables in simple random sampling. Further, Tripathi (1989) extended the results to the situation of two phase sampling. Ahmed (2003) proposed chain based general estimators using multi-auxiliary information under multiphase sampling and studied the properties of proposed estimators and suggested the optimum sample sizes using a modified cost.

Ahmad et al. (2010a) proposed generalized multivariate ratio estimators using multi auxiliary variables under partial information case for multi-phase sampling. Further contribution is by Hanif et al. (2009), Ahmad et al. (2009, 2010b, 2013 and 2014), Butt et al. (2011) and Ngesa et al. (2012).After studying the available work, it was observed that limited literature is available on generalized multivariate exponential estimators for the estimation of population mean. In this paper, our aim is to construct generalized multivariate exponential type estimator for estimating the population mean vector using multi-auxiliary variables in two-phase sampling.

In Section 2, the proposed generalized multivariate exponential type estimator is presented along with providing the expression of variance covariance matrix up to first order of approximation. Some special cases of our proposed estimator are also given in this section. In Section 3, an empirical study is conducted to check the superiority of proposed estimator. In Section 4, a simulation study has been conducted to demonstrate the performance of our proposed estimator. Discussion, conclusion and suggestions for future research work are presented in Section 5, 6 and 7 respectively.

2. GENERALIZED MULTIVARIATE EXPONENTIAL ESTIMATOR IN TWO PHASE SAMPLING

Suppose a first phase sample of size n_1 is selected with SRSWOR from a finite population of size N and $x_{(1)i}$ denote the observations of ith auxiliary variable collected at first phase sample where $i=1,...,q$. Then a sample of size n_2 is selected from first phase sample with SRSWOR and $y_{(2)j}$ denote the observations of jth study variable collected at second phase sample where $j=1,...,p$ and $x_{(2)i}$ denote the observations of ith auxiliary variable collected at second phase sample where $i=1,...,q$. Let $f_1=\left(N-n_1\right)/Nn_1$ and $f_2=\left(N-n_2\right)/Nn_2$. Suppose $\bar{e}_{y_{(2)j}}=\bar{y}_{(2)j}-\bar{Y}_j$, $\bar{e}_{x_{(1)i}}=\bar{x}_{(1)i}-\bar{X}_i$ and $\bar{e}_{x_{(2)i}}=\bar{x}_{(2)i}-\bar{X}_i$ are sampling errors. It is assumed that $E\left(\bar{e}_{y_{(2)j}}\right)=E\left(\bar{e}_{x_{(1)i}}\right)$ $=E\left(\bar{e}_{x_{(2)i}}\right)=0$ where $\bar{Y}_j$ and $\bar{X}_i$ are respective population means of study and auxiliary variables.

Smaiuddin and Hanif (2007) was the first who use the terminology of full information, partial information and no information cases while suggesting the estimators for population mean in connection with the availability of auxiliary information. They gave the term to an estimator as full information case estimator in which the information on all auxiliary variables is known, a partial information case estimator in which information on some auxiliary variables is known and the no information case in which information on auxiliary variables is not known. We suggested the following generalized multivariate exponential estimator considering partial information case. We constructed this estimator in such a way that we can deduce its special cases that are full information case as well as no information case and these special cases are discussed below. Therefore, this estimator is three in one. Suppose we have q auxiliary variables from which the information on first r auxiliary variables in known for population and for remaining $s=q-r$ is not then the multivariate estimator can be constructed as

$$\mathrm{t}_p=\left[t_j\right]_{p\times 1},$$

with

$$t_j=\bar{y}_{(2)j}\exp\left\{\sum_{i=1}^{r}\gamma_{ij}\left(\frac{\bar{x}_{(1)i}-\bar{x}_{(2)i}}{\bar{x}_{(1)i}+\bar{x}_{(2)i}}\right)+\sum_{i=1}^{r}\delta_{ij}\left(\frac{\bar{X}_i-\bar{x}_{(2)i}}{\bar{X}_i+\bar{x}_{(2)i}}\right)+\sum_{i=r+1}^{r+s=q}\gamma_{ij}\left(\frac{\bar{x}_{(1)i}-\bar{x}_{(2)i}}{\bar{x}_{(1)i}+\bar{x}_{(2)i}}\right)\right\}. \quad (1)$$

where γ_{ij} and δ_{ij} are unknown constant to be determined by minimizing the variance covariance matrix of t_p.

To proceed for variance covariance matrix for first order of approximation, we start with univariate case using multi-auxiliary variables as

$$t_j=\bar{y}_{(2)j}\exp\left\{\sum_{i=1}^{r}\gamma_{ij}\left(\frac{\bar{x}_{(1)i}-\bar{x}_{(2)i}}{\bar{x}_{(1)i}+\bar{x}_{(2)i}}\right)+\sum_{i=1}^{r}\delta_{ij}\left(\frac{\bar{X}_i-\bar{x}_{(2)i}}{\bar{X}_i+\bar{x}_{(2)i}}\right)+\sum_{i=r+1}^{r+s=q}\gamma_{ij}\left(\frac{\bar{x}_{(1)i}-\bar{x}_{(2)i}}{\bar{x}_{(1)i}+\bar{x}_{(2)i}}\right)\right\}.$$

or

$$t_j=\left(\bar{Y}_j+\bar{e}_{y_{(2)j}}\right)\exp\sum_{i=1}^{r}\gamma_{ij}\frac{\left(\bar{e}_{x_{(1)i}}-\bar{e}_{x_{(2)i}}\right)}{2\bar{X}_i+\left(\bar{e}_{x_{(1)i}}+\bar{e}_{x_{(2)i}}\right)}$$

$$\exp\sum_{i=1}^{r}\delta_{ij}\frac{-\bar{e}_{x_{(2)i}}}{2\bar{X}_i+\bar{e}_{x_{(2)i}}}\exp\sum_{i=r+1}^{r+s=q}\gamma_{ij}\frac{\left(\bar{e}_{x_{(1)i}}-\bar{e}_{x_{(2)i}}\right)}{2\bar{X}_i+\left(\bar{e}_{x_{(1)i}}+\bar{e}_{x_{(2)i}}\right)}, \quad (2)$$

or

$$t_j=\left(\bar{Y}_j+\bar{e}_{y_{(2)j}}\right)\exp\sum_{i=1}^{r}\frac{\gamma_{ij}\left(\bar{e}_{x_{(1)i}}-\bar{e}_{x_{(2)i}}\right)}{2\bar{X}_i}\left[1+\frac{\left(\bar{e}_{x_{(1)i}}+\bar{e}_{x_{(2)i}}\right)}{2\bar{X}_i}\right]^{-1}$$

$$\exp\sum_{i=1}^{r}-\frac{\delta_{ij}\bar{e}_{x_{(2)i}}}{2\bar{X}_i}\left[1+\frac{\bar{e}_{x_{(2)i}}}{2\bar{X}_i}\right]^{-1}\exp\sum_{i=r+1}^{r+s=q}\frac{\gamma_{ij}\left(\bar{e}_{x_{(1)i}}-\bar{e}_{x_{(2)i}}\right)}{2\bar{X}_i}\left[1+\frac{\left(\bar{e}_{x_{(1)i}}+\bar{e}_{x_{(2)i}}\right)}{2\bar{X}_i}\right]^{-1}. \quad (3)$$

Using Binomial expansion and ignoring second and higher order terms, we have

$$t_j = \left(\bar{Y}_j + \bar{e}_{y_{(2)j}}\right) \exp \sum_{i=1}^{r} \frac{\gamma_{ij}\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right)}{2\bar{X}_i}\left[1 - \frac{\left(\bar{e}_{x_{(1)i}} + \bar{e}_{x_{(2)i}}\right)}{2\bar{X}_i} + \ldots\right]$$

$$\exp \sum_{i=1}^{r} - \frac{\delta_{ij}\bar{e}_{x_{(2)i}}}{2\bar{X}_i}\left[1 - \frac{\bar{e}_{x_{(2)i}}}{2\bar{X}_i} + \ldots\right] \exp \sum_{i=r+1}^{r+s=q} \frac{\gamma_{ij}\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right)}{2\bar{X}_i}\left[1 - \frac{\left(\bar{e}_{x_{(1)i}} + \bar{e}_{x_{(2)i}}\right)}{2\bar{X}_i} + \ldots\right]. \quad (4)$$

Ignoring second and higher order terms, we have

$$t_j = \left(\bar{Y}_j + \bar{e}_{y_{(2)j}}\right) \exp\left\{\sum_{i=1}^{r} \frac{\gamma_{ij}\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right)}{2\bar{X}_i} - \sum_{i=1}^{r} \frac{\delta_{ij}\bar{e}_{x_{(2)i}}}{2\bar{X}_i} + \sum_{i=r+1}^{r+s=q} \frac{\gamma_{ij}\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right)}{2\bar{X}_i}\right\}. \quad (5)$$

Approximating the exponential function up to order one, we have

$$t_j = \left(\bar{Y}_j + \bar{e}_{y_{(2)j}}\right)\left\{1 + \sum_{i=1}^{r} \frac{\gamma_{ij}\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right)}{2\bar{X}_i} - \sum_{i=1}^{r} \frac{\delta_{ij}\bar{e}_{x_{(2)i}}}{2\bar{X}_i} + \sum_{i=r+1}^{r+s=q} \frac{\gamma_{ij}\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right)}{2\bar{X}_i}\right\}. \quad (6)$$

or

$$t_j = \bar{Y}_j + \bar{e}_{y_{(2)j}} + \sum_{i=1}^{r} \frac{\gamma_{ij}\bar{Y}_j}{2\bar{X}_i}\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right) - \sum_{i=1}^{r} \frac{\delta_{ij}\bar{Y}_j}{2\bar{X}_i}\bar{e}_{x_{(2)i}} + \sum_{i=r+1}^{r+s=q} \frac{\gamma_{ij}\bar{Y}_j}{2\bar{X}_i}\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right). \quad (7)$$

or

$$t_j = \bar{Y}_j + \bar{e}_{y_{(2)j}} + \sum_{i=1}^{r} a_{ij}^{*}\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right) - \sum_{i=1}^{r} b_{ij}\bar{e}_{x_{(2)i}} + \sum_{i=r+1}^{r+s=q} a_{ij}^{**}\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right). \quad (8)$$

For multivariate case of $j = 1, 2, ..., p$, we can write the above equation in matrix from as

$$\mathrm{t}_p = \bar{\mathrm{Y}} + \mathrm{d}_y + \mathrm{A}_1\mathrm{d}_{1r} - \mathrm{B}\mathrm{d}_{2r} + \mathrm{A}_2\mathrm{d}_s, \quad (9)$$

where

$$\bar{\mathrm{Y}} = [\bar{Y}_j]_{p\times 1},\ \mathrm{d}_y = [\bar{e}_{y_{(2)j}}]_{p\times 1},\ \mathrm{d}_{1r} = \left[\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right)\right]_{r\times 1},\ \mathrm{d}_{2r} = \left[\bar{e}_{x_{(2)i}}\right]_{r\times 1},$$

$$\mathrm{d}_s = \left[\left(\bar{e}_{x_{(1)i}} - \bar{e}_{x_{(2)i}}\right)\right]_{s\times 1},\ \mathrm{A}_1 = [a_{ij}^{*}]_{p\times r},\ \mathrm{B} = [b_{ij}]_{p\times r} \text{ and } \mathrm{A}_2 = [a_{ij}^{**}]_{p\times s}.$$

From (9), the variance covariance matrix of t_p can be written as

$$\Sigma_{t_p} = E\left[(\mathrm{t}_p - \bar{\mathrm{Y}})(\mathrm{t}_p - \bar{\mathrm{Y}})'\right]$$

$$= E\left[\left(\mathrm{d}_y + \mathrm{A}_1\mathrm{d}_{1r} - \mathrm{B}\mathrm{d}_{2r} + \mathrm{A}_2\mathrm{d}_s\right)\left(\mathrm{d}_y + \mathrm{A}_1\mathrm{d}_{1r} - \mathrm{B}\mathrm{d}_{2r} + \mathrm{A}_2\mathrm{d}_s\right)'\right].$$

(10)

or

$$\Sigma_{t_p} = E\Big[\mathrm{d}_y\left(\mathrm{d}'_y + \mathrm{d}'_{1r}\mathrm{A}'_1 - \mathrm{d}'_{2r}\mathrm{B}' + \mathrm{d}'_s\mathrm{A}'_2\right) + \mathrm{A}_1\mathrm{d}_{1r}\left(\mathrm{d}'_y + \mathrm{d}'_{1r}\mathrm{A}'_1 - \mathrm{d}'_{2r}\mathrm{B}' + \mathrm{d}'_s\mathrm{A}'_2\right)$$
$$-\mathrm{B}\mathrm{d}_{2r}\left(\mathrm{d}'_y + \mathrm{d}'_{1r}\mathrm{A}'_1 - \mathrm{d}'_{2r}\mathrm{B}' + \mathrm{d}'_s\mathrm{A}'_2\right) + \mathrm{A}_2\mathrm{d}_s\left(\mathrm{d}'_y + \mathrm{d}'_{1r}\mathrm{A}'_1 - \mathrm{d}'_{2r}\mathrm{B}' + \mathrm{d}'_s\mathrm{A}'_2\right)\Big]. \quad (11)$$

We can have

$$E\left(\mathrm{d}_y\mathrm{d}'_y\right) = f_2\Sigma_{y_p},\ E\left(\mathrm{d}_y\mathrm{d}'_{1r}\right) = \left(f_1 - f_2\right)\Sigma_{y_px_r},\ E\left(\mathrm{d}_y\mathrm{d}'_{2r}\right) = f_2\Sigma_{y_px_r},$$
$$E\left(\mathrm{d}_y\mathrm{d}'_s\right) = \left(f_1 - f_2\right)\Sigma_{y_px_s},\ E\left(\mathrm{d}_{1r}\mathrm{d}'_{1r}\right) = \left(f_2 - f_1\right)\Sigma_{x_r},$$
$$E\left(\mathrm{d}_{1r}\mathrm{d}'_{2r}\right) = \left(f_1 - f_2\right)\Sigma_{x_r},\ E\left(\mathrm{d}_{1r}\mathrm{d}'_s\right) = \left(f_2 - f_1\right)\Sigma_{x_rx_s},\ E\left(\mathrm{d}_{2r}\mathrm{d}'_{2r}\right) = f_2\Sigma_{x_r},$$
$$E\left(\mathrm{d}_{2r}\mathrm{d}'_s\right) = \left(f_1 - f_2\right)\Sigma_{x_rx_s} \text{ and } E\left(\mathrm{d}_s\mathrm{d}'_s\right) = \left(f_2 - f_1\right)\Sigma_{x_s}.$$

Then

$$\Sigma_{t_p} = f_2\Sigma_{y_p} + \left(f_1 - f_2\right)\Sigma_{y_px_r}\mathrm{A}'_1 - f_2\Sigma_{y_px_r}\mathrm{B}' + \left(f_1 - f_2\right)\Sigma_{y_px_s}\mathrm{A}'_2$$
$$+\left(f_1 - f_2\right)\mathrm{A}_1\Sigma'_{y_px_r} + \left(f_2 - f_1\right)\mathrm{A}_1\Sigma_{x_r}\mathrm{A}'_1 - \left(f_1 - f_2\right)\mathrm{A}_1\Sigma_{x_r}\mathrm{B}'$$
$$+(f_2 - f_1)\mathrm{A}_1\Sigma_{x_rx_s}\mathrm{A}'_2 - f_2\mathrm{B}\Sigma'_{y_px_r} - \left(f_1 - f_2\right)\mathrm{B}\Sigma'_{x_r}\mathrm{A}'_1 + f_2\mathrm{B}\Sigma_{x_r}\mathrm{B}'$$
$$-\left(f_1 - f_2\right)\mathrm{B}\Sigma_{x_rx_s}\mathrm{A}'_2 + \left(f_1 - f_2\right)\mathrm{A}_2\Sigma'_{y_px_s} + \left(f_2 - f_1\right)\mathrm{A}_2\Sigma'_{x_rx_s}\mathrm{A}'_1$$
$$-\left(f_1 - f_2\right)\mathrm{A}_2\Sigma'_{x_rx_s}\mathrm{B}' + \left(f_2 - f_1\right)\mathrm{A}_2\Sigma_{x_s}\mathrm{A}'_2 \quad (12)$$

For optimum values of unknown matrices A_1, B and A_2, differentiating (11) with respect to these matrices one by one and equating to zero and then solving the resulting normal equations we can have

$$\left.\begin{aligned}
&\mathrm{A}_1 = \left(\Sigma_{y_px_r}\Sigma_{x_r}^{-1}\Sigma_{x_rx_s} - \Sigma_{y_px_s}\right)\mathrm{G}_s^{-1}\Sigma'_{x_rx_s}\Sigma_{x_r}^{-1},\ \mathrm{B} = \Sigma_{y_px_r}\Sigma_{x_r}^{-1}\text{and}\\
&\mathrm{A}_2 = \left(\Sigma_{y_px_s} - \Sigma_{y_px_r}\Sigma_{x_r}^{-1}\Sigma_{x_rx_s}\right)\mathrm{G}_s^{-1},\ \text{where } \mathrm{G}_s = \left(\Sigma_{x_s} - \Sigma'_{x_rx_s}\Sigma_{x_r}^{-1}\Sigma_{x_rx_s}\right)
\end{aligned}\right\} \quad (13)$$

Using normal equations used to get optimum values of unknown matrices A_1, B and A_2, from (11), we can write

$$\Sigma_{t_p} = E\left[\mathrm{d}_y\left(\mathrm{d}'_y + \mathrm{d}'_{1r}\mathrm{A}'_1 - \mathrm{d}'_{2r}\mathrm{B}' + \mathrm{d}'_s\mathrm{A}'_2\right)\right]$$
$$= f_2\Sigma_{y_p} + \left(f_1 - f_2\right)\Sigma_{y_px_r}\mathrm{A}'_1 - f_2\Sigma_{y_px_r}\mathrm{B}' + \left(f_1 - f_2\right)\Sigma_{y_px_s}\mathrm{A}'_2 \quad (14)$$

Now substituting the optimum values of unknown matrices A_1, B and A_2, we get after simplification

$$\Sigma_{t_p} = f_2\left(\Sigma_{y_p} - \Sigma_{y_p x_r}\Sigma_{x_r}^{-1}\Sigma'_{y_p x_r}\right) - \left(f_2 - f_1\right)$$
$$\left(\Sigma_{y_p x_s} - \Sigma_{y_p x_r}\Sigma_{x_r}^{-1}\Sigma_{x_r x_s}\right) G_s^{-1}\left(\Sigma'_{y_p x_s} - \Sigma'_{x_r x_s}\Sigma_{x_r}^{-1}\Sigma'_{y_p x_r}\right). \quad (15)$$

Singh and Majhi (2014) proposed chain ratio estimator based on information of two auxiliary variables as,

$$t = \bar{y}_2 \exp\left(\frac{\bar{x}_{1(1)} - \bar{x}_{1(2)}}{\bar{x}_{1(1)} + \bar{x}_{1(2)}}\right)\left(\frac{\bar{X}_2}{\bar{x}_{2(1)}}\right) \quad (16)$$

with mean square error

$$MSE\left(t\right) = \bar{Y}^2\left[f_1 C_y^2 + \frac{f_3}{4}\left(C_{x_1}^2 - 4\rho_{yx_1} C_y C_{x_1}\right) + f_2\left(C_{x_2}^2 - 2\rho_{yx_2} C_y C_{x_2}\right)\right]. \quad (17)$$

We are dealing with multivariate estimators, then (16) can be modified to multivariate case for p study variables. The multivariate version of (16) can be written as

$$\mathrm{t}_g = \left[t_j\right]_{1\times p},$$

where

$$t_j = \bar{y}_{j(2)} \exp\sum_{i=1}^{r}\alpha_{ij}\left(\frac{\bar{x}_{i(1)} - \bar{x}_{i(2)}}{\bar{x}_{i(1)} + \bar{x}_{i(2)}}\right)\prod_{i=r+1}^{r+s=q}\left(\frac{\bar{X}_i}{\bar{x}_{i(1)}}\right)^{\omega_{ij}}, \text{ for } j = 1, \ldots, p. \quad (18)$$

The matrix of optimum values of $\alpha_{ij} = 2\bar{X}_i \tau_{ij} / \bar{Y}_j$ where $\tau_{ij(r\times p)}$ is the ij^{th} element of $\mathrm{A} = \Sigma_{y_p x_r}\Sigma_{x_r}^{-1}$ and of $\omega_{ij} = \bar{X}_i \xi_{ij} / \bar{Y}_j$ where ξ_{ij} is ij^{th} element of $\mathrm{C} = \Sigma_{y_p x_s}\Sigma_{x_s}^{-1}$. Notice that the minimum MSE of (18) for univariate case using two auxiliary variables will be identical to the MSE of (16) given in (17). The expression of minimum variance covariance matrix up to first order of approximation can be written as

$$\Sigma_g = f_2\left(\Sigma_{y_p} - \Sigma_{y_p x_r}\Sigma_{x_r}^{-1}\Sigma'_{y_p x_r}\right) + f_1\left(\Sigma_{y_p x_r}\Sigma_{x_r}^{-1}\Sigma'_{y_p x_r} - \Sigma_{y_p x_s}\Sigma_{x_s}^{-1}\Sigma'_{y_p x_s}\right). \quad (19)$$

The proposed estimator t_p can be compared with modified estimator t_g. For comparison empirical and simulation study is given in the following sections.

2.1. Special Cases

As the proposed estimator given in (1) is general in nature and many special members of this general class can be deduced. Some important multivariate members using multi auxiliary variables are given in the following Table 1. From this list further univariate members can be deduced for different number of auxiliary variables. The optimum values and MSE/variance covariance matrices of any member can easily be deduced from (13) and (15).

S#	Expression of t_j	Optimum Matrices	Variance Covariance Matrices
t_{p_1}	$\bar{y}_{(2)j}\exp\sum_{i=1}^{q}\gamma_{ij}\left(\frac{\bar{x}_{(1)i}-\bar{x}_{(2)i}}{\bar{x}_{(1)i}+\bar{x}_{(2)i}}\right)$ (MEE for NIC)	$\Sigma_{y_p x_q}\Sigma_{x_q}^{-1}$	$\left(f_2\Sigma_{y_p}-\left(f_2-f_1\right)\Sigma_{y_p x_q}\Sigma_{x_q}^{-1}\Sigma'_{y_p x_q}\right)$
t_{p_2}	$\bar{y}_{(2)j}\exp\sum_{i=1}^{q}\delta_{ij}\left(\frac{\bar{X}_i-\bar{x}_{(2)i}}{\bar{X}_i+\bar{x}_{(2)i}}\right)$ (MEE for FIC)	$\Sigma_{y_p x_q}\Sigma_{x_q}^{-1}$	$f_2\left(\Sigma_{y_p}-\Sigma_{y_p x_q}\Sigma_{x_q}^{-1}\Sigma'_{y_p x_q}\right)$
t_{p_3}	$\bar{y}_{(2)j}\exp\left\{\sum_{i=1}^{r}\gamma_{ij}\left(\frac{\bar{x}_{(1)i}-\bar{x}_{(2)i}}{\bar{x}_{(1)i}+\bar{x}_{(2)i}}\right)+\sum_{i=r+1}^{r+s=q}\delta_{ij}\left(\frac{\bar{X}_i-\bar{x}_{(2)i}}{\bar{X}_i+\bar{x}_{(2)i}}\right)\right\}$ (MEE for PIC)	$\Sigma_{y_p x_r}\Sigma_{x_r}^{-1}$, $\Sigma_{y_p x_s}\Sigma_{x_s}^{-1}$	$f_2\left(\Sigma_{y_p}-\Sigma_{y_p x_r}\Sigma_{x_r}^{-1}\Sigma'_{y_p x_r}\right)-f_1\left(\Sigma_{y_p x_r}\Sigma_{x_r}^{-1}\Sigma'_{y_p x_r}-\Sigma_{y_p x_s}\Sigma_{x_s}^{-1}\Sigma'_{y_p x_s}\right)$

MEE*: Multivariate Exponential Estimator;* NIC*: No Information Case;*
FIC*: Full Information Case;* PIC*: Partial Information Case*

It is obvious that the general class will always be efficient than its members, however, by reducing the number of exponential function based on auxiliary variables, the bias can possibly be reduced. So the special cases are not included in the empirical and simulation study.

3. EMPIRICAL STUDY

For empirical study, we use the data of 1998 census reports of five districts of Punjab, a province of Pakistan, Jhang, Faisalabad, Gujrat, Kasur and Sialkot. The detail of populations and variables description is given in Table A1 and A2 respectively of Appendix-A. We consider three variables of interests denoted by Y's and five auxiliary variables denoted by X's for computing the determinants of variance covariance matrices of proposed multivariate exponential estimator and modified Singh and Majhi (2014) estimator. The variance covariance matrices of study and auxiliary variables used to compute the variance covariance matrices are given in Table A3 for all five populations. The variance covariance matrices of t_p and t_g along with percent relative efficiency (PRE) based on their determinants and trace are given in Table A4.

4. SIMULATION STUDY

In this section, a simulation study based on artificial data of size 3000 is conducted to assess the performance of proposed estimator given in (1) over modified Singh and Majhi (2014) estimator given in (20). Five auxiliary variables are generated using normal and lognormal distribution so that the performance can be assessed for both symmetric and skewed data. Further three response variables are generated using linear regression model with different regression coefficients. For symmetric data the auxiliary variables are generated by; $x_1 \sim N(14, 2.5)$; $x_2 \sim N(15, 2)$; $x_3 \sim N(18, 3)$; $x_4 \sim N(17, 4)$ $x_5 \sim N(16, 3)$ and for skewed data; $x_1 \sim LogN(2,1)$; $x_2 \sim LogN(2, 2)$; $x_3 \sim LogN(2,1.75)$; $x_4 \sim LogN(1, 2)$; $x_5 \sim LogN(1, 2.5)$.

The three response variables are generated by model $y_{ij} = \sum_{i}^{5} \beta_i x_i + \varepsilon_{ij}$. The regression coefficients for y_{1j} are all 1, for y_{2j} are $(0.1, 0.2, 0.3, 0.4, 0.5)$ and y_{3j} are $(1.2, 1.3, 1.4, 1.5, 1.7)$.

The sizes of first phase and second phase samples are assumed to be respectively $n_1 = 500$ and $n_2 = 100$. The number of replications for simulation study is $S = 3000$. We generated five auxiliary variables and assumed $r = 3$ for first three variables (x_1, x_2, x_3) and $s = 2$ for last two variables (x_4, x_5). The sample values of these variables are required to compute the estimators t_p and t_g.

We computed the vector of proposed estimators by taking the log of (1) as

$$\log(\mathrm{t}_p) = \log(\bar{\mathrm{y}}_{(2)}) + \mathrm{A}_1 \mathrm{x}_{r_1} + \mathrm{B}\mathrm{x}_{r_2} + \mathrm{A}_2 \mathrm{x}_s, \quad (20)$$

where A_1, B and A_2 are adjusted versions of matrices of optimum values given in (12), the need of adjustment can easily be seen from (7) and

$$\mathrm{x}_{r_1} = (x_{r_{11}}, x_{r_{12}}, x_{r_{13}})^T, \text{ where } x_{r_{1k}} = \left(\bar{x}_{(1)k} - \bar{x}_{(2)k}\right) \Big/ \left(\bar{x}_{(1)k} + \bar{x}_{(2)k}\right),$$
$$\mathrm{x}_{r_2} = (x_{r_{21}}, x_{r_{22}}, x_{r_{23}})^T, \text{ where } x_{r_{2k}} = \left(\bar{X}_k - \bar{x}_{(2)k}\right) \Big/ \left(\bar{X}_k + \bar{x}_{(2)k}\right) \text{ and}$$
$$\mathrm{x}_s = (x_{s_4}, x_{s_5})^T, \text{ where } x_{s_k} = \left(\bar{x}_{(1)k} - \bar{x}_{(2)k}\right) \Big/ \left(\bar{x}_{(1)k} + \bar{x}_{(2)k}\right). \quad (21)$$

Similarly, the modified estimator given (18) can be computed by

$$\log(\mathrm{t}_g) = \log(\bar{\mathrm{y}}_{(2)}) + \mathrm{A}\mathrm{x}_{r_1} + \mathrm{C}\log(\mathrm{x}_{s_1}), \quad (22)$$

where $\mathrm{x}_{r_1} = (x_{r_{11}}, x_{r_{12}}, x_{r_{13}})^T$; for $x_{r_{1k}} = \left(\bar{x}_{(1)k} - \bar{x}_{(2)k}\right) \Big/ \left(\bar{x}_{(1)k} + \bar{x}_{(2)k}\right)$, $\mathrm{x}_{s_1} = (x_{s_{14}}, x_{s_{15}})^T$, for $x_{s_{1k}} = \left(\bar{X}_k - \bar{x}_{(1)k}\right) \Big/ \left(\bar{X}_k + \bar{x}_{(1)k}\right)$ and A and C are also adjusted version of the ones that are given below (18).The values of optimum matrices are computed from population as well as from first-phase sample because practically population parameters are usually not available. These matrices are used to compute these matrices of optimum values. Finally, we need to take anti-log of these vectors of estimators given in (20) and (22) to obtain the correct vector of estimators.

After computing the vector t_p and t_g, with three elements each, we obtained the mean and variance-covariance matrix by,

$$\bar{y} = \left[\bar{t}_k\right]_{(1\times 3)} \text{ and } \Sigma_{\bar{y}} = \left[\text{cov}(t_k, t_l)\right]_{3\times 3}$$

where

$$\bar{t}_k = \frac{1}{S}\sum_{i=1}^{S} t_{ki} \text{ and } \text{cov}(t_k, t_l) = \frac{1}{S}\sum_{i=1}^{S}(t_{ki} - \bar{t}_k)\left(t_{li} - \bar{t}_l\right). \tag{23}$$

The performance of univariate estimator using multi-auxiliary variable can be observed from the diagonal of variance covariance matrix of a multivariate estimator. The variance covariance matrices of study and auxiliary variables that are necessary to find the matrices of optimum values based on population symmetric and skewed data are given in Table B1 and B2 respectively of Appendix B. These matrices are also computed from first phase sample for each replication and for one of the replication based on symmetric and skewed data are given in Table B3 and B4 respectively. Table B5 and B6 contains the vector of parameters and bias of both estimators t_p and t_g for symmetric and skewed data using both cases of finding matrices of optimum values. The Table B7 contains the matrices of optimum values computed from population and first phase sample data that are used to compute t_p and t_g for symmetric and skewed data. Finally, the variance covariance matrices based on 3000 simulation of t_p and t_g along with their PRE based on their determinants and trace are given in Table B8.

5. DISCUSSION

Considering the results of empirical study, from Table A4, it can be seen that proposed multivariate estimator is efficient than modified Singh and Majhi (2014) estimator. From the results of simulation study the proposed multivariate estimator is efficient than modified Singh and Majhi (2014) estimator for symmetric as well as for skewed data. The performance is even better for skewed data. If we want to compare the performance of univariate estimators for each study variable separately using multi-auxiliary variables, then it can be assessed from the diagonals of variance covariance matrices given in Table A4 of empirical study and from Table B8 of simulation study. From both tables, each univariate estimator deduced from proposed multivariate estimator is efficient than its counterpart of modified Singh and Majhi (2014) estimator.

Comparing the performance of proposed and modified Singh and Majhi (2014) estimator based on bias computed in simulation study given in Table B5 and B6, the bias is less for proposed estimator for symmetric data for fist study variable whereas the performance of modified Singh and Majhi (2014) is better for other two study variables for symmetric data as well as for skewed data when optimum values are computed from population data. But when optimum values are computed from first phase sample data that is actually possible practically, the proposed estimator has less bias than modified Singh and Majhi (2014) estimator for both symmetric and skewed for all study variables.

6. CONCLUSIONS

In this paper, a generalized multivariate exponential estimator is developed for two-phase sampling using multi-auxiliary variables. As the proposed class is general in nature, some special cases are deduced along with their expressions of mean square errors. The proposed estimator has potential of utilizing information of any number of auxiliary variables and either this auxiliary information on all auxiliary variables is available for population, partially available or even not available. Because the estimator has such special cases those can handle all three situations. On the basis of empirical and simulation studies the performance of proposed estimator is far better than its competitor in term of bias and mean square errors for both multivariate and univariate cases. The performance is even better for highly skewed data that is usually the type of business and economic data. Hence the estimator has a large scope of application.

This paper also fills the gap in the literature for practitioners those want to estimate population mean vector of correlated response variables either with symmetric or skewed nature but the set of study variables needs to depend on the same set of auxiliary variables as usually in the multivariate regression case. For independent response variables the univariate version can be used for real life applications. Further the estimated vector along with estimated variance covariance matrix can be used for hypothesis testing and confidence region for the vector of population mean using large sample properties.

7. FUTURE RESEARCH WORK

Currently we estimated the vector of mean using same set of auxiliary variables for each study variables. But practically it is possible that each study variable can depend on different set of auxiliary variables. Then how multivariate estimators can be constructed and how the variance covariance matrix can be derived, etc.

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APPENDIX-A

Table A1
Detail of Populations

S#	Source of Populations
1	Population census report of Jhang district (1998), Pakistan
2	Population census report of Faisalabad district (1998), Pakistan.
3	Population census report of Gujrat district (1998), Pakistan.
4	Population census report of Kasur (1998) Pakistan
5	Population census report of Sialkot district (1998), Pakistan.

Table A2
Description of Variables (Each variable is taken from Rural Locality)

Y_1	Literacy ratio	X_2	Population of primary but below matric
Y_2	Population of currently married	X_3	Population of matric and above
Y_3	Total household	X_4	Population of 18 years old and above
X_1	Population of both sexes	X_5	Population of women 15-49 years old

Table A3
Variance Covariance Matrices for Different Real Population

	Population-1: Jhang							
	Y_1	Y_2	Y_3	X_1	X_2	X_3	X_4	X_5
Y_1	64.34662	746.4499	603.8044	5898.349	1678.78	749.7782	3640.971	1098.924
Y_2	746.4499	262050.3	172473.6	1232946	212534	57534.33	606356.1	231539.5
Y_3	603.8044	172473.6	211454.9	1226646	153144.4	58724.61	629385.4	239175.8
X_1	5898.349	1232946	1226646	31656918	1064708	404717.2	4378867	1644823
X_2	1678.78	212534	153144.4	1064708	207081.3	63983.5	529310.4	209420.7
X_3	749.7782	57534.33	58724.61	404717.2	63983.5	29126.96	246602.6	84299.8
X_4	3640.971	606356.1	629385.4	4378867	529310.4	246602.6	6011764	848465
X_5	1098.924	231539.5	239175.8	1644823	209420.7	84299.8	848465	1133120
	Population-1: Faisalabad							
Y_1	27208.63	9137.748	10148.32	64829.89	6926.907	2859.612	40291.92	10372.21
Y_2	9137.748	621542.7	547234.7	3963615	792130.4	303104	2481865	823818
Y_3	10148.32	547234.7	542278	3864714	759920.4	289118.5	2408090	896759.6
X_1	64829.89	3963615	3864714	29441849	5836183	2230312	17507724	6588651
X_2	6926.907	792130.4	759920.4	5836183	2815407	624113.7	3788277	1390088
X_3	2859.612	303104	289118.5	2230312	624113.7	276331.9	1795351	533963.3
X_4	40291.92	2481865	2408090	17507724	3788277	1795351	39560446	4060606
X_5	10372.21	823818	896759.6	6588651	1390088	533963.3	4060606	2196837

Table A3
Variance Covariance Matrices for Different Real Population (Cont…)

	Population-1: Gujrat							
	Y_1	Y_2	Y_3	X_1	X_2	X_3	X_4	X_5
Y_1	69.95666	244.6617	252.589	2687.361	2625.045	1733.004	4713.955	715.5435
Y_2	244.6617	284132.9	283000.5	1860371	471505.4	155355.8	2126371	440706.8
Y_3	252.589	283000.5	288622.8	1854593	471285.4	153488.2	2130101	439717.5
X_1	2687.361	1860371	1854593	12300162	3146925	1065225	14526885	2900732
X_2	2625.045	471505.4	471285.4	3146925	884510	320289.3	3824186	747634.9
X_3	1733.004	155355.8	153488.2	1065225	320289.3	145685.6	1303769	252585.4
X_4	4713.955	2126371	2130101	14526885	3824186	1303769	66254461	3410751
X_5	715.5435	440706.8	439717.5	2900732	747634.9	252585.4	3410751	688916.1
	Population-1: Kasur							
Y_1	567.5286	5390.125	5506.205	39254.71	6657.904	2997.443	19502.98	8079.445
Y_2	5390.125	589265.4	582861	4223569	637439.4	241415.5	2065147	831935.6
Y_3	5506.205	582861	589511.4	4194681	633004	241274.2	2062780	824664.6
X_1	39254.71	4223569	4194681	30419800	4618159	1755314	14886161	5993412
X_2	6657.904	637439.4	633004	4618159	1200530	313024	2275052	911440.9
X_3	2997.443	241415.5	241274.2	1755314	313024	128082.4	872333.8	348998.6
X_4	19502.98	2065147	2062780	14886161	2275052	872333.8	7394115	2939038
X_5	8079.445	831935.6	824664.6	5993412	911440.9	348998.6	2939038	1837755
	Population-1: Sialkot							
Y_1	58.37917	1694.029	1555.411	11831.97	3818.858	2124.136	6361.846	2750.949
Y_2	1694.029	469251.1	440380.8	3274827	789524.6	384698.5	1678884	740882.5
Y_3	1555.411	440380.8	415877.7	3076130	741371.7	362801.6	1578600	696488.8
X_1	11831.97	3274827	3076130	22917782	5519189	2688860	11744903	5176758
X_2	3818.858	789524.6	741371.7	5519189	1375257	678639.3	2842578	1253310
X_3	2124.136	384698.5	362801.6	2688860	678639.3	363872.7	1394183	616035.4
X_4	6361.846	1678884	1578600	11744903	2842578	1394183	6059402	2657890
X_5	2750.949	740882.5	696488.8	5176758	1253310	616035.4	2657890	1325529

Table A4
Variance Covariance Matrices and their Determinants for Proposed Estimator t_p and Modified Version of Singh and Majhi (2014) t_g

Popu-lations		Σ_p			Σ_g			$\left\|\Sigma_p\right\|$	$\left\|\Sigma_g\right\|$	PRE	$tr\left(\Sigma_p\right)$	$tr\left(\Sigma_g\right)$	PRE
Jhang	$N=368$	**1.715904**	-26.491	-30.542	**1.945207**	-21.6797	-26.836	3550720	12358573	**384%**	4293.033	6215.816	**145%**
	$n_1=92$	-26.491	**1272.237**	731.6099	-21.6797	**2613.8**	1442.951						
	$n_2=23$	-30.542	731.6099	**3019.08**	-26.836	1442.951	**3600.071**						
Faisalabad	$N=283$	**1431.423**	9.915648	87.95819	**1433.584**	68.75117	123.9724	7628212455	16262814635	**213%**	7579.807	11328.28	**149%**
	$n_1=71$	9.915648	**4391.521**	1537.597	68.75117	**6735.288**	3144.86						
	$n_2=18$	87.95819	1537.597	**1756.863**	123.9724	3144.86	**3159.411**						
Gujrat	$N=204$	**1.279806**	-0.16334	0.42261	**2.049714**	-3.77887	-3.22324	56933	146199	**257%**	600.8388	731.8696	**122%**
	$n_1=51$	-0.16334	**95.1932**	59.12798	-3.77887	**156.4312**	122.6345						
	$n_2=13$	0.42261	59.12798	**504.3658**	-3.22324	122.6345	**573.3887**						
Kasur	$N=181$	**41.09988**	0.216741	9.808454	**41.44006**	-0.83586	8.400774	7570051	14394652	**190%**	1106.794	1386.626	**125%**
	$n_1=45$	0.216741	**219.9524**	36.38985	-0.83586	**379.8907**	136.516						
	$n_2=11$	9.808454	36.38985	**845.742**	8.400774	136.516	**965.2957**						
Sialkot	$N=269$	**1.762236**	-0.77912	-2.67069	**1.987819**	-1.40068	-3.31411	15821	25971	**164%**	230.2871	282.7104	**123%**
	$n_1=67$	-0.77912	**70.87266**	44.25511	-1.40068	**101.2038**	68.45794						
	$n_2=17$	-2.67069	44.25511	**157.6522**	-3.31411	68.45794	**179.5188**						

APPENDIX-B

Table B1
Variance Covariance Matrices for Population Symmetric Data

	Y_1	Y_2	Y_3	X_1	X_2	X_3	X_4	X_5
Y_1	45.35775	15.21981	64.73709	5.886159	3.871583	8.994559	16.20042	9.158527
Y_2	15.21981	6.912563	23.10148	0.534825	0.71647	2.679124	6.633	4.550391
Y_3	64.73709	23.10148	96.5379	7.003653	4.938654	12.4321	24.46592	15.51921
X_1	5.886159	0.534825	7.003653	6.107956	0.003288	-0.06796	-0.19847	0.058006
X_2	3.871583	0.71647	4.938654	0.003288	3.99322	0.1426	-0.22676	-0.06407
X_3	8.994559	2.679124	12.4321	-0.06796	0.1426	8.925396	0.077209	-0.17382
X_4	16.20042	6.633	24.46592	-0.19847	-0.22676	0.077209	16.42683	0.05982
X_5	9.158527	4.550391	15.51921	0.058006	-0.06407	-0.17382	0.05982	9.209475

Table B2
Variance Covariance Matrices for Population Skewed Data

	Y_1	Y_2	Y_3	X_1	X_2	X_3	X_4	X_5
Y_1	712526.1	296809.2	1106862	-50.2508	56832.55	31197.2	361659.6	262866.8
Y_2	296809.2	128646.7	468302	-89.9575	11494.91	9693.212	143987.4	131713.4
Y_3	1106862	468302	1730562	-159.812	73951.29	44022.57	541725.3	447289.8
X_1	-50.2508	-89.9575	-159.812	226.0211	-65.1695	4.130317	-67.0677	-147.897
X_2	56832.55	11494.91	73951.29	-65.1695	55698.46	552.394	1294.646	-641.233
X_3	31197.2	9693.212	44022.57	4.130317	552.394	26943.02	3460.358	229.1756
X_4	361659.6	143987.4	541725.3	-67.0677	1294.646	3460.358	357878.4	-904.764
X_5	262866.8	131713.4	447289.8	-147.897	-641.233	229.1756	-904.764	264311.7

Table B3
Variance Covariance Matrices for Symmetric Data of First Phase Sample of Single Replication

	Y_1	Y_2	Y_3	X_1	X_2	X_3	X_4	X_5
Y_1	48.58365	16.89561	69.27827	5.244022	4.602702	9.187083	17.63002	10.64294
Y_2	16.89561	7.563587	25.40424	0.385033	0.847428	3.428716	6.704852	5.383115
Y_3	69.27827	25.40424	103.1441	5.975325	5.944287	13.41969	26.07778	17.58903
X_1	5.244022	0.385033	5.975325	5.691019	0.32899	-0.20434	-0.26643	-0.32198
X_2	4.602702	0.847428	5.944287	0.32899	4.135329	-0.72001	0.430378	0.344678
X_3	9.187083	3.428716	13.41969	-0.20434	-0.72001	8.92293	0.58677	0.730399
X_4	17.63002	6.704852	26.07778	-0.26643	0.430378	0.58677	16.09141	0.563694
X_5	10.64294	5.383115	17.58903	-0.32198	0.344678	0.730399	0.563694	9.304631

Table B4
Variance Covariance Matrices for Skewed Data of First Phase Sample of Single Replication

	Y_1	Y_2	Y_3	X_1	X_2	X_3	X_4	X_5
Y_1	1702628	356541.6	2233560	1058.834	1637684	8669.284	13449.92	41706.79
Y_2	356541.6	78721.33	473079.8	277.6068	328743.4	2389.812	3415.399	21704.05
Y_3	2233560	473079.8	2937155	1469.676	2129823	11960.19	18148.48	72084.98
X_1	1058.834	277.6068	1469.676	316.9161	431.0041	-61.6477	80.55651	290.9828
X_2	1637684	328743.4	2129823	431.0041	1628094	2995.38	9474.038	-3364.32
X_3	8669.284	2389.812	11960.19	-61.6477	2995.38	5396.785	-75.3832	410.5995
X_4	13449.92	3415.399	18148.48	80.55651	9474.038	-75.3832	4519.123	-549.029
X_5	41706.79	21704.05	72084.98	290.9828	-3364.32	410.5995	-549.029	44920.44

Table B5
Vector of Parameters and Bias (Optimum Values from Population Data)

	Parameters		**Bias(t_p)**		**Bias(t_g)**		$Bias(t_g) \times 100 / Bias(t_p)$	
	Sym Data	**Shew Data**	**Sym Data**	**Shew Data**	**Sym Data**	**Shew Data**	**Sym Data**	**Shew Data**
$\bar{Y}_1$	80.14688	166.6264	0.021686	7.586573	0.023542	3.551182	108%	46%
$\bar{Y}_2$	24.64037	52.67132	0.001489	3.065257	0.000308	1.479961	20%	48%
$\bar{Y}_3$	114.4182	240.4233	0.05641	35.86344	0.054581	26.26223	96%	73%

Table B6
Vector of Parameters and Bias (Optimum Values from First Phase Sample Data)

	Parameters		**Bias(t_p)**		**Bias(t_g)**		$Bias(t_g) \times 100 / Bias(t_p)$	
	Sym Data	**Shew Data**	**Sym Data**	**Shew Data**	**Sym Data**	**Shew Data**	**Sym Data**	**Shew Data**
$\bar{Y}_1$	80.16346	226.5144	0.015778	14.52294	0.022781	34.57441	144%	238%
$\bar{Y}_2$	24.70791	74.74683	0.002354	4.28135	0.004425	10.82014	188%	253%
$\bar{Y}_3$	114.5124	331.8108	0.056101	54.48384	0.068919	72.54067	123%	133%

Table B7
Matrices of Optimum Values of Unknown Constants $(A_1, B \& A_2)$ and $(A \& C)$

A_1			B			A_2		A			C	
Population Symmetric data												
-0.01382	-0.07408	0.001829	0.362085	1.216901	0.244298	0.424426	0.398859	0.362085	1.216901	0.244298	0.215209	0.204654
-0.00659	-0.03444	0.000666	0.041984	0.293905	0.077315	0.548079	0.664154	0.041984	0.293905	0.077315	0.275017	0.334996
-0.02798	-0.14818	0.003287	0.569516	2.043672	0.436598	0.446618	0.473884	0.569516	2.043672	0.436598	0.225832	0.241617
First Phase Sample Symmetric data												
-0.06553	-0.23382	-0.01185	0.412245	1.414538	0.254794	0.428332	0.390287	0.412245	1.414538	0.254794	0.237632	0.212892
-0.02951	-0.10996	-0.0058	0.068573	0.318665	0.087348	0.564687	0.621326	0.068573	0.318665	0.087348	0.293814	0.322329
-0.13164	-0.48639	-0.02548	0.662866	2.390535	0.465015	0.454792	0.482675	0.662866	2.390535	0.465015	0.247665	0.257335
Population Skewed data												
-0.14081	-0.00039	0.002167	0.290114	0.472348	0.101239	0.233037	0.535757	0.290114	0.472348	0.101239	0.119678	0.266755
-0.329	-0.00022	0.004737	0.396917	0.431924	0.137156	0.295002	0.847399	0.396917	0.431924	0.137156	0.149038	0.422317
-0.64911	-0.00098	0.009607	1.127583	1.644287	0.378073	0.242258	0.631111	1.127583	1.644287	0.378073	0.123876	0.314394
First Phase Sample Skewed data												
0.076579	-0.00092	-0.01208	0.032426	0.331547	0.086568	0.167906	0.694907	0.032426	0.331547	0.086568	0.083974	0.536257
0.251462	-0.00314	-0.04057	-0.17808	0.45141	0.192372	0.202904	1.053447	-0.17808	0.45141	0.192372	0.102389	0.646471
0.076579	-0.00092	-0.01208	0.008397	1.140046	0.329453	0.171822	0.806619	0.008397	1.140046	0.329453	0.086135	0.572047

Table B8
Variance Covariance Matrices and their Determinants for Proposed Estimator t_p and Modified Version of Singh and Majhi (2014) t_g

Populations	Σ_p			Σ_g			$\lvert\Sigma_p\rvert$	$\lvert\Sigma_g\rvert$	PRE	$tr(\Sigma_p)$	$tr(\Sigma_g)$	PRE
When Optimum Values are Computed from Population Parameters												
Symmetric Data	**0.243527**	0.028502	0.570622	**0.417131**	0.102696	0.833885	0.001123718	0.003075868	**274%**	1.910153	2.493858	**131%**
	0.028502	**0.023377**	0.027774	0.102696	**0.054808**	0.14958						
	0.570622	0.027774	**1.643249**	0.833885	0.14958	**2.021919**						
Skewed Data	502.2181	157.9712	8.00496	997.3593	376.4787	1189.706	6972658	57488866	**824%**	4882.047	7553.766	**155%**
	157.9712	62.18631	202.6978	376.4787	165.0411	686.5081						
	8.00496	202.6978	4317.643	1189.706	686.5081	6391.366						
When Optimum Values are Computed from First Phase Sample												
Symmetric Data	**0.236689**	0.029745	0.559355	0.425511	0.105514	0.844511	0.00113232	0.003175403	**280%**	1.915537	2.549942	**133%**
	0.029745	**0.024151**	0.02547	0.105514	0.054833	0.147618						
	0.559355	0.02547	**1.654697**	0.844511	0.147618	2.069598						
Skewed Data	26105.46	8408.184	32670.16	**41580.45**	11109.08	47025.24	37598837293	2.11187e+11	**562%**	91338.47	124712.4	**137%**
	8408.184	3120.512	13226.8	11109.08	**3583.825**	15903.93						
	32670.16	13226.8	62112.5	47025.24	15903.93	**79548**.09						