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PROPERTIES OF THE FOUR-PARAMETER WEIBULL DISTRIBUTION AND ITS APPLICATIONS

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ABSTRACT

In this article, we study the so-called the Weibull Weibull distribution. General explicit expressions for the quantile function, expansion of its density function, ordinary and incomplete moments, moments of the residual and reversed residual lifes, order statistics and Rényi and q–entropies are derived. The model parameters are estimated using the maximum likelihood method. Simulation results are provided to assess the accuracy and performance of the maximum likelihood estimators. The usefulness and flexibility of the Weibull Weibull model are illustrated using real data sets.

KEY WORDS

Entropy, Maximum Likelihood, Moments, Order Statistics, Weibull Distribution, Weibull-G Family.

1. INTRODUCTION

The Weibull (W) distribution is a very popular distribution for modeling lifetime data in reliability where the hazard rate function is monotone. However, in many applied areas such as lifetime analysis, the two-parameter W distribution is inadequate when the true hazard shape is of unimodal or bathtub shape. Many generalizations of the W distribution have been proposed in the statistical literature to handle with bathtub shaped failure rates. Mudholkar and Srivastava (1993) and Mudholkar et al. (1996) pioneered exponentiated W (EW) distribution to analyze bathtub failure data. Xie et al. (2002) proposed a three-parameter modified W distribution with a bathtub shaped hazard function. Carrasco et al. (2008) suggested the generalized modified W distribution. Cordeiro et al. (2016) proposed the Kumaraswamy exponential W distribution, among others.

Recently, new generated families of continuous distributions have attracted several statisticians to develop new models. These families are obtained by introducing one or more additional shape parameter(s) to the baseline distribution. Some of the generated families are: the beta-G (Eugene et al., 2002), gamma-G (Zografos and Balakrishanan, 2009), Kumaraswamy-G (Cordeiro and de Castro, 2011), McDonald-G (Alexander et al., 2012), transformed-transformer (Alzaatreh et al., 2013), Kumaraswamy odd log-logistic (Alizadeh et al., 2015), type 1 half-logistic (Cordeiro et al., 2016), Garhy generated family (Elgarhy et al., 2016), Kumaraswamy Weibull-G (Hassan and Elgarhy, 2016), additive Weibull-G (Hassan and Hemeda, 2016), type II half logistic-G (Hassan et al. 2017) and Weibull-G (W-G) (Bourguignon et al., 2014) families, among others.

The cumulative distribution function (cdf) of the W-G family is given by

$$F(x) = 1 - \exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]^{\beta}\right\}, x > 0, \alpha, \beta > 0, \tag{1}$$

where α and β are two positive shape parameters. The cdf (1) provides a wider family of continuous distributions. The probability density function (pdf) corresponding to (1) is given by

$$f(x) = \frac{\alpha\beta g(x) G(x)^{\beta-1}}{[1-G(x)]^{\beta+1}} \exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]^{\beta}\right\}. \tag{2}$$

Recently, many authors constructed generalizations based on the W-G family. For example, Tahir et al. (2015a) introduced the W Lomax, Merovci and Elbatal (2015) defined the W Rayleigh, Tahir et al. (2016) proposed the W Pareto, Tahir et al. (2015b) studied the W Dagum, Afify et al. (2016b) pioneered the W Fréchet, Hassan et al. (2016) introduced the W quasi Lindley, Afify et al. (2016a) proposed the W Burr XII distributions.

Bourguignon et al. (2014) defined the Weibull-Weibull (WW) distribution. However, they do not investigate its several properties. Therefore, the main objective of this paper is to study the WW distribution defined from the W-G family and give a comprehensive account of some of its mathematical properties. Further, we prove empirically that the WW distribution provides better fits than at least three other competitive models in two applications.

The WW distribution contains several lifetime distributions as special cases (see Table 1). We are motivated to study the WW distribution because (i) It contains a number of known lifetime sub models listed in Table 1; (ii) The WW distribution exhibits buthtab hazard rate which makes this distribution to be superior to other lifetime distributions, which exhibit only monotonically increasing/decreasing, or constant hazard rates. (iii) It is shown in Section 3.1 that the WW distribution can be viewed as a mixture of Weibull distribution introduced by Weibull (1951); and (iv) The WW distribution outperforms several of the well-known lifetime distributions with respect to two real data sets.

The remainder of the paper is organized as follows: In Section 2, we define the WW distribution and provide its special models. In Section 3, we derive a very useful representation for the WW density and distribution functions. Further, we derive some mathematical properties of the proposed distribution. The maximum likelihood method is used to estimate the model parameters in Section 4. In Section 5, simulation results to assess the performance of the proposed maximum likelihood estimation procedure are discussed. In Section 6, we prove imperially the importance of the WW distribution using two real data sets. Finally, we give some concluding remarks in Section 7.

2. THE WW DISTRIBUTION

The cdf of the W distribution with scale parameter $\lambda > 0$ and shape parameter $\gamma > 0$ is given (for $x > 0$) by

$$G(x;\lambda,\gamma) = 1 - \exp\left(-\lambda x^{\gamma}\right). \tag{3}$$

The corresponding pdf of (3) is given by

$$g(x;\lambda,\gamma) = \lambda\gamma x^{\gamma-1}\exp\left(-\lambda x^{\gamma}\right). \tag{4}$$

The random variable X is said to have a WW distribution, denoted by $X \sim$ WW $\left(\alpha,\beta,\lambda,\gamma\right)$, if its cdf is given (for $x > 0$) by

$$x > 0 \;\; F(x;\alpha,\beta,\lambda,\gamma) = 1 - \exp\left\{-\alpha\left[\exp(\lambda x^{\gamma}) - 1\right]^{\beta}\right\}, \tag{5}$$

where α,β and γ are shape parameters and λ is a scale parameter.

The corresponding pdf of X is

$$f(x;\alpha,\beta,\lambda,\gamma) = \alpha\beta\lambda\gamma x^{\gamma-1}\left[\exp\left(\lambda x^{\gamma}\right)-1\right]^{\beta-1}\exp\left\{-\alpha\left[\exp\left(\lambda x^{\gamma}\right)-1\right]^{\beta}+\lambda x^{\gamma}\right\}. \tag{6}$$

The hazard rate function (hrf), reversed hazard rate function and cumulative hazard rate function of X are, respectively, given by

$$h(x;\alpha,\beta,\lambda,\gamma) = \alpha\beta\lambda\gamma x^{\gamma-1}\left[\exp\left(\lambda x^{\gamma}\right)-1\right]^{\beta-1}\exp\left(\lambda x^{\gamma}\right),$$

$$\tau(x;\alpha,\beta,\lambda,\gamma) = \frac{\alpha\beta\lambda\gamma x^{\gamma-1}\left[\exp\left(\lambda x^{\gamma}\right)-1\right]^{\beta-1}\exp\left\{-\alpha\left[\exp\left(\lambda x^{\gamma}\right)-1\right]^{\beta}+\lambda x^{\gamma}\right\}}{1-\exp\left\{-\alpha\left[\exp\left(\lambda x^{\gamma}\right)-1\right]^{\beta}\right\}}$$

and

$$H(x;\alpha,\beta,\lambda,\gamma) = -\ln\left(\exp\left\{-\alpha\left[\exp\left(\lambda x^{\gamma}\right)-1\right]^{\beta}\right\}\right).$$

Plots of the WW density for some selected parameter values are displayed in Figure 1. Figure 2 displays some possible shapes of the hrf of the WW model for selected parameter values. The plots in Figure 1 reveal that the pdf of the WW distribution can be reversed J-shape, right skewed, left skewed or concave down. It can be seen, from Figure 2, that the hrf can be decreasing, increasing or bathtub failure rate shapes.

The WW model is a very flexible distribution that approaches different distributions when its parameters are changed. Table 1 lists the special sub-models of the WW distribution.

Table 1
Special Models of the WW Distribution

S#	Model	λ	γ	α	β	Author
1	Weibull exponential	λ	1	α	β	Oguntunde et al. (2015)
2	Weibull Rayleigh	λ	2	α	β	Merovci and Elbatal (2015)
3	Exponential Weibull	λ	γ	α	1	New
4	Exponential Exponential	λ	1	α	1	New
5	Exponential Rayleigh	λ	2	α	1	New
6	Rayleigh Weibull	λ	γ	α	2	New
7	Rayleigh Exponential	λ	1	α	2	New
8	Rayleigh Rayleigh	λ	2	α	2	New

3. STATISTICAL PROPERTIES

This section discusses some important statistical properties of the WW distribution. Let Z be a random variable having the W distribution with cdf (3) and pdf (4). Then, the rth ordinary and incomplete moments of Z are given by

$$\mu'_{r,Z} = \lambda^{-r/\gamma}\Gamma\left(1+r/\gamma\right) \text{ and } \varphi_{r,Z}\left(t\right) = \lambda^{-r/\gamma}\,\gamma\left(1+r/\gamma, \lambda^{1/\gamma}t^{\gamma}\right),$$

respectively, where $\gamma\left(w,t\right) = \int_0^t x^{w-1}e^{-x}dx$ is the lower incomplete gamma function.

3.1 Useful Expansions

Now, we derive a useful mixture representation for the pdf and cdf of the WW distribution. The pdf (6) can be rewritten as

$$f(x) = \frac{\alpha\beta\lambda\gamma x^{\gamma-1}\exp\left(-\lambda x^{\gamma}\right)}{\left[\exp\left(-\lambda x^{\gamma}\right)\right]^{\beta+1}}\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]^{\beta-1}\exp\left\{-\alpha\left[\frac{1-\exp\left(-\lambda x^{\gamma}\right)}{\exp\left(-\lambda x^{\gamma}\right)}\right]^{\beta}\right\}.$$

Using the exponential series, we can write

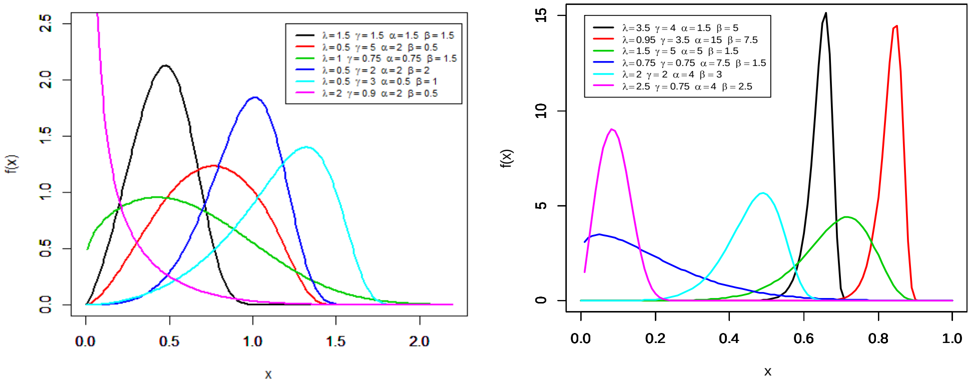

Figure 1: Some Possible Shapes for the pdf of the WW Distribution

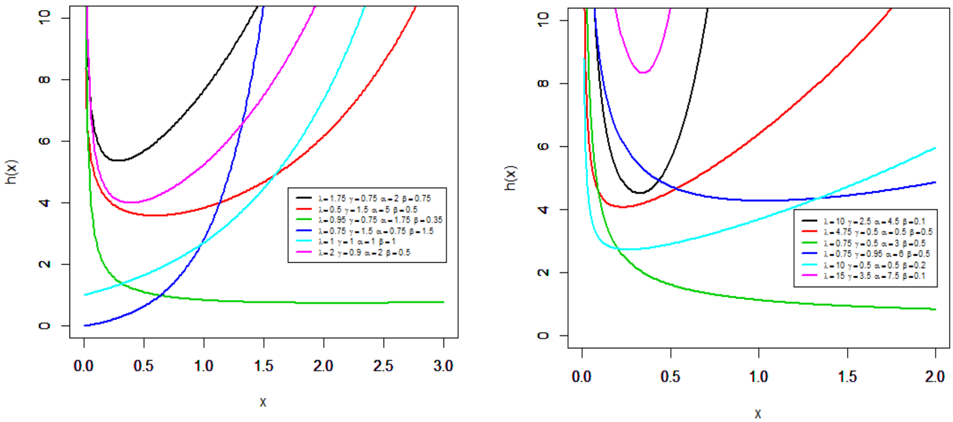

Figure 2: Some Possible Shapes for the hrf of the WW Distribution

$$\exp\left\{-\alpha\left[\frac{1-\exp\left(-\lambda x^{\gamma}\right)}{\exp\left(-\lambda x^{\gamma}\right)}\right]^{\beta}\right\}=\sum_{k=0}^{\infty}\frac{(-1)^{k}\alpha^{k}}{k!}\frac{\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]^{k\beta}}{\left[\exp\left(-\lambda x^{\gamma}\right)\right]^{k\beta}}. \tag{7}$$

Using (7), the WW density function reduces to

$$f(x)=\beta\lambda\gamma x^{\gamma-1}\exp\left(-\lambda x^{\gamma}\right)\sum_{k=0}^{\infty}\frac{(-1)^{k}\alpha^{k+1}}{k!}\frac{\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]^{\beta(k+1)-1}}{\left[1-\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]\right]^{\beta(k+1)+1}}. \tag{8}$$

Using the generalized binomial series, we can write

$$\left[1-\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]\right]^{-[\beta(k+1)+1]}=\sum_{j=0}^{\infty}\frac{\Gamma\left(\beta(k+1)+j+1\right)}{j!\,\Gamma\left(\beta(k+1)+1\right)}\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]^{j}.$$

Then, equation (8) reduces to

$$f(x)=\beta\lambda\gamma x^{\gamma-1}\exp\left(-\lambda x^{\gamma}\right)\sum_{k,j=0}^{\infty}\frac{(-1)^{k}\alpha^{k+1}\Gamma\left(\beta(k+1)+j+1\right)}{k!\,j!\,\Gamma\left(\beta(k+1)+1\right)}\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]^{\beta(k+1)+j-1}.$$

Consider the generalized binomial expansion, for $b>0$ is real non integer and $|z|<1$,

$$\left(1-z\right)^{b-1}=\sum_{i=0}^{\infty}(-1)^{i}\binom{b-1}{i}z^{i}. \tag{9}$$

Applying expansion (9) to the last equation gives

$$f(x)=\beta\sum_{k,j,s=0}^{\infty}\frac{(-1)^{k+s}\alpha^{k+1}\left(\beta(k+1)+j+1\right)}{k!\,j!\left(\beta(k+1)+1\right)}\binom{\beta(k+1)+j-1}{s}\lambda\gamma x^{\gamma-1}\exp\left(-\lambda(s+1)x^{\gamma}\right).$$

or, equivalently, we can write

$$f(x)=\sum_{s=0}^{\infty}\eta_{s}\,g_{(s+1)}\left(x\right), \tag{10}$$

where $\eta_{s}=\beta\sum_{k,j=0}^{\infty}\frac{(-1)^{k+s}\alpha^{k+1}\Gamma(\beta(k+1)+j+1)}{k!\,j!(s+1)\left(\beta(k+1)+1\right)}\binom{\beta(k+1)+j-1}{s}$ and $g_{(s+1)}\left(x\right)$ is the pdf of the W distribution with shape parameter γ and scale parameter $\lambda\left(s+1\right)$. Thus, the WW density function can be expressed as a linear combination of W densities. Then, several of its structural properties can be obtained from equation (10) and those properties of the W distribution.

By integrating equation (10), the cdf of X can be given in the mixture form

$$F(x)=\sum_{s=0}^{\infty}\eta_{s}\,G_{(s+1)}\left(x\right),$$

where $G_{(s+1)}\left(x\right)$ is the W cdf with with shape parameter γ and scale parameter $\lambda\left(s+1\right)$.

3.2 Quantile Function

The quantile function, say $Q(u)=F^{-1}(u)$ of X can be obtained by inverting (5) as follows

$$u=1-\exp\left(-\alpha\left(\left(e^{\lambda Q(u)^{\gamma}}-1\right)\right)^{\beta}\right).$$

After some simplifications, it reduces to

$$Q(u) = \sqrt[\gamma]{\ln\left\{1+\left[\ln(1-u)^{\frac{-1}{\alpha}}\right]^{1/\beta}\right\}^{1/\lambda}}, \tag{11}$$

where u is a uniform random variable on the unit interval $(0,1)$. In particuler, the median can be derived from (11) by setting $u = 0.5$.

3.3 Moments

The rth moment of X can be obtained from (10) as

$$\mu_r' = E(X^r) = \sum_{s=0}^{\infty} \eta_s \int_{-\infty}^{\infty} x^r g_{(s+1)}(x)\,dx.$$

Then, we have

$$\mu_r' = \sum_{s=0}^{\infty} \eta_s \left[\lambda(s+1)\right]^{-r/\gamma} \Gamma\left(1+r/\gamma\right).$$

The nth incomplete moment of X can be expressed, based on (10), as

$$\varphi_n(t) = \int_{-\infty}^{t} x^n f(x)\,dx = \sum_{s=0}^{\infty} \eta s \int_{-\infty}^{t} x^n g_{(s+1)}(x)\,dx \ .$$

Hence, we have

$$\varphi_n(t) = \sum_{s=0}^{\infty} \eta_s \left[\lambda(s+1)\right]^{-n/\gamma} \gamma\left(1+\frac{r}{\gamma},\left[\lambda(s+1)\right]^{1/\gamma} t^{\gamma}\right).$$

The nth moment of the residual lifetime is defined (for $t > 0$ and $n = 1,2,...$) by

$$m_n(t) = E[(X-t)^n \mid X > t] = \frac{1}{R(t)} \int_t^{\infty} (x-t)^n f(x)\,dx,$$

where $R(t)$ is the reliability function. Then, we have

$$m_n(t) = \frac{1}{R(t)} \sum_{s=0}^{\infty} \sum_{d=0}^{n} (-t)^{n-d} \binom{n}{d} \eta_s \left[\lambda(s+1)\right]^{-d/\gamma} \Gamma\left(1+\frac{d}{\gamma},\lambda^{1/\gamma} t^{\gamma}\right),$$

where $\Gamma(a,x) = \int_x^{\infty} y^{a-1} e^{-y}\,dy$ is the upper incomplete gamma function.

The nth moment of the reversed residual life is defined (for $t > 0, = 1,2,...$) by

$$M_n(t) = E\left[(t-X)^n \mid X \le t\right] M_n(t) = \frac{1}{F(t)} \int_0^t (t-x)^n f(x)\,dx \, .$$

Using equation (10), we can write

$$M_n(t)=\frac{1}{F(t)}\sum_{d=0}^{n}\sum_{s=0}^{\infty}(-1)^d t^{n-d}\binom{n}{d}\eta_s\left[\lambda(s+1)\right]^{-d/\gamma}\gamma\left(1-\frac{d}{\gamma},\lambda^{1/\gamma}t^{\gamma}\right).$$

The mean inactivity time (MIT) of X follows from the above equation with $n=1$.

3.4 Order Statistics

Let $X_{1:n}<X_{2:n}<\cdots<X_{n:n}$ be the order statistics of a random sample of size n following the WW distribution as given in (5) and (6), respectively. Then, the pdf of the kth order statistic, $X_{k:n}$, denoted by $f_{k:n}(x)$, is defined by

$$f_{k:n}(x)=\frac{f(x)}{B(k,n-k+1)}\sum_{v=0}^{n-k}(-1)^v\binom{n-k}{v}F(x)^{v+k-1}, \tag{12}$$

where $B(.,.)$ is the beta function. Hence, we can write

$$F(x)^{v+k-1}=\sum_{m=0}^{\infty}(-1)^m\binom{v+k-1}{m}\exp\left(\frac{-\alpha m\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]^{\beta}}{\left\{1-\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]\right\}^{\beta}}\right).$$

Using (6), we have

$$f(x)F(x)^{v+k-1}=\sum_{m=0}^{\infty}(-1)^m\binom{v+k-1}{m}\frac{\alpha\beta\lambda\gamma x^{\gamma-1}\exp\left(-\lambda x^{\gamma}\right)}{\left\{1-\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]\right\}^{\beta+1}}\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]^{\beta-1}$$
$$\times\exp\left(\frac{-\alpha(m+1)\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]^{\beta}}{\left\{1-\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]\right\}^{\beta}}\right).$$

Applying the exponential series and the generalized binomial expansion, we have

$$f(x)F(x)^{v+k-1}=\sum_{m,r,j,s=0}^{\infty}\frac{(-1)^{m+r+s}\beta m^r\alpha^{r+1}\Gamma\left(\beta(r+1)+j+1\right)}{r!\,j!\,\Gamma\left(\beta(r+1)+1\right)}\binom{v+k-1}{m}$$
$$\times\binom{\beta(r+1)+j-1}{s}\lambda\gamma x^{\gamma-1}\exp\left(-\lambda(s+1)x^{\gamma}\right).$$

By inserting the last equation in (12), we obtain

$$f_{k:n}(x)=\sum_{m,r,j,s=0}^{\infty}\sum_{v=0}^{n-k}\frac{(-1)^{v+m+r+s}\beta m^r\alpha^{r+1}\Gamma\left(\beta(r+1)+j+1\right)}{r!\,j!\,B(k,n-k+1)\Gamma\left(\beta(r+1)+1\right)}\binom{n-k}{v}\binom{v+k-1}{m}$$
$$\times\binom{\beta(r+1)+j-1}{s}\lambda\gamma x^{\gamma-1}\exp\left(-\lambda(s+1)x^{\gamma}\right).$$

Then, The pdf of $X_{k:n}$ reduces to

$$f_{k:n}(x)=\sum_{s=0}^{\infty}\eta_s^{*}\,g_{(s+1)}\left(x\right), \tag{13}$$

where

$$\eta_s^{*}=\sum_{m,r,j=0}^{\infty}\sum_{v=0}^{n-k}\frac{(-1)^{v+m+r+s}\beta m^{r}\alpha^{r+1}\Gamma\left(\beta(r+1)+j+1\right)}{r!\,j!\left(s+1\right)B(k,n-k+1)\Gamma\left(\beta(r+1)+1\right)}\binom{n-k}{v}\binom{v+k-1}{m}\binom{\beta\left(r+1\right)+j-1}{s}$$

and, as before, $g_{(s+1)}\left(x\right)$ is the W pdf with shape parameter γ and scale parameter $\lambda\left(s+1\right)$. So, the density function of the WW order statistics is a linear combination of W densities. Based on equation (13), we can obtain some structural properties of $X_{k:n}$ from those W properties. For example, the qth moment of $X_{k:n}$ is given by

$$E\left(X_{k:n}^{q}\right)=\sum_{s=0}^{\infty}\eta_s^{*}\left[\lambda\left(s+1\right)\right]^{-q/\gamma}\Gamma\left(1+q/\gamma\right).$$

3.5 Rényi and q-Entropies

The Rényi entropy (Rényi, 1961) of X is defined by

$$I_{\delta}(X)=\frac{1}{1-\delta}\log\int_{-\infty}^{\infty}f(x)^{\delta}dx,\qquad \delta>0,\ \delta\neq 1.$$

Using the pdf (6) and the exponential series, we can write

$$f(x)^{\delta}=\left(\alpha\beta\lambda\gamma\right)^{\delta}x^{\delta(\gamma-1)}\exp\left(-\lambda\delta x^{\gamma}\right)\sum_{k=0}^{\infty}\frac{(-1)^{k}(\alpha\delta)^{k}}{k!}\frac{\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]^{\beta(\delta+k)-\delta}}{\left\{1-\left[1-\exp\left(-\lambda x^{\gamma}\right)\right]\right\}^{\beta(\delta+k)+\delta}}.$$

Applying the generalized binomial expansion and after some algebra, the above equation reduces to

$$f(x)^{\delta}=(\alpha\beta\lambda\gamma)^{\delta}\sum_{k,i,j=0}^{\infty}\frac{(-1)^{k+j}(\alpha\delta)^{k}\Gamma\left(\beta(\delta+k)+\delta+i\right)}{k!\,i!\,\Gamma\left(\beta(\delta+k)+\delta\right)}\binom{\beta(\delta+k)+i-\delta}{j}x^{\delta(\gamma-1)}\exp\left(-\lambda(\delta+j)x^{\gamma}\right).$$

Then, $I_{\delta}(X)$ reduces to

$$I_{\delta}(X)=\frac{1}{1-\delta}\log\left\{\sum_{j=0}^{\infty}d_j\left[\lambda(\delta+j)\right]^{\frac{\delta(1-\gamma)-1}{\gamma}}\Gamma\left(\frac{\delta(\gamma-1)+1}{\gamma}\right)\right\}, \tag{14}$$

where $d_j=\sum_{k,i=0}^{\infty}\frac{(-1)^{k+j}\alpha^{k+\delta}\delta^{k}\gamma^{\delta-1}(\beta\lambda)^{\delta}\Gamma\left(\beta(\delta+k)+\delta+i\right)}{k!\,i!\,\Gamma\left(\beta(\delta+k)+\delta\right)}\binom{\beta(\delta+k)+i-\delta}{j}.$

The q-entropy is defined (for $q>0$ and $q\neq 1$) by

$$H_q(X)=\frac{1}{1-q}\log\left(1-J_q\right),$$

where $J_q=\int_{-\infty}^{\infty} f(x)^q dx,$ follows from (14) as $J_q=(1-q)I_q(X)$.

4. MAXIMUM LIKELIHOOD ESTIMATION

The maximum likelihood estimates (MLEs) of the unknown parameters for the WW distribution are determined based on complete samples. Let $X_1,\dots,X_n$ be a random sample of size n from the this distribution with vector of parameters $\Psi=(\lambda,\gamma,\alpha,\beta)^T$. The total log-likelihood function for Ψ can be expressed as

$$\ell(\Psi)=n\ln\alpha+n\ln\beta+n\ln\lambda+n\ln\gamma+(\gamma-1)\sum_{i=1}^{n}\ln\left(x_i\right)+(\beta-1)\sum_{i=1}^{n}\ln\left(e^{\lambda x_i^{\gamma}}-1\right)$$
$$+\lambda\sum_{i=1}^{n}x_i^{\gamma}-\alpha\sum_{i=1}^{n}\left[\left(e^{\lambda x_i^{\gamma}}-1\right)\right]^{\beta}.$$

The score vector elements come out as

$$U_\lambda=\frac{n}{\lambda}+\left(\beta-1\right)\sum_{i=1}^{n}\frac{x_i^{\gamma}e^{\lambda x_i^{\gamma}}}{e^{\lambda x_i^{\gamma}}-1}+\sum_{i=1}^{n}x_i^{\gamma}-\alpha\beta\sum_{i=1}^{n}x_i^{\gamma}\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta-1}e^{\lambda x_i^{\gamma}},$$

$$U_\gamma=\frac{n}{\gamma}+\sum_{i=1}^{n}\ln\left(x_i^{\gamma}\right)+\lambda\left(\beta-1\right)\sum_{i=1}^{n}\frac{x_i^{\gamma}e^{\lambda x_i^{\gamma}}\ln\left(x_i\right)}{e^{\lambda x_i^{\gamma}}-1}+\lambda\sum_{i=1}^{n}x_i^{\gamma}\ln\left(x_i\right)$$
$$-\alpha\beta\lambda\sum_{i=1}^{n}x_i^{\gamma}\ln\left(x_i\right)\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta-1}e^{\lambda x_i^{\gamma}},$$

$$U_\alpha=\frac{n}{\alpha}-\sum_{i=1}^{n}\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta}$$

and

$$U_\beta=\frac{n}{\beta}+\sum_{i=1}^{n}\ln\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta}-\alpha\sum_{i=1}^{n}\left(e^{\lambda x_i^{\gamma}}-1\right)^{\beta}\ln\left(e^{\lambda x_i^{\gamma}}-1\right).$$

The MLEs $\hat{\psi}$ of ψ can be determined by maximizing $\ell\left(\psi\right)$ (for a given x) either directly by using the Mathcad, R (optim function), SAS (PROC NLMIXED), Ox program (sub-routine MaxBFGS) or by solving the above nonlinear system obtained by differentiating this equation and equating its four components to zero.

5. SIMULATION STUDY

In this section, an extensive numerical investigation is carried out to assess on the finite sample behavior of the MLEs of α, β, λ and γ. We evaluate the performance of MLEs through their biases and mean square errors (MSEs) for sample sizes =10, 30, 50 and 100. All results are obtained from 3000 Monte Carlo replications. The means, MSEs and biases for the different estimators will be reported from these experiments. Table 2 presents the means of the MLEs of the parameters of the WW distribution and the corresponding biases and MSEs. It can be verified that the estimates are stable and quite close the true parameter values for all sample sizes. Further, the MSEs decrease when the sample size increases in all cases.

Table 2
The Parameter Estimation from WW Distribution using MLE

n		Init	MLE	Bias	MSE	Init	MLE	Bias	MSE
10	α	0.5	0.5576	0.0576	0.0444	0.5	0.5544	0.0544	0.0412
	β	0.5	0.6442	0.1442	0.2001	0.75	0.9879	0.2379	0.5166
	λ	0.5	0.6109	0.1109	0.0801	0.5	0.6687	0.1687	0.1801
	γ	0.5	0.6035	0.1035	0.8862	0.5	0.5818	0.0818	0.1835
30	α	0.5	0.5145	0.0145	0.0097	0.5	0.5175	0.0175	0.0103
	β	0.5	0.5198	0.0198	0.0330	0.75	0.8072	0.0572	0.0769
	λ	0.5	0.5370	0.0370	0.0242	0.5	0.5556	0.0556	0.0393
	γ	0.5	0.4517	-0.0483	0.0382	0.5	0.4495	-0.0505	0.0372
50	α	0.5	0.5104	0.0104	0.0055	0.5	0.5101	0.0101	0.0052
	β	0.5	0.5043	0.0043	0.0165	0.75	0.7771	0.0271	0.0374
	λ	0.5	0.5261	0.0261	0.0138	0.5	0.5331	0.0331	0.0187
	γ	0.5	0.4302	-0.0698	0.0233	0.5	0.4245	-0.0755	0.0226
100	α	0.5	0.5056	0.0056	0.0026	0.5	0.5049	0.0049	0.0026
	β	0.5	0.4866	-0.0134	0.0075	0.75	0.7530	0.0030	0.0171
	λ	0.5	0.5121	0.0121	0.0064	0.5	0.5166	0.0166	0.0087
	γ	0.5	0.4060	-0.0940	0.0170	0.5	0.4047	-0.0953	0.0170

6. DATA ANALYSIS

In this section, we use two real data sets to illustrate the importance and flexibility of the WW distribution. We compare the fits of the WW model with some models namely: the beta Weibull (BW) (Lee et al., 2007), Mcdonald Weibull (McW) (Cordeiro et al., 2014) and exponentiated Weibull (EW) (Mudholkar and Srivastava, 1993) dsitributions.

The maximized log-likelihood (-2ℓ), Akaike information criterion (AIC), the corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Anderson-Darling $\left(A^*\right)$ and Cramér-Von Mises (W^*) statistics are used for model selection.

Table 2
The Parameter Estimation from WW Distribution using MLE (Continued)

n		Init	MLE	Bias	MSE	Init	MLE	Bias	MSE
10	α	0.5	0.6397	0.1397	0.1895	0.5	0.5485	0.0485	0.0371
	β	0.5	0.6090	0.1090	0.0788	0.5	0.6354	0.1354	0.1924
	λ	0.75	0.5593	-0.1907	0.1449	0.5	0.6052	0.1052	0.0768
	γ	0.5	0.5274	0.0274	0.0168	1.5	0.5580	-0.9420	0.9309
30	β	0.5	0.5397	0.0397	0.0235	0.5	0.5186	0.0186	0.0307
	λ	0.75	0.4754	-0.2746	0.0974	0.5	0.5368	0.0368	0.0237
	γ	0.5	0.5103	0.0103	0.0056	1.5	0.5084	-0.9916	0.9951
50	α	0.5	0.5038	0.0038	0.0169	0.5	0.5097	0.0097	0.0054
	β	0.5	0.5251	0.0251	0.0138	0.5	0.5015	0.0015	0.0165
	λ	0.75	0.4601	-0.2899	0.0962	0.5	0.5235	0.0235	0.0137
	γ	0.5	0.5082	0.0082	0.0036	1.5	0.5024	-0.9976	1.0023
100	α	0.5	0.4877	-0.0123	0.0077	0.5	0.5057	0.0057	0.0027
	β	0.5	0.5124	0.0124	0.0065	0.5	0.4881	-0.0119	0.0079
	λ	0.75	0.4466	-0.3034	0.0980	0.5	0.5129	0.0129	0.0067
	γ	0.5	0.5547	0.0547	0.0407	1.5	0.4959	-1.0041	1.0118

Example 1:

The data have been obtained from Nicholas and Padgett (2006). The data represent tensile strength of 100 observations of carbon fibers and they are:

3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

For the data in Example 1, Table 3 gives the MLEs of the fitted models and their standard errors (SEs) in parenthesis. The values of goodness-of-fit statistics are listed in Table 4.

It is noted, from Table 4, that the WW distribution provides a better fit than other competitive fitted models. It has the smallest values for goodness-of-fit statistics among all fitted models. Plots of the histogram, fitted densities and estimated cdfs are shown in Figures 3 and 4, respectively. These figures supported the conclusion drawn from the numerical values in Table 4.

Example 2:

The second data set is obtained from Tahir et al. (2015) and represents failure times of 84 Aircraft Windshield. The data are:

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

Table 5 lists the MLEs of the fitted models and their SEs in parenthesis. The values of goodness-of-fit statistics are presented in Table 6.

Table 3
The MLEs and SEs of the Model Parameters for First Data Set

Model	**Estimates (SEs)**				
WW $(\alpha,\beta,\lambda,\gamma)$	20.394 (1.582)	13.273 (0.236)	0.493 (0.073)	0.159 (0.086)	
BW (a,b,λ,γ)	34.051 (0.961)	14.541 (0.19)	0.833 (0.11)	0.427 (0.077)	
McW (a,b,λ,γ,c)	35.28 (0.916)	18.125 (0.254)	0.813 (0.13)	0.399 (0.085)	1.548 (6.993)
EW (λ,γ,a)	5.77 (0.103)	0.295 (0.057)	1135 (0.662)		

Table 4
Goodness-of-Fit Statistics for First Data Set

Model	-2ℓ	**AIC**	**CAIC**	**BIC**	**HQIC**	A^*	W^*
WW	299.747	307.747	309.347	305.656	309.54	0.45081	0.06256
BW	317.214	325.214	326.814	325.214	329.431	1.22496	0.23356
McW	308.116	318.116	319.716	318.116	323.388	1.22090	0.23286
EW	373.861	377.861	378.305	376.815	378.757	2.81959	0.51324

Table 5
The MLEs and SEs for Second Data Set

Model	Estimates (SEs)				
WW $(\alpha,\beta,\lambda,\gamma)$	24.862 (1.44)	3.752 (0.298)	0.199 (0.069)	0.545 (0.113)	
BW (a,b,λ,γ)	53.874 (2.717)	20.528 (0.278)	1.076 (0.278)	0.231 (0.184)	
McW (a,b,λ,γ,c)	51.321 (5.329)	19.762 (0.605)	1.119 (0.48)	0.23 (0.424)	1.525 (38.539)
EW (λ,γ,a)	7.017 (0.134)	0.144 (0.063)	1773 (0.827)		

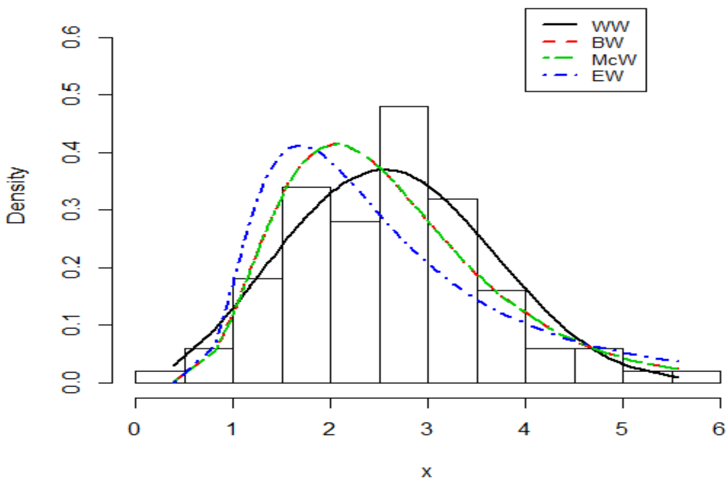

Figure 3: Estimated Densities of the Fitted Models for Data Set 1

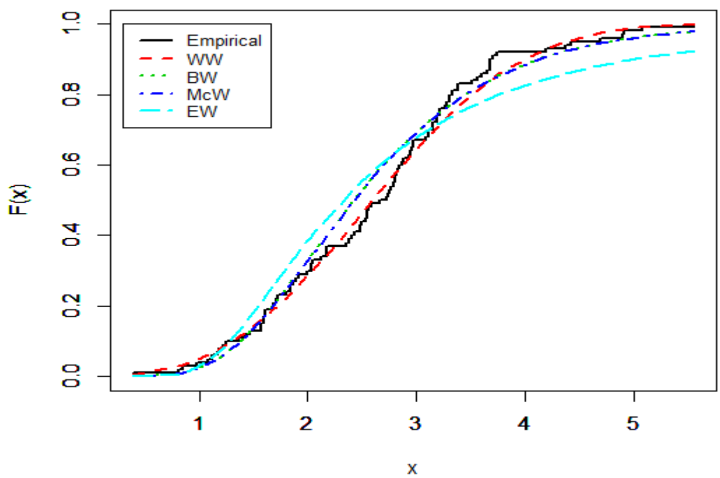

Figure 4: Estimated cdfs of for Data Set 1

Table 6
Goodness-of-Fit Statistics for Second Data Set

Model	-2ℓ	AIC	CAIC	BIC	HQIC	A^*	W^*
WW	261.389	269.389	269.895	269.086	273.298	0.65619	0.07529
BW	289.948	297.948	298.455	297.645	301.857	3.34711	0.48715
McW	283.983	293.983	294.752	293.604	298.869	3.33313	0.4847
EW	320.347	326.347	326.647	324.196	326.302	32.74879	7.04167

It is observed, from Table 6, that the WW distribution gives a better fit than other fitted models. Plots of the histogram, fitted densities and estimated cdfs are displayed in Figures 5 and 6, respectively.

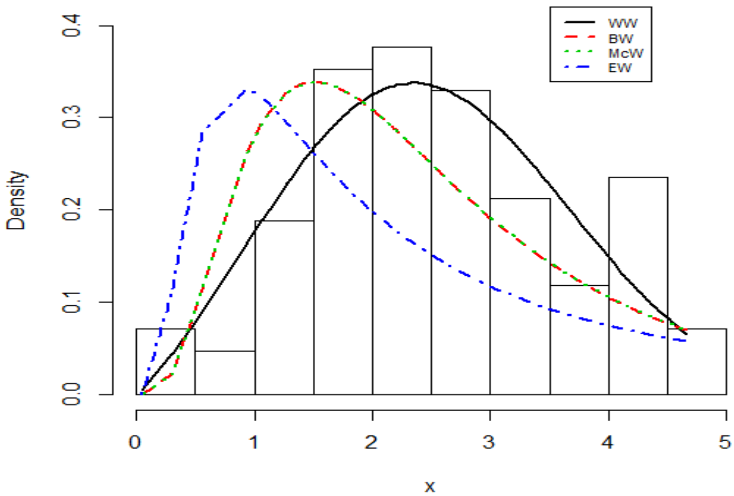

Figure 5: Estimated pdfs for Data Set 2

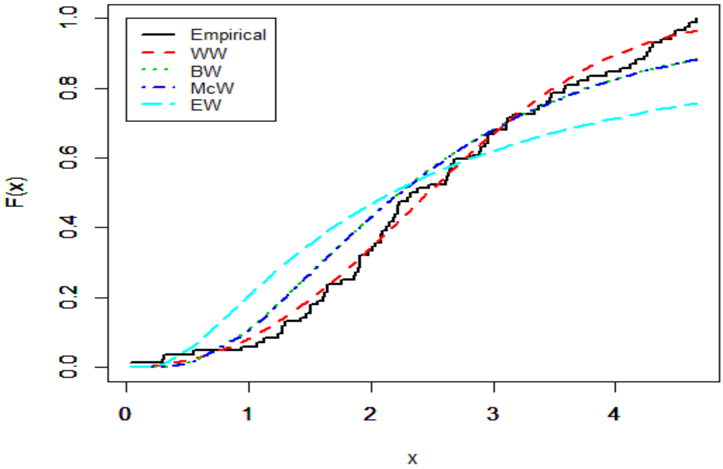

Figure 6: Estimated cdfs for Data Set 2

7. CONCLUSIONS

In this paper, we study a four-parameter model, named the Weibull Weibull (WW) distribution. The WW model is motivated by the wide use of the Weibull distribution in practice. The WW pdf can be expressed as a mixture of Weibull densities. We derive explicit expressions for the quantile function, ordinary and incomplete moments, moments of the residual and reversed residual function, order statistics and Rényi and q-entropies. The maximum likelihood estimation method is used to estimate the model parameters. We provide some simulation results to assess the performance of the proposed model. The practical importance of the WW distribution is demonstrated by means of two real data sets.

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