Pak. J. Statist.
2017 Vol. 33(6), 419-440

BAYESIAN ANALYSIS OF MAXWELL-BOLTZMANN DISTRIBUTION UNDER DIFFERENT LOSS FUNCTIONS AND PRIOR DISTRIBUTIONS

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ABSTRACT

Present paper discusses Bayesian estimation of Maxwell-Boltzmann distribution under different combinations of loss functions and prior distributions. Rate parameter of Maxwell- Boltzmann distribution is assumed to follow extension of Jeffrey's prior and gamma prior. Loss functions considered in this paper are squared error, precautionary, Stein's and Al-Bayyati's loss function. We have derived different estimators of rate parameter under the mutual combinations of considered loss functions and prior distributions. These estimators are then compared in terms of mean square error (MSE) which is computed by using the programming language R.

KEYWORDS

Extension of Jeffrey's prior, Gamma prior, squared error loss function, precautionary loss function, Al-Bayyati's loss function, Stein's loss function.

1. INTRODUCTION

Maxwell Boltzmann distribution is widely used in physics especially in statistical mechanics, in order to describe the speed of a particle in an idealized gas. This distribution can also be used to describe the distribution of $\sqrt{e_1+e_2+e_3}$ where $e_1, e_2 \,\&\, e_3$ are the measurement errors in position coordinates of the particle in a 3-dimensional space.This distribution was initially set forth in 1859 by the Scottish physicist James Clerk Maxwell and in 1871, Maxwell's result was generalized by a German physicist Ludwig Boltzmann to express the distribution of energies among molecules. This distribution is often used to describe the speed of a particle moving in a 3-dimensional space such that it's movement along the three coordinate axes are independently and Normally distributed random variables with mean zero and variance equal to the inverse of rate parameter. However, instead of 3-dimensional space if a particle moves in a 2-dimensional space it's speed is better described by Rayleigh distribution. The Maxwell-Boltzmann distribution can be used to find the distribution of particle's kinetic energy (E) which is related to particle's speed (v) by the formula

$E = mv^2/2$, provided the distribution of speed is known. The *Pdf* of Maxwell-Boltzmann distribution is given by:

$$f(v) = \sqrt{\left(\frac{m}{2\pi kT}\right)^3} 4\pi v^2 exp\left(-\frac{mv^2}{2kT}\right) \quad (1)$$

where, $m =$ mass of particle, $k =$ Boltzmann's constant and $T =$ thermodynamic temperature. Re-parameterizing equation (1) by $m/kT = \theta$, we will have the resulting *Pdf* as given in (2).

$$f(v;\theta) = \sqrt{\frac{2}{\pi}}\theta^{3/2} v^2 exp\left(-\frac{\theta v^2}{2}\right); v \geq 0, \theta > 0 \quad (2)$$

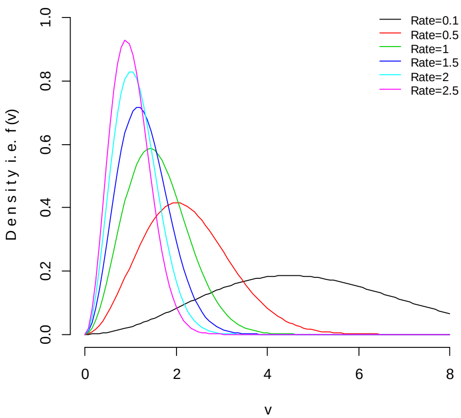

Fig. 1(a): Pdf Curve of Maxwell Distribution at Different Values of Rate Paramter

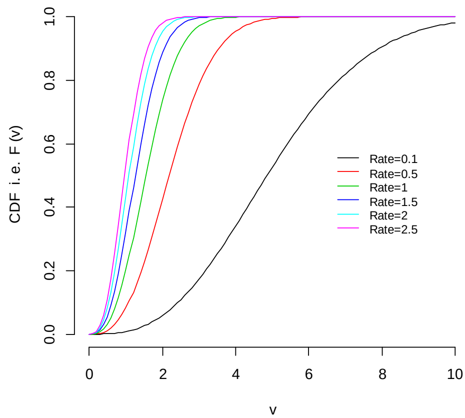

Fig. 1(b): Pdf Curve of Maxwell Distribution at Different Values of Rate Paramter

It is due to the practical applicability of Maxwell Distribution in Statistical mechanics, which lead the statisticians and reliability engineers to analyze it's different statistical properties. See for instance, Bekker and Roux (2005), Huang and Chen (2016), Krishna and Malik (2012), Marc Delphin (2011), Tomer *et al.* (2015) and Zhang *et al.* (2008). Various researchers have carried out the Bayesian estimation of Maxwell- Boltzmann distribution by using different types of loss functions and prior distributions, e.g. Chaturvedi *et al.* (1998), Al-Baldawi (2013), Dey *et al.* (2010), Dey *et al.* (2013), Kazmi *et al.* (2012), Podder and Roy (2003), Rasheed (2013), Tyagi and Bhattacharya (1989). In this paper, we have derived the maximum likelihood estimator and several other Bayes' estimators by considering the squared error, precautionary, Stein's and Al-Bayyati's loss functions in connection with the extension of Jeffrey's and gamma prior. These estimators are then compared in terms of their MSE. Overall, twenty-nine estimators including MLE have been derived for the parameter θ and are given in Table1.

2. MATERIALS AND METHODS

2.1 Prior Distribution (Prior-Information)

In comparison to classical approach, Bayesian approach is considered to be fair enough in estimating the parameters of a distribution provided that the prior distribution describes nicely the random behavior of a parameter. Since in Bayesian setup, parameter of a distribution is considered to be a random variable having the sample space denoted by Θ. Hence, it seeks the use of prior information about the parameter. This prior information about the parameter is described in terms of probability distribution usually known as prior distribution. Thus the most important element in Bayesian estimation is the specification of a prior distribution which a parameter as a random variable is supposed to follow. Therefore, an important pre-requisite in Bayesian estimation is the appropriate choice of prior(s) for the estimation of parameter θ. However, Bayesian analysts have pointed out that there isn't any hard and fast rule by which one can conclude that one prior is better than the other. Very often, priors are chosen according to one's subjective knowledge and beliefs that is why Bayesian approach is sometimes called as subjective approach. If one has adequate information about the parameter(s), one should use informative prior(s); otherwise it is preferable to use non-informative prior(s) (also known as vague prior). Usually, Uniform prior is preferred as a non-informative prior. However, Aslam *et al.* (2003) have shown an application of prior predictive distribution to elicit the prior density. In this paper, we assume the parameter (θ) to follow extension of Jeffrey's prior (proposed by Al-Kutubi in 2005) and gamma prior which are given by:

2.1.1 Extension of the Jeffrey's-Prior:

Prior proposed by Al-Kutubi, known as extension ofJeffrey's prior is given by:

$$g\left(\theta\right) \propto \left[I\left(\theta\right)\right]^{c_1} ; c_1 \in R^+ \tag{3}$$

where $I(\theta) = -n\,E\left[\frac{\partial^2}{\partial\theta^2}\log f(v;\theta)\right]$ is known as Fisher's Information matrix. Therefore $I(\theta)$ for Maxwell distribution will be:

$$I(\theta) = -n\,E\left[\frac{\partial^2}{\partial\theta^2}\left\{\frac{1}{2}\log\left(\frac{2}{\pi}\right) + \frac{3}{2}\log(\theta) + 2\log(v) - \frac{\theta v^2}{2}\right\}\right] = \frac{3n}{2\theta^2} \tag{4}$$

Thus the resulting extension of Jeffrey's- prior will be:

$$g_1\left(\theta\right) \propto \left(\frac{1}{\theta^2}\right)^{c1} \tag{5}$$

Remark 1:

If $c_1 = 1/2$, we will have Jeffrey's-prior i.e.

$$g_{11}\left(\theta\right) \propto \frac{1}{\theta} \tag{6}$$

Remark 2:

If $c_1 = 3/2$, we will have Hartigan's-prior i.e.

$$g_{12}(\theta) \propto \frac{1}{\theta^3} \tag{7}$$

Remark 3:

If $c_1 = 0$, we will have $U(0, k = 1/p)$ prior i.e.

$$g_{13}(\theta) \propto 1 \tag{8}$$

$g_{13}(\theta) = p$, where p is the constant of proportionality.

2.1.2 The Gamma (α, β) Prior:

The density of parameter (θ) on assuming it to follow Gamma(α, β) distribution is given by:

$$g_2(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{(\alpha-1)} e^{-\beta\theta}, \theta > 0, \alpha > 0, \beta > 0. \tag{9}$$

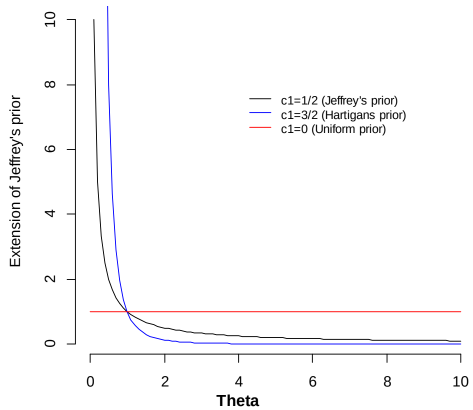

Fig. 2(a): Curve of Extension of Jaffery's Prior at Different Values of c1

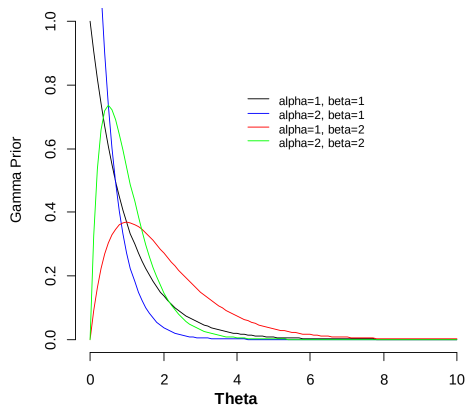

Fig. 2(a): Curve of Gamma Prior at Different Values of Alpha and Beta

2.2 Loss Function

The concept of loss function is as old as Laplace, and was reintroduced in statistics by Abraham Wald in the middle of 20^{th} Century see; A.Wald, "Statistical Decision Functions", Wiley (1950). Sound statistical practice requires selecting an estimator consistent with the actual acceptable variation experienced in the context of a particular applied problem. Thus, in the applied use of loss functions, selecting which statistical method to use to model an applied problem depends on knowing the losses that will be experienced from being wrong under the problem's particular circumstances. One of the consequences of Bayesian inference is that, in addition to experimental data, the loss function does not in itself wholly determine a decision. Moreover, the relationship between the loss function and the posterior probability is important. Thus the choice of loss function is an integral part of Bayesian inference. As there is no specific analytical procedure that allows us to identify the appropriate loss function to be used. Most of the works on point estimation and point prediction assume the underlying loss function to be squared error which is symmetric in nature. However, indiscriminate use of squared error loss function is not appropriate particularly in the cases, where losses are not symmetric. Thus, in order to make the statistical inferences more practical and applicable, we often need to choose an asymmetric loss function. A number of symmetric and asymmetric loss functions have been shown to be functional, see Dey *et al.* (2010), Norstrom *et al.* (1996), Podder *et al.* (2003), Rasheed *et al.* (2013), Spiring *et al.* (1998), Zellner *et al.* (1986), Reshi *et al. (2014)* and Ahmed *et al. (2013)* etc. In the present paper, we have considered squared error, precautionary, Al-Bayyati's and Stein's loss functions for better comparison of Bayes' estimators.

2.2.1 Squared error loss function is given by:

$$l_{sq}(\hat{\theta},\theta) = c(\hat{\theta}-\theta)^2 \tag{10}$$

2.2.2 Precautionary loss function is given by:

$$l_{pr}(\hat{\theta},\theta) = \frac{c(\hat{\theta}-\theta)^2}{\hat{\theta}} \tag{11}$$

2.2.3 Al-Bayyati's loss function is given by:

$$l_{Al}(\hat{\theta},\theta) = \theta^{c_2}(\hat{\theta}-\theta)^2; c_2 \in R^+ \tag{12}$$

2.2.4 Stein's loss function is given by:

$$l_{St}(\hat{\theta},\theta) = \frac{\hat{\theta}}{\theta} - \log\frac{\hat{\theta}}{\theta} - 1 \tag{13}$$

where $\hat{\theta}$ is the estimated value of parameter θ.

3. ESTIMATION

3.1 Maximum Likelihood Estimation

Let $\underline{v} = (v_1, v_2, v_3, ..., v_n)$ be a random sample of size n from Maxwell distribution. Let this vector represents the speeds of n randomly selected particles. Therefore the likelihood function will be given by:

$$L(\theta \mid v) = \frac{n}{2}\log\left(\frac{2}{\pi}\right) + \frac{3n}{2}\log\theta + 2\sum_{i=1}^{n}\log v_i - \frac{\theta}{2}\sum_{i=1}^{n} v_i^2 \tag{14}$$

Now, solving $\frac{\partial}{\partial\theta}\log\left\{L\left(\theta\,|\,v\right)\right\}=0$, i.e.

$$\frac{\partial}{\partial\theta}\left\{\frac{n}{2}\log\left(\frac{2}{\pi}\right)+\frac{3n}{2}\log\theta+2\sum_{i=1}^{n}\log v_i-\frac{\theta}{2}\sum_{i=1}^{n}v_i^2\right\}=0 \text{ for } \theta \text{ we obtain:}$$

$$\hat{\theta}_{mle}=3n\Big/\sum_{i=1}^{n}v_i^2 \tag{15}$$

3.2 Bayesian Estimation

3.2.1 Bayesian Estimation Assuming Parameter to Follow Extension of Jeffery's Prior:

The joint density of v and θ is given by:

$$f_1\left(v,\theta\right)=L\left(\theta\,|\,v\right)g_1\left(\theta\right)$$

$$f_1\left(v,\theta\right)=\left(\frac{2}{\pi}\right)^{\frac{n}{2}}\theta^{\left(\frac{3n}{2}-2c_1\right)}\prod_{i=1}^{n}v_i^2\exp\left(-\frac{\theta}{2}\sum_{i=1}^{n}v_i^2\right) \tag{16}$$

Marginal density of v is given by:

$$f_1\left(v\right)=\int_0^\infty f_1\left(v,\theta\right)d\theta$$

$$f_1\left(v\right)=\int_0^\infty\left(\frac{2}{\pi}\right)^{\frac{n}{2}}\theta^{\left(\frac{3n}{2}-2c_1\right)}\prod_{i=1}^{n}v_i^2\exp\left(-\frac{\theta}{2}\sum_{i=1}^{n}v_i^2\right)d\theta$$

$$f_1\left(v\right)=\frac{2^{\left(2n-2c_1+1\right)}\prod_{i=1}^{n}v_i^2\,\Gamma\left(\frac{3n}{2}-2c_1+1\right)}{\pi^{\frac{n}{2}}\left(\sum_{i=1}^{n}v_i^2\right)^{\left(\frac{3n}{2}-2c_1+1\right)}} \tag{17}$$

Posterior density of $\theta\,|\,v$ is given by:

$$\pi_1\left(\theta\,|\,v\right)\propto f_1\left(v,\theta\right)$$

$$\pi_1\left(\theta\,|\,v\right)=\frac{\theta^{\left(\frac{3n}{2}-2c_1\right)}\left(\sum_{i=1}^{n}v_i^2\right)^{\left(\frac{3n}{2}-2c_1+1\right)}\exp\left(-\frac{\theta}{2}\sum_{i=1}^{n}v_i^2\right)}{2^{\left(\frac{3n}{2}-2c_1+1\right)}\Gamma\left(\frac{3n}{2}-2c_1+1\right)} \tag{18}$$

3.2.1.1 Bayes' estimator under Squared-Error Loss Function using Extension of Jeffrey's-Prior:

Risk function under squared-error loss function using extension of Jeffrey's-prior is given by:

$$R_{(sq,\,EJ)}(\hat{\theta}) = \int_0^{\infty} c\,(\hat{\theta}-\theta)^2\,\pi_1\left(\theta \mid v\right) d\theta$$

$$R_{(sq,\,EJ)}(\hat{\theta}) = \int_0^{\infty} c\,(\hat{\theta}-\theta)^2 \frac{\theta^{\left(\frac{3n}{2}-2c_1\right)} \left(\sum_{i=1}^{n} v_i^2\right)^{\left(\frac{3n}{2}-2c_1+1\right)} \exp\left(-\frac{\theta}{2}\sum_{i=1}^{n} v_i^2\right)}{2^{\left(\frac{3n}{2}-2c_1+1\right)}\,\Gamma\left(\frac{3n}{2}-2c_1+1\right)} d\theta$$

$$R_{(sq,\,EJ)}(\hat{\theta}) = c\hat{\theta}^2 + \frac{4c\left(\frac{3n}{2}-2c_1+2\right)\left(\frac{3n}{2}-2c_1+1\right)}{\left(\sum_{i=1}^{n} v_i^2\right)^2} - \frac{4c\left(\frac{3n}{2}-2c_1+1\right)\hat{\theta}}{\sum_{i=1}^{n} v_i^2} \qquad (19)$$

Now, on solving $\frac{\partial R_{(sq,\,EJ)}(\hat{\theta})}{\partial \hat{\theta}} = 0$, for $\hat{\theta}$ we will have required Bayes' estimator given by:

$$\hat{\theta}_{(sq,\,EJ)} = \frac{3n-4c_1+2}{\sum_{i=1}^{n} v_i^2} \qquad (20)$$

3.2.1.2 Bayes' Estimator under Precautionary-Loss Function using Extension of Jeffrey's-Prior:

Risk function on assuming the combination of precautionary loss and extension of Jeffery's prior is given by:

$$R_{(pr,\,EJ)}(\hat{\theta}) = \int_0^{\infty} \frac{c\,(\hat{\theta}-\theta)^2}{\hat{\theta}}\,\pi_1\left(\theta \mid v\right) d\theta.$$

$$R_{(pr,\,EJ)}(\hat{\theta}) = \frac{1}{\hat{\theta}} \int_0^{\infty} c\,(\hat{\theta}-\theta)^2\,\pi_1\left(\theta \mid v\right) d\theta$$

$$R_{(pr,\,EJ)}(\hat{\theta}) = \frac{1}{\hat{\theta}} R_{(sq,EJ)}(\hat{\theta})$$

using eqn. (19), we get:

$$R_{(pr,\,EJ)}(\hat{\theta}) = c\hat{\theta} + \frac{4c\left(\frac{3n}{2}-2c_1+2\right)\left(\frac{3n}{2}-2c_1+1\right)}{\hat{\theta}\left(\sum_{i=1}^{n} v_i^2\right)^2} - \frac{4c\left(\frac{3n}{2}-2c_1+1\right)}{\sum_{i=1}^{n} v_i^2} \qquad (21)$$

Now, the solution of $\frac{\partial R_{(pr,EJ)}(\hat{\theta})}{\partial \hat{\theta}} = 0$ is the required Bayes' estimator and is given by:

$$\hat{\theta}_{(pr,EJ)} = \frac{\sqrt{(3n-4c_1+4)(3n-4c_1+2)}}{\sum_{i=1}^{n} v_i^2} \quad (22)$$

3.2.1.3 Bayes' Estimator under the Combination of Al-Bayyati's Loss Function and Extension of Jeffrey's-Prior:

Risk function using Al-Bayyati's loss and assuming θ to follow extension of Jeffrey's prior is given by:

$$R_{(Al,EJ)}(\hat{\theta}) = \int_0^\infty \theta^{c_2} (\hat{\theta}-\theta)^2 \pi_1(\theta \mid v)\, d\theta$$

$$R_{(Al,EJ)}(\hat{\theta}) = \frac{2^{c_2}\,\hat{\theta}^2\,\Gamma\left(\frac{3n}{2}-2c_1+c_2+1\right)}{\left(\sum_{i=1}^{n} v_i^2\right)^{c_2}\Gamma\left(\frac{3n}{2}-2c_1+1\right)} + \frac{2^{(c_2+2)}\,\Gamma\left(\frac{3n}{2}-2c_1+c_2+3\right)}{\left(\sum_{i=1}^{n} v_i^2\right)^{(c_2+2)}\Gamma\left(\frac{3n}{2}-2c_1+1\right)} - \frac{\hat{\theta}\,2^{(c_2+2)}\,\Gamma\left(\frac{3n}{2}-2c_1+c_2+2\right)}{\left(\sum_{i=1}^{n} v_i^2\right)^{(c_2+1)}\Gamma\left(\frac{3n}{2}-2c_1+1\right)}$$

Now, solving $\frac{\partial R_{(Al,EJ)}(\hat{\theta})}{\partial \hat{\theta}} = 0$ for $\hat{\theta}$, yields the required Bayes' estimator under the assumed combination of loss function and prior distribution and is given by:

$$\hat{\theta}_{(Al,EJ)} = \frac{3n-4c_1+2c_2+2}{\sum_{i=1}^{n} v_i^2} \quad (23)$$

Remark 4:

Estimator derived under Al-Bayyati's loss function and extension of Jeffrey's prior with $c_2 = 0$ is same as the estimators derived under the combination of Squared error loss function and extension of Jeffery's prior.

3.2.1.4 Bayes' Estimator under the Combination of Stein's Loss Function and Extension of Jeffrey's Prior:

Risk function using the combination of Stein's loss and extension of Jeffrey's prior is given by:

$$R_{(St,EJ)}(\hat{\theta}) = \int_0^\infty l_{St}(\hat{\theta},\theta)\pi_1(\theta \mid v)\, d\theta$$

$$R_{(St,\,EJ)}(\hat{\theta}) = \frac{\hat{\theta}\sum_{i=1}^{n} v_i^2}{2\left(\frac{3n}{2} - 2c_1\right)} - 1 - \log\hat{\theta} + m\,,$$

where m is the constant of integration.

Now, the solution of $\frac{\partial R_{(St,\,EJ)}(\hat{\theta})}{\partial\hat{\theta}} = 0$ will be the required Bayes' estimator under the considered combination of loss and prior information and is given by:

$$\hat{\theta}_{(St,\,EJ)} = (3n - 4c_1)\Big/\sum_{i=1}^{n} v_i^2 \tag{24}$$

3.2.2 Bayesian Estimation Assuming Parameter to follow Gamma (α,β) Prior:

The joint density of v and θ is given by:

$$f_2(v,\theta) = L(\theta \mid v)\, g_2(\theta)$$

$$f_2(v,\theta) = \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \theta^{\left(\frac{3n}{2}+\alpha-1\right)} \prod_{i=1}^{n} v_i^2 \exp\left(-\frac{\theta}{2}\sum_{i=1}^{n} v_i^2 - \beta\theta\right) \frac{\beta^{\alpha}}{\Gamma(\alpha)} \tag{25}$$

Marginal density of v is given by:

$$f_2(v) = \int_0^{\infty} f_2(v,\theta)\, d\theta$$

$$f_2(v) = \frac{\left(\frac{2}{\pi}\right)^{\frac{n}{2}} \prod_{i=1}^{n} v_i^2 \frac{\beta^{\alpha}}{\Gamma(\alpha)} \Gamma\left(\frac{3n}{2}+\alpha\right)}{\left(\frac{1}{2}\sum_{i=1}^{n} v_i^2 + \beta\right)^{\left(\frac{3n}{2}+\alpha\right)}} \tag{26}$$

Posterior density of $\theta \mid v$ is given by:

$$\pi_2(\theta \mid v) \propto f_2(v,\theta)$$

$$\pi_2(\theta \mid v) = \frac{f_2(v,\theta)}{f_2(v)}$$

$$\pi_2(\theta \mid v) = \frac{\theta^{\left(\frac{3n}{2}+\alpha-1\right)} \left(\frac{1}{2}\sum_{i=1}^{n} v_i^2 + \beta\right)^{\left(\frac{3n}{2}+\alpha\right)} \exp\left\{-\left(\frac{1}{2}\sum_{i=1}^{n} v_i^2 + \beta\right)\theta\right\}}{\Gamma\left(\frac{3n}{2}+\alpha\right)} \tag{27}$$

3.2.2.1 Bayes' Estimator under the Combination of Squared-Error Loss Function and Gamma-Prior:

Risk function is given by:

$$R_{(sq,\,gp)}(\hat{\theta}) = \int_0^\infty c\,(\hat{\theta}-\theta)^2\,\pi_2\left(\theta \mid v\right)d\theta$$

$$R_{(sq,\,gp)}(\hat{\theta}) = \int_0^\infty c\,(\hat{\theta}-\theta)^2\,\frac{\theta^{\left(\frac{3n}{2}+\alpha-1\right)}\left(\frac{1}{2}\sum_{i=1}^{n}v_i^2+\beta\right)^{\left(\frac{3n}{2}+\alpha\right)}\exp\left\{-\left(\frac{1}{2}\sum_{i=1}^{n}v_i^2+\beta\right)\theta\right\}}{\Gamma\left(\frac{3n}{2}+\alpha\right)}d\theta$$

$$R_{(sq,\,gp)}(\hat{\theta}) = c\hat{\theta}^2 + c\,\frac{\left(\frac{3n}{2}+\alpha+1\right)\left(\frac{3n}{2}+\alpha\right)}{\left(\frac{1}{2}\sum_{i=1}^{n}v_i^2+\beta\right)^2} - 2c\hat{\theta}\,\frac{\left(\frac{3n}{2}+\alpha\right)}{\left(\frac{1}{2}\sum_{i=1}^{n}v_i^2+\beta\right)} \tag{28}$$

Now, the required Bayes estimator will be obtained by solving $\frac{\partial R_{(sq,\,gp)}(\hat{\theta})}{\partial\hat{\theta}} = 0$ for $\hat{\theta}$ and is given by:

$$\hat{\theta}_{(sq,\,gp)} = \frac{3n+2\alpha}{\sum_{i=1}^{n}v_i^2+2\beta} \tag{29}$$

3.2.2.2 Bayes' Estimator under the Combination of Precautionary Loss Function and Gamma-Prior:

Risk function is given by:

$$R_{(pr,\,gp)}(\hat{\theta}) = \int_0^\infty \frac{c\,(\hat{\theta}-\theta)^2}{\hat{\theta}}\,\pi_2\left(\theta \mid v\right)d\theta$$

$$R_{(pr,\,gp)}(\hat{\theta}) = \frac{1}{\hat{\theta}}\,R_{(sq,\,gp)}(\hat{\theta})$$

$$R_{(pr,gp)}(\hat{\theta}) = c\hat{\theta} + c\,\frac{\left(\frac{3n}{2}+\alpha+1\right)\left(\frac{3n}{2}+\alpha\right)}{\hat{\theta}\left(\frac{1}{2}\sum_{i=1}^{n}v_i^2+\beta\right)^2} - 2c\,\frac{\frac{3n}{2}+\alpha}{\frac{1}{2}\sum_{i=1}^{n}v_i^2+\beta} \tag{30}$$

Now, the required Bayes' estimator is obtained on solving the following equation for $\hat{\theta}$ and is given by (31).

$$\frac{\partial R_{(pr,gp)}(\hat{\theta})}{\partial \hat{\theta}} = 0$$

$$\hat{\theta}_{(pr,gp)} = \frac{\sqrt{(3n+2\alpha+2)(3n+2\alpha)}}{\sum_{i=1}^{n} v_i^2 + 2\beta} \tag{31}$$

3.2.2.3 Bayes' Estimator under the Combination of Al-Bayyati's Loss Function and Gamma-Prior:

Risk function on considering Al-Bayyati's loss and Gamma prior is given by:

$$R_{(Al,gp)}(\hat{\theta}) = \int_0^\infty \theta^{c_2} (\hat{\theta}-\theta)^2 \pi_2(\theta \mid v)\, d\theta$$

$$R_{(Al,gp)}(\hat{\theta}) = \int_0^\infty \theta^{c_2} (\hat{\theta}-\theta)^2 \frac{\theta^{\left(\frac{3n}{2}+\alpha-1\right)} \left(\frac{1}{2}\sum_{i=1}^{n} v_i^2 + \beta\right)^{\left(\frac{3n}{2}+\alpha\right)} \exp\left\{-\left(\frac{1}{2}\sum_{i=1}^{n} v_i^2 + \beta\right)\theta\right\}}{\Gamma\left(\frac{3n}{2}+\alpha\right)} d\theta$$

$$R_{(Al,gp)}(\hat{\theta}) = \left(\sum_{i=1}^{n} v_i^2 + 2\beta\right)^2 \left(\frac{\hat{\theta}}{2}\right)^2 + \frac{(3n+2\alpha+2c_2+2)(3n+2\alpha+2c_2)}{4} - \hat{\theta}\frac{\left(\sum_{i=1}^{n} v_i^2 + 2\beta\right)(3n+2\alpha+2c_2)}{2} \tag{32}$$

Solving $\frac{\partial R_{(Al,gp)}(\hat{\theta})}{\partial \hat{\theta}} = 0$ for $\hat{\theta}$, we will have the required Bayes' estimator given by (33):

$$\hat{\theta}_{(Al,gp)} = \frac{3n+2\alpha+2c_2}{\sum_{i=1}^{n} v_i^2 + 2\beta} \tag{33}$$

3.2.2.4 Bayes' Estimator using Stein's Loss Function and Gamma-Prior:

Risk function is given by:

$$R_{(St,gp)}(\hat{\theta}) = \int_0^\infty l_{St}(\hat{\theta},\theta) \pi_2(\theta \mid v)\, d\theta$$

$$R_{(St,gp)}(\hat{\theta}) = \int_0^\infty \left(\frac{\hat{\theta}}{\theta} - \log\frac{\hat{\theta}}{\theta} - 1\right) \frac{\theta^{\left(\frac{3n}{2}+\alpha-1\right)} \left(\frac{1}{2}\sum_{i=1}^{n} v_i^2 + \beta\right)^{\left(\frac{3n}{2}+\alpha\right)} \exp\left\{-\left(\frac{1}{2}\sum_{i=1}^{n} v_i^2 + \beta\right)\theta\right\}}{\Gamma\left(\frac{3n}{2}+\alpha\right)} d\theta$$

$$R_{(St,\,gp)}(\hat{\theta}) = \frac{\hat{\theta}\left(\frac{1}{2}\sum_{i=1}^{n} v_i^2 + \beta\right)}{\left(\frac{3n}{2} + \alpha - 1\right)} - 1 - \log\hat{\theta} + m,$$

where m is the constant of integration.

On solving $\frac{\partial R_{(St,\,gp)}(\hat{\theta})}{\partial\hat{\theta}} = 0$ for $\hat{\theta}$, we will have required Bayes' estimator and is given by:

$$\hat{\theta}_{(St,\,gp)} = (3n + 2\alpha - 2)\Big/\left(\sum_{i=1}^{n} v_i^2 + 2\beta\right) \tag{34}$$

Now, on summarizing the above derived Bayes' estimators and their special cases, we can have the following table of estimators.

Table 1
Bayes' Estimators under Different Combinations of Loss Functions and Prior Distributions

Prior	Loss Function	Estimator
Extension of Jeffrey's	*Squared-error*	$\hat{\theta}_{(sq,\,EJ)} = \frac{3n - 4c_1 + 2}{\sum_{i=1}^{n} v_i^2}$
	Precautionary	$\hat{\theta}_{(pr,\,EJ)} = \frac{\sqrt{(3n - 4c_1 + 4)(3n - 4c_1 + 2)}}{\sum_{i=1}^{n} v_i^2}$
	Al-Bayyati's	$\hat{\theta}_{(Al,\,EJ)} = \frac{3n - 4c_1 + 2c_2 + 2}{\sum_{i=1}^{n} v_i^2}$
	Stein's	$\hat{\theta}_{(St,\,EJ)} = \frac{3n - 4c_1}{\sum_{i=1}^{n} v_i^2}$
Hartigan's ($i.e.\ c_1 = 3/2$)	*Squared-error*	$\hat{\theta}_{(sq,\,Hp)} = \frac{3n - 4}{\sum_{i=1}^{n} v_i^2}$
	Precautionary	$\hat{\theta}_{(pr,\,Hp)} = \frac{\sqrt{(3n - 2)(3n - 4)}}{\sum_{i=1}^{n} v_i^2}$
	Al-Bayyati's	$\hat{\theta}_{(Al,\,Hp)} = \frac{3n + 2c_2 - 4}{\sum_{i=1}^{n} v_i^2}$
	Stein's	$\hat{\theta}_{(St,\,Hp)} = \frac{3n - 6}{\sum_{i=1}^{n} v_i^2}$

Prior	Loss Function	Estimator
Jeffrey's ($i.e. c_1 = 1/2$)	*Squared-error*	$\hat{\theta}_{(sq, Jp)} = \frac{3n}{\sum_{i=1}^{n} v_i^2}$
	Precautionary	$\hat{\theta}_{(pr, Jp)} = \frac{\sqrt{3n(3n+2)}}{\sum_{i=1}^{n} v_i^2}$
	Al-Bayyati's	$\hat{\theta}_{(Al, Jp)} = \frac{3n+2c_2}{\sum_{i=1}^{n} v_i^2}$
	Stein's	$\hat{\theta}_{(St, Jp)} = \frac{3n-2}{\sum_{i=1}^{n} v_i^2}$
Uniform ($i.e. c_1 = 0$)	*Squared-error*	$\hat{\theta}_{(sq, Up)} = \frac{3n+2}{\sum_{i=1}^{n} v_i^2}$
	Precautionary	$\hat{\theta}_{(pr, Up)} = \frac{\sqrt{(3n+4)(3n+2)}}{\sum_{i=1}^{n} v_i^2}$
	Al-Bayyati's	$\hat{\theta}_{(Al, Up)} = \frac{3n+2c_2+2}{\sum_{i=1}^{n} v_i^2}$
	Stein's	$\hat{\theta}_{(St, Up)} = \frac{3n}{\sum_{i=1}^{n} v_i^2}$
Gamma (α, β)	*Squared-error*	$\hat{\theta}_{(sq, gp)} = \frac{3n+2\alpha}{\sum_{i=1}^{n} v_i^2 + 2\beta}$
	Precautionary	$\hat{\theta}_{(pr, gp)} = \frac{\sqrt{(3n+2\alpha+2)(3n+2\alpha)}}{\sum_{i=1}^{n} v_i^2 + 2\beta}$
	Al-Bayyati's	$\hat{\theta}_{(Al, gp)} = \frac{3n+2\alpha+2c_2}{\sum_{i=1}^{n} v_i^2 + 2\beta}$
	Stein's	$\hat{\theta}_{(St, gp)} = \frac{3n+2\alpha-2}{\sum_{i=1}^{n} v_i^2 + 2\beta}$
Chi-squared ($2\alpha\, d.f. \,\&\, \beta = 1/2$)	*Squared-error*	$\hat{\theta}_{(sq, Chi)} = \frac{3n+2\alpha}{\sum_{i=1}^{n} v_i^2 + 1}$
	Precautionary	$\hat{\theta}_{(pr, Chi)} = \frac{\sqrt{(3n+2\alpha+2)(3n+2\alpha)}}{\sum_{i=1}^{n} v_i^2 + 1}$
	Al-Bayyati's	$\hat{\theta}_{(Al, Chi)} = \frac{3n+2\alpha+2c_2}{\sum_{i=1}^{n} v_i^2 + 1}$
	Stein's	$\hat{\theta}_{(St, Chi)} = \frac{3n+2\alpha-2}{\sum_{i=1}^{n} v_i^2 + 1}$

Prior	Loss Function	Estimator
Exponential ($i.e. \alpha = 1$)	*Squared-error*	$\hat{\theta}_{(sq,\,Ep)} = \dfrac{3n+2}{\sum_{i=1}^{n} v_i^2 + 2\beta}$
	Precautionary	$\hat{\theta}_{(pr,\,Ep)} = \dfrac{\sqrt{(3n+4)(3n+2)}}{\sum_{i=1}^{n} v_i^2 + 2\beta}$
	Al-Bayyati's	$\hat{\theta}_{(Al,\,Ep)} = \dfrac{3n+2+2c_2}{\sum_{i=1}^{n} v_i^2 + 2\beta}$
	Stein's	$\hat{\theta}_{(St,\,Ep)} = \dfrac{3n}{\sum_{i=1}^{n} v_i^2 + 2\beta}$

Note: Subscript in the form of ordered pair in each estimator represents the combination of loss function and prior distribution used in the derivation of Baye's estimator. First element of the ordered pair represents loss function whereas the second element represents prior distribution.

3. SIMULATION STUDY OF MAXWELL-BOLTZMANN DISTRIBUTION

In the simulation study, data sets of size $n = 100, 200 \& 500$ have been generated from Maxwell distribution with rate parameter (θ) equal to 5 & 10. Values of hyper-parameter c_1 are taken to be 0, 0.5, 1 & 1.5. Whereas, values of another hyper-parameter c_2 are considered to be 1 & 5. We have also taken $\alpha = 1, 5$ and $\beta = 1, 5$ as the values of hyper-parameter present in Gamma prior. After carrying out simulation study with the help of programming language R, results are presented in the tables below given. Table 2 and Table 4 present estimated value of rate parameter (θ) along with it's standard error ($S.E.$) and variance respectively under two different priors. Whereas, Table 3 and Table 5 present the bias and MSE of estimators derived under extension of Jeffrey's prior and Gamma (α, β) prior respectively.

Table 3, reveals that the Baye's estimator derived under Stein's loss function totally dominates all the other estimators in terms of having least MSE. Since the bias is positive in most of the cases except under Stein's loss function, which implies that the parameter is mostly overestimated on using the extension of Jeffrey's prior. It can also be seen that on considering the Hartigan's prior (i.e. $c_1 = 1.5$), usually parameter gets underestimated and starts overestimating it as we go on increasing the parameter c_2 from 1 to 5.

Table 2
Estimates of Estimators alongwith their Standard Errors and Variances under Extension of Jeffrey's Prior

n	θ	MLE			c_1	$\hat{\theta}_{(sq,EJ)}$			$\hat{\theta}_{(pr,EJ)}$			$\hat{\theta}_{(St,EJ)}$			$\hat{\theta}_{(Al,EJ)}$ $c_2=1$			$\hat{\theta}_{(Al,EJ)}$ $c_2=5$		
		Est.	*S.E.*	*Var*		*Est.*	*S.E.*	*Var*	*Est.*	*S.E.*	*Var*	*Est.*	*S.E.*	*Var*	*Est.*	*S.E.*	*Var*	*Est.*	*S.E.*	*Var*
100	5	5.005	0.413	0.171	0	5.064	0.408	0.166	5.080	0.409	0.168	5.030	0.405	0.164	5.097	0.411	0.169	5.254	0.424	0.180
					0.5	5.005	0.413	0.171	5.022	0.415	0.172	4.972	0.410	0.168	5.039	0.416	0.173	5.182	0.408	0.167
					1	5.001	0.393	0.154	5.018	0.394	0.155	4.968	0.390	0.152	5.035	0.395	0.156	5.179	0.443	0.197
					1.5	4.958	0.392	0.154	4.975	0.394	0.155	4.925	0.390	0.152	4.992	0.395	0.156	5.122	0.439	0.193
	10	10.087	0.828	0.685	0	10.125	0.824	0.679	10.159	0.827	0.684	10.058	0.819	0.670	10.192	0.830	0.688	10.476	0.860	0.739
					0.5	10.087	0.828	0.685	10.120	0.830	0.690	10.019	0.822	0.676	10.154	0.833	0.694	10.399	0.859	0.738
					1	9.973	0.806	0.649	10.006	0.808	0.653	9.906	0.800	0.640	10.040	0.811	0.658	10.351	0.834	0.695
					1.5	9.950	0.825	0.680	9.984	0.828	0.685	9.883	0.819	0.671	10.017	0.830	0.690	10.270	0.832	0.692
200	5	5.004	0.284	0.081	0	5.042	0.296	0.088	5.050	0.297	0.088	5.025	0.295	0.087	5.059	0.297	0.088	5.116	0.290	0.084
					0.5	5.004	0.284	0.081	5.012	0.285	0.081	4.987	0.283	0.080	5.021	0.285	0.081	5.098	0.290	0.084
					1	4.998	0.291	0.085	5.006	0.291	0.085	4.981	0.290	0.084	5.014	0.292	0.085	5.091	0.283	0.080
					1.5	4.990	0.291	0.084	4.999	0.291	0.085	4.974	0.290	0.084	5.007	0.292	0.085	5.075	0.294	0.086
	10	10.016	0.608	0.369	0	10.064	0.572	0.327	10.080	0.573	0.328	10.030	0.570	0.325	10.097	0.574	0.329	10.265	0.599	0.358
					0.5	10.016	0.608	0.369	10.032	0.609	0.370	9.982	0.606	0.367	10.049	0.610	0.372	10.188	0.603	0.364
					1	9.959	0.554	0.307	9.975	0.555	0.308	9.926	0.552	0.305	9.992	0.556	0.309	10.149	0.555	0.308
					1.5	9.936	0.571	0.326	9.952	0.572	0.327	9.902	0.569	0.323	9.969	0.573	0.328	10.138	0.593	0.352
500	5	5.004	0.181	0.033	0	5.017	0.190	0.036	5.020	0.190	0.036	5.010	0.190	0.036	5.024	0.190	0.036	5.043	0.188	0.035
					0.5	5.004	0.181	0.033	5.007	0.181	0.033	4.997	0.181	0.033	5.011	0.182	0.033	5.045	0.183	0.033
					1	5.002	0.186	0.035	5.005	0.186	0.035	4.995	0.186	0.034	5.008	0.186	0.035	5.032	0.188	0.035
					1.5	5.000	0.187	0.035	5.004	0.187	0.035	4.994	0.186	0.035	5.007	0.187	0.035	5.019	0.188	0.035
	10	10.011	0.379	0.143	0	10.056	0.362	0.131	10.063	0.362	0.131	10.042	0.362	0.131	10.069	0.363	0.131	10.087	0.384	0.148
					0.5	10.011	0.379	0.143	10.018	0.379	0.144	9.998	0.378	0.143	10.025	0.379	0.144	10.077	0.376	0.141
					1	9.984	0.369	0.137	9.991	0.370	0.137	9.971	0.369	0.136	9.997	0.370	0.137	10.086	0.370	0.137
					1.5	9.985	0.361	0.130	9.992	0.361	0.131	9.972	0.361	0.130	9.998	0.362	0.131	10.057	0.381	0.145

Table 3
Bias and MSE of Estimators Derived under Extension of Jeffrey's Prior

n	θ	MLE		c_1	$\hat{\theta}_{(sq,EJ)}$		$\hat{\theta}_{(pr,EJ)}$		$\hat{\theta}_{(St,EJ)}$		$\hat{\theta}_{(Al,EJ)}$			
											$c_2=1$		$c_2=5$	
		Bias	*MSE*		*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*
100	5	0.005167	0.171027	0	0.063678	0.170055	0.080417	0.174467	0.030144	0.164909	0.097212	0.178450	0.253802	0.244415
				0.5	0.005167	0.171027	0.021823	0.172476	-0.028201	0.168795	0.038535	0.174485	0.182442	0.200285
				1	0.001217	0.154001	0.017972	0.155323	-0.032348	0.153046	0.034783	0.157210	0.178996	0.229040
				1.5	-0.041988	0.155763	-0.025266	0.155638	-0.075488	0.157698	-0.008488	0.156072	0.122400	0.207982
	10	0.100493	0.695099	0	0.130229	0.695960	0.163717	0.710803	0.063141	0.673987	0.197316	0.726934	0.436386	0.929433
				0.5	0.100493	0.695099	0.134106	0.707984	0.033157	0.677099	0.167830	0.722167	0.412831	0.908429
				1	0.049541	0.651454	0.083208	0.659924	-0.017906	0.640321	0.116987	0.671686	0.345990	0.814709
				1.5	-0.091365	0.688348	-0.057947	0.688358	-0.158316	0.696064	-0.024415	0.690596	0.264087	0.761742
200	5	0.003879	0.081015	0	0.042033	0.089767	0.050401	0.090540	0.025282	0.087639	0.058783	0.091455	0.115570	0.097356
				0.5	0.003879	0.081015	0.012212	0.081149	-0.012800	0.080164	0.020559	0.081423	0.097969	0.093598
				1	-0.002335	0.085005	0.006016	0.085036	-0.019049	0.084363	0.014380	0.085207	0.091363	0.088347
				1.5	-0.009725	0.084095	-0.001360	0.085002	-0.026471	0.084701	0.007020	0.085049	0.075321	0.091673
	10	0.027390	0.369750	0	0.108296	0.338728	0.125073	0.343643	0.074713	0.330582	0.141878	0.349129	0.204873	0.399973
				0.5	0.027390	0.369750	0.044089	0.371944	-0.006034	0.367036	0.060815	0.375698	0.196392	0.402570
				1	-0.000291	0.307000	0.016417	0.308270	-0.033735	0.306138	0.033153	0.310099	0.117086	0.321709
				1.5	-0.047383	0.328245	-0.030698	0.327942	-0.080781	0.329526	-0.013985	0.328196	0.126542	0.368013
500	5	0.003954	0.033016	0	0.016834	0.036283	0.020173	0.036407	0.010154	0.036103	0.023514	0.036553	0.042789	0.036831
				0.5	0.003954	0.033016	0.007289	0.033053	-0.002718	0.033007	0.010626	0.033113	0.044633	0.034992
				1	0.001583	0.035003	0.004921	0.035024	-0.005095	0.034026	0.008261	0.035068	0.032404	0.036050
				1.5	0.000491	0.035000	0.003833	0.035015	-0.006194	0.035038	0.007176	0.035051	0.019322	0.035373
	10	0.013990	0.143196	0	0.013990	0.131196	0.020655	0.131427	0.000656	0.131000	0.027324	0.131747	0.072354	0.153235
				0.5	0.013990	0.143196	0.020655	0.144427	0.000656	0.143000	0.027324	0.144747	0.067339	0.145535
				1	0.014518	0.137211	0.021201	0.137449	0.001148	0.136001	0.027889	0.137778	0.049880	0.139488
				1.5	-0.014973	0.130224	-0.008301	0.131069	-0.028322	0.130802	-0.001624	0.131003	0.046391	0.147152

Table 4
Estimates of Estimators alongwith their Standard Errors and Variances under Gamma (α,β) Prior

n	θ	MLE			α	β	$\hat{\theta}_{(sq,gp)}$			$\hat{\theta}_{(pr,gp)}$			$\hat{\theta}_{(St,gp)}$			$\hat{\theta}_{(Al,gp)}$					
																$c_2=1$			$c_2=5$		
		Est.	*S.E.*	*Var*			*Est.*	*S.E.*	*Var*	*Est.*	*S.E.*	*Var*	*Est.*	*S.E.*	*Var*	*Est.*	*S.E.*	*Var*	*Est.*	*S.E.*	*Var*
100	5	5.028	0.433	0.187	1	1	4.897	0.387	0.150	4.914	0.388	0.151	4.865	0.384	0.148	4.930	0.389	0.152	5.085	0.407	0.165
						5	4.328	0.299	0.089	4.342	0.300	0.090	4.299	0.297	0.088	4.356	0.301	0.090	4.494	0.312	0.098
					5	1	5.026	0.418	0.175	5.042	0.420	0.176	4.993	0.416	0.173	5.058	0.421	0.177	5.193	0.425	0.180
						5	4.443	0.312	0.098	4.457	0.313	0.098	4.414	0.310	0.096	4.471	0.314	0.099	4.599	0.320	0.103
	10	10.064	0.823	0.677	1	1	9.489	0.746	0.557	9.521	0.749	0.560	9.426	0.741	0.549	9.552	0.751	0.564	9.779	0.782	0.611
						5	7.581	0.468	0.219	7.606	0.469	0.220	7.530	0.465	0.216	7.631	0.471	0.222	7.828	0.495	0.245
					5	1	9.742	0.746	0.556	9.773	0.748	0.560	9.679	0.741	0.549	9.804	0.750	0.563	10.071	0.761	0.579
						5	7.786	0.474	0.225	7.811	0.476	0.226	7.736	0.471	0.222	7.836	0.477	0.228	8.025	0.490	0.240
200	5	5.017	0.292	0.085	1	1	4.951	0.281	0.079	4.960	0.282	0.079	4.935	0.280	0.079	4.968	0.282	0.080	5.034	0.278	0.077
						5	4.648	0.245	0.060	4.656	0.245	0.060	4.633	0.244	0.060	4.664	0.246	0.060	4.731	0.250	0.062
					5	1	5.017	0.287	0.082	5.025	0.287	0.082	5.000	0.286	0.082	5.033	0.288	0.083	5.100	0.290	0.084
						5	4.710	0.255	0.065	4.718	0.256	0.065	4.695	0.255	0.065	4.726	0.256	0.066	4.781	0.245	0.060
	10	10.012	0.567	0.321	1	1	9.733	0.542	0.294	9.749	0.543	0.295	9.701	0.540	0.292	9.766	0.544	0.296	9.904	0.563	0.317
						5	8.594	0.409	0.167	8.609	0.409	0.168	8.566	0.407	0.166	8.623	0.410	0.168	8.767	0.441	0.194
					5	1	9.849	0.539	0.291	9.865	0.540	0.292	9.817	0.538	0.289	9.881	0.541	0.293	10.000	0.559	0.312
						5	8.714	0.425	0.181	8.728	0.426	0.182	8.685	0.424	0.180	8.742	0.427	0.182	8.888	0.435	0.189
500	5	5.012	0.183	0.033	1	1	4.443	0.312	0.098	4.457	0.313	0.098	4.972	0.180	0.032	4.985	0.180	0.033	5.016	0.179	0.032
						5	4.443	0.312	0.098	4.457	0.313	0.098	4.841	0.168	0.028	4.853	0.169	0.028	4.888	0.177	0.031
					5	1	4.443	0.312	0.098	4.457	0.313	0.098	5.005	0.181	0.033	5.018	0.182	0.033	5.030	0.180	0.032
						5	4.878	0.174	0.030	4.882	0.174	0.030	4.872	0.174	0.030	4.885	0.174	0.030	4.912	0.177	0.031
	10	10.007	0.364	0.132	1	1	9.885	0.350	0.123	9.892	0.350	0.123	9.872	0.350	0.122	9.898	0.351	0.123	9.953	0.361	0.130
						5	9.397	0.322	0.104	9.403	0.322	0.104	9.385	0.321	0.103	9.410	0.322	0.104	9.478	0.325	0.106
					5	1	9.941	0.357	0.127	9.948	0.357	0.127	9.928	0.356	0.127	9.954	0.357	0.128	10.019	0.3517	0.123
						5	9.437	0.324	0.105	9.443	0.324	0.105	9.425	0.324	0.105	9.450	0.325	0.105	9.504	0.334	0.111

Table 5
Bias and MSE of Estimators Derived under Gamma (α,β) Prior

n	θ	MLE		α	β	$\hat{\theta}_{(sq,gp)}$		$\hat{\theta}_{(pr,gp)}$		$\hat{\theta}_{(St,gp)}$		$\hat{\theta}_{(Al,gp)}$			
												$c_2=1$		$c_2=5$	
		Bias	*MSE*			*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*
100	5	0.027880	0.187777	1	1	-0.102515	0.160509	-0.086325	0.158452	-0.134949	0.166211	-0.070081	0.156911	0.084729	0.172179
					5	-0.672420	0.541149	-0.658114	0.523114	-0.701080	0.579513	-0.643761	0.504428	-0.505877	0.353912
				5	1	0.025809	0.175666	0.041995	0.177764	-0.006615	0.173044	0.058234	0.180391	0.193495	0.217440
					5	-0.557386	0.408679	-0.543078	0.392934	-0.586048	0.439452	-0.528724	0.378549	-0.401404	0.264125
	10	0.063752	0.681064	1	1	-0.510816	0.817933	-0.479446	0.789868	-0.573658	0.878084	-0.447973	0.764680	-0.221390	0.660014
					5	-2.419357	6.072288	-2.394297	5.952658	-2.469560	6.314727	-2.369154	5.834891	-2.172315	4.963952
				5	1	-0.258452	0.622797	-0.227078	0.611564	-0.321300	0.652234	-0.195603	0.601261	0.070609	0.583986
					5	-2.214125	5.127350	-2.189050	5.017940	-2.264357	5.349313	-2.163894	4.910437	-1.975190	4.141376
200	5	0.017237	0.085297	1	1	-0.048552	0.081357	-0.040334	0.080627	-0.065002	0.083225	-0.032102	0.081031	0.033504	0.078123
					5	-0.351774	0.183745	-0.344059	0.178377	-0.367217	0.194848	-0.336332	0.173119	-0.269276	0.134510
				5	1	0.016680	0.082278	0.024897	0.082620	0.000231	0.082000	0.033128	0.084097	0.099879	0.093976
					5	-0.289877	0.149029	-0.282161	0.144615	-0.305320	0.158220	-0.274434	0.141314	-0.218609	0.107790
	10	0.011994	0.321144	1	1	-0.266737	0.365149	-0.250582	0.357791	-0.299073	0.381445	-0.234400	0.350943	-0.095684	0.326155
					5	-1.405638	2.142818	-1.391374	2.103922	-1.434191	2.222904	-1.377085	2.064363	-1.233067	1.714454
				5	1	-0.150855	0.313757	-0.134722	0.310150	-0.183148	0.322543	-0.118563	0.307057	0.000119	0.312000
					5	-1.286163	1.835215	-1.271890	1.799704	-1.314733	1.908523	-1.257593	1.763540	-1.112365	1.426356
500	5	0.011962	0.033143	1	1	-0.557386	0.408679	-0.543078	0.392934	-0.028223	0.032797	-0.014965	0.033224	0.015850	0.032251
					5	-0.557386	0.408679	-0.543078	0.392934	-0.159473	0.053432	-0.146565	0.049481	-0.111961	0.043535
				5	1	-0.557386	0.408679	-0.543078	0.392934	0.005200	0.033027	0.018477	0.033341	0.030386	0.032923
					5	-0.121537	0.044771	-0.118307	0.043997	-0.127999	0.046384	-0.115076	0.043242	-0.088322	0.038801
	10	0.007298	0.132053	1	1	-0.115035	0.136233	-0.108456	0.134763	-0.128198	0.138435	-0.101873	0.133378	-0.046530	0.132165
					5	-0.602890	0.467476	-0.596636	0.459975	-0.615403	0.481721	-0.590377	0.452545	-0.522289	0.378786
				5	1	-0.058806	0.130458	-0.052225	0.129727	-0.071973	0.132180	-0.045639	0.130083	0.018505	0.123342
					5	-0.562816	0.421762	-0.556568	0.414768	-0.575315	0.435987	-0.550316	0.407848	-0.495736	0.356754

From Table 5, it is apparent from bias being negative that the parameter is underestimated on assuming it to follow Gamma prior. It can also be seen that the estimators derived under the Al-Bayyati's loss function possess the least MSE followed by Precautionary, Squared error and Stein's loss function respectively.

4. FITTING TO REAL LIFE DATA SET

In this section, we have fitted Maxwell distribution to a real life data set consisting of 55 observations related to burning velocity of different chemicals. Burning velocity is the velocity of a laminar flame under stated conditions of composition, temperature, and pressure. It decreases with increasing inhibitor concentration and can be determined by analyzing the pressure-time profiles in the spherical vessel and checked by direct observation of flame propagation. The data related to the burning velocity (in cm/sec) of different chemical materials is given below:

68.2, 61.1, 64.5, 55.4, 51.4, 68.9, 44.6, 82.6, 60.2, 89.3, 61.2, 54.3, 166.1, 66.2, 50.6, 87.5, 48.3, 42.3, 58.4, 46.6, 67.3, 46.2, 46.1, 44.0, 48.3, 56.3, 47.1, 54.02, 47.7, 80.7, 38.2, 108.1, 46.2, 40.3, 44.0, 312.21, 41.2, 31.1, 40.9, 41.5, 40.2, 56.3, 45.2, 43.2, 46.6, 46.7, 46.0, 46.21, 52.3, 58.1, 82.5, 71.4, 48.6, 39.8, 41.3.

After the fitting of Maxwell distribution to the burning velocity data set, estimates of different derived estimators along with their MSE are presented in Table 6 and Table 7 respectively for two different prior distributions (i.e. extension of Jeffrey's prior and Gamma prior).

Table 6

Bias and MSE of Estimators Derived under Extension of Jeffrey's prIor for real Life Data Set (Burning Velocity)

n	MLE		c_1	$\hat{\theta}_{(sq,EJ)}$		$\hat{\theta}_{(pr,EJ)}$		$\hat{\theta}_{(St,EJ)}$		$\hat{\theta}_{(Al,EJ)}$			
										$c_2=1$		$c_2=5$	
	Est.	*MSE*		*Est.*	*MSE*	*Est.*	*MSE*	*Est.*	*MSE*	*Est.*	*MSE*	*Est.*	*MSE*
55	0.000558	0.1858322	0	0.000565	0.1851893	0.000569	0.1848773	0.000558	0.1858322	0.000572	0.1845691	0.000599	0.1823074
			0.5	0.000558	0.1858322	0.000562	0.1855089	0.000552	0.1864983	0.000565	0.1851893	0.000592	0.1828408
			1	0.000552	0.1864983	0.000555	0.1861633	0.000545	0.1871878	0.000558	0.1858322	0.000585	0.1833952
			1.5	0.000545	0.1871878	0.000548	0.1868411	0.000538	0.1879014	0.000552	0.1864983	0.000579	0.1839712

Table 7

Bias and MSE of Estimators Derived under Gamma(α,β) Prior for Real Life Data Set (Burning Velocity)

n	MLE		α	β	$\hat{\theta}_{(sq,gp)}$		$\hat{\theta}_{(pr,gp)}$		$\hat{\theta}_{(St,gp)}$		$\hat{\theta}_{(Al,gp)}$			
											$c_2=1$		$c_2=5$	
	Est.	*MSE*			*Est.*	*MSE*	*Est.*	*MSE*	*Est.*	*MSE*	*Est.*	*MSE*	*Est.*	*MSE*
55	0.000558	0.185832	1	1	0.00056516	0.185190	0.00056853	0.184878	0.00055839	0.185833	0.00057192	0.184569	0.00059900	0.182308
				5	0.00056514	0.185191	0.00056851	0.184879	0.00055837	0.185834	0.00057191	0.184571	0.00059898	0.182309
			5	1	0.00059223	0.182841	0.00059560	0.182573	0.00058546	0.183396	0.00059900	0.182308	0.00062607	0.180378
				5	0.00059221	0.182842	0.00059559	0.182574	0.00058545	0.183397	0.00059898	0.182309	0.00062605	0.180379

Table [6-7] show that for fitting Maxwell distribution to a considered real life data set, Al-Bayyati's loss function proves to be better in comparison to the other three loss functions considered due to it's least possession of MSE.

6. CONCLUSIONS

Estimators of rate parameter (θ) of Maxwell distribution derived under different combinations of loss functions and prior distributions have been compared in terms of MSE after the fitting of Maxwell distribution to a simulated and a real life data set. Programming language R has been used to derive the results which are presented in Table [2-7]. It is revealed that on assuming θ to follow Extension of Jeffrey's prior, Stein's loss function totally dominates all the other estimators in terms of having least MSE. Whereas, on considering the Gamma (α, β) prior, Al-Bayyati's loss function provides the least MSE followed by precautionary, squared error and Stein's loss function respectively. It is also observed that on fitting Maxwell distribution to considered real life data set, it is better to consider Al-Bayyati's loss function in comparison to the other three considered loss functions due to it's least possession of MSE. Decrease in MSE with increase in sample size is also apparent. It can also be concluded that for simulated data sets, parameter gets overestimated on assuming it to follow Extension of Jeffrey's prior. Whereas parameter is underestimated on assuming it to follow Gamma prior. Overall twenty eight different Bayes' estimators have been derived for the rate parameter (θ) of Maxwell distribution and are given in Table 1.

ACKNOWLEDGMENT

The authors are highly thankful to referees and the editor for their valuable suggestions.

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